Math 105: Sequences Practice Problem Solutions

1. Find the next few terms of each of the following sequences. For each, briefly describe the recursive rule for the sequence:

Answers:
3, 7, 11, 15, 19, 23, 27, $\mathbf{31, 35, 39, \dots}$
Recursive Rule: Each term is four more than the preceding term.
Equation: $x_n = x_{n-1} + 4$

2, 5, 11, 23, 47, 95, $\mathbf{191, 383, 767, \dots}$
Recursive Rule: Each term is one more than double the preceding term.
Equation: $x_n = 2x_{n-1} + 1$

1, 3, 4, 7, 11, 18, 29, $\mathbf{47, 76, 123, \dots}$
Recursive Rule: Each term is the sum of the two preceding terms. (This is similar to the Fibonacci sequence; in fact, the recursive rule is actually the same, but the first two terms are different.)
Equation: $x_n = x_{n-1} + x_{n-2}$

1, 3, 6, 10, 15, 21, $\mathbf{28, 36, 45, \dots}$
Recursive Rule: The difference between consecutive terms is increasing by one at each step. (Alternative rule: the n^{th} term of this sequence is the sum of the first n counting numbers – e.g., 6 = 1+2+3, 10=1+2+3+4, 15=1+2+3+4+5, etc.)
The equation here is a little tricky. Here's the pattern: the 2^{nd} term is 2 more than the 1^{st} term; the 3^{rd} term is 3 more than the 2^{nd} term; the 4^{th} term is 4 more than the 3^{rd} term; etc. In general, then, the nth term is n more than the preceding term. This gives us the equation: $x_n = x_{n-1} + n$.

2. Write the first ten terms of each sequence, given the recursive rule for the sequence

1, 1, 3, 5, 11, 21, … where $x_n = x_{n-1} + 2x_{n-2}$
Answer: The recursive rule is telling us that each term is the sum of the preceding term and twice the term that came before that. So, for example, the next term in the sequence after the terms 11, 21, would be $21 + 2 \cdot 11 = 43$. The next term, which comes after 21, 43, would be $43 + 2 \cdot 21 = 85$. Continuing with this rule, here are the first ten terms of the sequence… 1, 1, 3, 5, 11, 21, 43, 85, 171, 341

1, 2, 5, 14, … where $x_n = 3x_{n-1} - 1$
Answer: The recursive rule here is that each term is one less than three times the preceding term. (Note that the only subscript involved is n-1 – there is no n-2 term this time – so each new term in the sequence depends only on the immediately preceding term, and not on any terms that came before that.) So, for example, the term following 14 in this sequence would be $3 \cdot 14 - 1 = 41$. The term after 41 would be $3 \cdot 41 - 1 = 122$. Continuing with this rule, here are the first ten terms of the sequence… 1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842

1, 1, 1, 3, 5, 9, …, where $x_n = x_{n-1} + x_{n-2} + x_{n-3}$
Answer: The recursive rule here is that each new term is the sum of the preceding three terms. (That's what the subscripts n-1, n-2, and n-3 are telling us.) So, for example, the next term after 3,5,9 in the sequence will be $3 + 5 + 9 = 17$. The next term after that, which will come after 5, 9, 17, will be $5 + 9 + 17 = 31$. And so on; continuing with this rule, here are the first ten terms of the sequence… 1, 1, 1, 3, 5, 9, 17, 31, 57, 105

3. Let b_n stand for the number of n-beat rhythms (not melodies – just rhythms this time) we can write under the rule that every note is either a quarter note (one beat) or a dotted half note (3 beats).

a) Write out all possible rhythms for n = 1, 2, 3, 4, 5 and 6. Use your results to find the values of b_1, b_2, b_3, b_4, b_5 and b_6. (Hint: none of these should be a very large number.)

Answers: 1, 1, 2, 3, 4, 6

b) See if you can figure out, and explain, a recursive rule for the sequence of numbers b_n. (Hint: the reasoning for this one will be *similar* to the reasoning for the example from class in which we restricted ourselves to only quarter notes and half notes.)

Answer: $b_n = b_{n-1} + b_{n-3}$ -- that is, each term is equal to the preceding term plus the term that came three places earlier. (This is similar to the Fibonacci rule, except that instead of using the term that came two places earlier in the sequence, we use the term that came three places earlier.

Rationale: to write an n-beat rhythm, either add a quarter note to the end of an n-1 beat rhythm, or add a dotted half note to the end of a n-3 beat rhythm.

4. Let a_n stand for the number of n-beat melodies we can write with the following rules:

- every note is a quarter note (one beat) or a half note (two beats), and
- every note is a C or a D.

a. Find all possible 1-beat, 2-beat and 3-beat melodies under these rules.
This will give you the values of $a_1, a_2,$ and a_3 .
Answers: $a_1 = 2, a_2 = 6, a_3 = 16$

Parts b. and c. will be answered together (see below)

b. Based on your results for #3, see if you can infer the recursive rule for the sequence of terms a_n – that is, a rule which will let you predict a_4, a_5, etc. without actually having to find all of the possible melodies.

c. Once you've found a recursive rule (if you find one) from #4, see if you can figure out *why* it works – that is, how and why the number of n-beat melodies depends on the number of n-1 beat melodies and/or n-2 beat melodies.

Answer: The recursive rule for this sequence, starting with the third term $(a_3 = 16)$, the rule is: $a_n = 2a_{n-1} + 2a_{n-2}$. This means that the next few terms will be:

$$a_4 = 2a_3 + 2a_2 = 2 \cdot 16 + 2 \ \cdot 6 = 44$$
$$a_5 = 2a_4 + 2a_3 = 2 \cdot 44 + 2 \cdot 16 = 120$$
$$a_6 = 2a_5 + 2a_4 = 2 \cdot 120 + 2 \cdot 44 = 328$$

…and so on.

The reasoning is as follows: suppose we wish to write an n-beat melody under the given rules. We know that, no matter how we write our melody, the last note of the melody must be either a quarter note or a half note, and it must be a C or a D. If the last note is a quarter note, then we have an n-1 beat melody followed by a C or a D quarter note; this gives us two new n-beat melodies for every n-1 beat melody we had found previously. If the last note in our melody is a half note, then we have an n-2 beat melody followed by a C or a D half note; this gives us two new n-beat melodies for every n-2 beat melody we had found previously. Therefore, to find the number of n-beat melodies, we just need to find two times the number of n-1 beat melodies, and add two times the number of n-2 beat melodies. This gives us the recursive rule stated above.