Math 105 – Spring 2017
Test #2 – Review Sheet

This document summarizes concepts you should understand, and some of the types of problems you should be able to solve, for the test that will be given on Wednesday, April 2.

Musical Variations

Definitions:
T_n = Transposition by n semitones. (upward if n is positive, downward if n is negative). This has the effect of preserving all intervals in the original melody, resulting in a variation that is recognizably similar to the original.
I = Inversion (centered at C) – "reflects" each note across the note C. This variation also preserves all intervals in the original melody, but reverses their directions.
R = Retrograde – simply reverses the order of the notes in the melody.

Octaves: by convention, we consider notes an octave apart to be "equivalent" when dealing with variations. That is, transposition up (or down) 12 semitones is considered to be the same as transposition by 0 semitones, since the results differ by an octave. This results in a "mod 12" rule for combination of transpositions, as described in class.

We developed a few basic rules that determine how variations can be combined:

- $T_n T_m = T_{n+m}$ -- where the addition is under mod 12 rules (corresponding to the 12-tone scale)
- $RR = T_0$ -- the retrograde is its own opposite
- $II = T_0$ -- the inversion is its own opposite
- $RT_n = T_n R$ – transpositions and retrogrades commute with one another.
- $IR = RI$ -- inversions and retrogrades commute with one another.
- $IT_n = T_{-n} I$ -- inversions and transpositions do *not* generally commute with one another.

With these rules, we may rearrange any combination of variations into one of the following four forms:

- T_n -- a transposition
- $T_n I$ -- a transposition followed by an inversion
- $T_n R$ -- a transposition followed by a retrograde
- $T_n IR$ -- a transposition followed by a retrograde inversion

… where, in each case, n may take on any value from 0 through 11.

This gives us four distinct sets of variations, each containing twelve distinct variations (one for each value of n from 0 through 11). Thus, there are $4 \times 12 = 48$ different possible variations on a theme, if we restrict ourselves to transpositions, retrogrades and inversions.

When asked to "simplify" a combination of several variations, this means we should write the combined result of these variations as a single variation, written in one of the forms listed above – $T_n, T_n R, T_n I$, or $T_n IR$.

"Musical Clock" –The 12 notes of the 12 tone scale can be organized into a circle, or "clock" (with C at the top, then proceeding clockwise to C#, D, D#, etc.) As demonstrated in class, this can be useful in finding transpositions and inversions.

Modular arithmetic

When working with arithmetic operations (such as addition or multiplication), the designation "mod n" (short for "modulo n") indicates that we perform arithmetic subject to the following rules: if our result is greater than n-1, we subtract n from the result (multiple times, if necessary); on the other hand, if our result is less than 0, we add n to the result (multiple times, if necessary). For any computation under "mod n" arithmetic, all results are considered equivalent to one of the numbers 0, 1, 2, … , n-1. For example, when we combine transpositions, the result is found using addition with "mod 12" rules.

Groups

In mathematics, a collection of objects (such as our variations) that can be combined according to one or more rules is called a "group" if it satisfies the following conditions:

- **Identity**: One of the objects in the collection serves as the "identity" object of the collection. This is an object which, when combined with any other object in the group, leaves that object unchanged.

- **Closure:** Whenever any two objects in the collection are combined, the result is always an object in the same collection. In other words, you can't "escape" from your collection by combining elements.

- **Opposites:** For any object in the collection, there is some "opposite" object in the collection that can be combined with the original object in order to end up with the collection's "identity" object.

(A fourth criterion, "associativity," is also technically required to be a group; however, all examples that we will consider have this property, so you will not need to deal with it directly.)

The set of 48 variations is an example of a group: it has an identity (T_0), it is closed, and every variation has an opposite, summarized below (each of the following is true for all values of n):

- The opposite of T_n is T_{12-n}.
- The opposite of T_nR is $T_{12-n}R$.
- T_nI is its own opposite.
- T_nIR is its own opposite.

Note on groups: Recall that we looked at several non-musical examples of "groups" in class. These may come up on the test as well. When asked whether a set, with a certain operation, is a "group," the question is whether the given example satisfies the three criteria for a "group" listed above – closure, identity, and opposites.

Subgroups and Cosets: A "subgroup" is a smaller group that is contained in a larger group. A "coset" of that subgroup is a different set, also contained in the larger group, which is found by combining some specific element of the group with each element of the subgroup. This is called the "coset generated by" the element that was combined with each element of the subgroup. Several examples of subgroups and cosets were discussed in class and given as practice problems.

Cyclic subgroup: Any element of a group can be combined with itself repeatedly to generate a subgroup of the group in which it resides. A subgroup found in this way is called a "cyclic subgroup," or "the subgroup generated by x" where x is the group element being combined with itself to generate the subgroup. The number of elements in the cyclic subgroup generated by a group element is called that element's "order" in the group. The notation $<x>$ is shorthand for the cyclic subgroup generated by x.

Two important connections between cosets and subgroups:

First, each coset of a subgroup has the same number of elements as the subgroup itself, and none of the cosets overlap; as a result, the number of cosets of a subgroup is the size (number of elements) of the larger group divided by the size of the subgroup.

Second: the above observation tells us that the size of a subgroup must be a divisor of the size of the larger group. In other words, if we select a set of elements from a group, and the number of elements in the set *isn't* a divisor of the number of elements in that group, then the set cannot be a subgroup of that group (since it would, as a result, end up with a fractional number of cosets, which is impossible.)

Summary of the above observations:

- If H is a subgroup of G, where G contains n elements and H contains k elements, then H has n/k cosets in G. (This implies n/k is a whole number; that is, k is a divisor of n.)

- If we select a set of k elements from a group, G, where k is *not* a divisor of n (so that n/k is *not* a whole number), then this set cannot possibly be a subgroup of G. (This gives us, in some cases, a quick way of ruling out possible subgroups.)

Test preparation: In addition to going over your notes, and reviewing the collected homework problems, you should work through all of the sets of practice exercises that have been given since we started the "Variations and Groups" unit of the course. These are all posted on the class notes section of the class web page. Test questions will generally be very similar to the problems found in these sets of practice exercises.

For example, you should be able to do each of the following types of problems:

- Given a melody (as a list of notes from the 12-tone scale), be able to find a requested variation of that melody. You should be familiar with, and be able to find, transpositions, inversions, and/or retrogrades for any given melody. Conversely, given an original melody and a variation on that melody, be able to work out what variation was applied to the original melody to get the new one.

- Given some combination of variations (transpositions, inversions, and/or retrogrades), be able to simplify the combination so that it can be written in the form T_n, T_nI, or T_nIR, where n is a number in the range 0, 1, 2, …, 10, 11.

- Be able to do "mod n" arithmetic (addition, subtraction, multiplication) for any given whole number n.

- Given a set, and an operation on that set, determine if the set is a group under that operation. (That is, show that the identity, closure, and opposites criteria are all satisfied.) If it is a group, be able to explain why; if it is not a group, give a specific example to show why.

- Given a group, find the identity, and find the opposite of any other element of the group.

- Given a group, find the cyclic subgroup generated by any element of that group.

- Given a subgroup, be able to find the cosets of the subgroup, and/or to figure out how many distinct cosets the subgroup must have.

This is not necessarily an exhaustive list – it's just what I found with a quick look through the practice exercise sets on the class web page – but it's a pretty good summary of the main ideas. If you're prepared for questions like those described above (equivalently, if you've worked through the sets of practice exercises), then you will be in good shape for the test.

Final thoughts:

- This document is meant to *summarize* what we've covered to this point since the first test. While it is (hopefully) pretty thorough, I do not guarantee that it is exhaustive. In general, anything we covered in class, or anything that was covered in the assigned reading, after the first test is fair game for the second test.

- Calculators will not be allowed for this test. There won't be any problems on the test for which a calculator should be necessary. (Note: this is the only test on which calculators will not be allowed; you will need your calculator for the third test and for the final exam.)