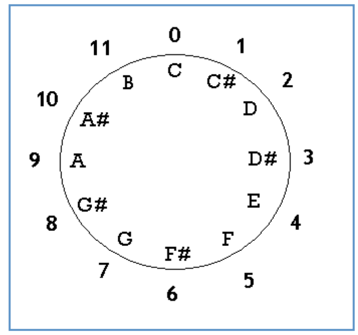
Math 105: Music & Mathematics
Test #2 Solutions & Comments

1. For each part of this problem, consider the melody consisting of the notes: **C E A G F D.**

For each of the following, find the result of applying the given variation to the above melody.

a) T_7



Answer: G B E D C A

b) IT_5

Answer: F C# G# A# C D#

Comment: A quick way to solve this would b to first rewrite IT_5 as T_7I, and then just find the inversion of your answer to part (a). (Finding first I, thenT_5, also works.)

c) T_{40}

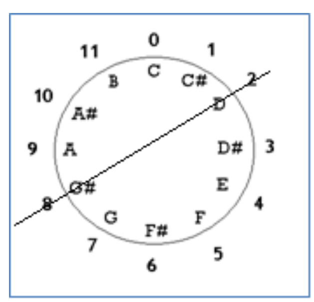
Answer: E G# C# B A F#

Comment: Since $40 = 3\times12 + 4$, T_{40} is equivalent to T_4.

d) I_D (an inversion centered at D rather than C)

Answer: E C G A B D

Comment: Recall that a way to find inversions about notes other than C is to draw a different "mirror" line through the middle of the musical clock; in this case, we'd draw it so that it passes through D instead of C.



2. Simplify each of the following combinations of variations. Write each answer in one of the following forms: $T_n, T_nR, T_nI,$ or T_nIR, with n between 0 and 11. Show your work.

a) $T_8RT_1RT_5R$

Answer: T_2R

b) $T_8IT_1IT_5$

Answer: T_0

Note: Don't forget the $IT_n = T_{-n}I$ rule…

$$T_8I\ T_1I\ T_5 = T_8T_{-1}IIT_5 = T_7T_0T_5 = T_{12} = T_0$$

c) IRT_2RIT_4

Answer: T_2

3. Determine whether each of the following sets, with the given operation, is a group. Justify your answers.

a) {0, 10, 5} under mod 15 addition

Answer: This is a group. A table demonstrates closure. The numbers 10 and 5 are opposites, while 0 is its own opposite.

Table:

+	0	5	10
0	0	5	10
5	5	10	0
10	10	0	5

b) $\{T_0,\ T_4R,\ T_8\}$ using the usual rules for combining variations

Answer: This is not a group. Here are a couple of examples illustrating why:

- The opposite of T_8 is T_4, which is not in this set.
- When we comine T_8 with T_4R, we get $T_{12}R = R$, which is not in this set.

Either of these examples would be sufficient to show that the set is *not* a group.

4. Consider the group {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} under mod 11 multiplication
(Note: this *does* turn out to be a group. You are *not* being asked to verify this!)

a) Find the cyclic subgroup ⟨3⟩.

Answer: Repeatedly multiplying by 3 (under mod 11 rules) gives us {3, 9, 5, 4, 1}.

b) Find the cyclic subgroup ⟨5⟩.

Answer: Repeatedly multiplying by 5 gives us {5, 3, 4, 9, 1}.

5. Consider the group of all transpositions: $\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}\}$

a. Show that $\{T_0, T_3, T_6\}$ is *not* a subgroup of this group.

Answer: This set is a not a subgroup because it is not closed: $T_3T_6 = T_9$, which is not in the set.

Alternative answer: The opposite of T_3 is T_9, but T_9 is not in the set.

(Note: If T_9 were added to the set, it would become a subgroup.)

b. Show that $\{T_0, T_8, T_4\}$ *is* a subgroup of this group.

Answer: This set is a group since it contains the identity (T_0), it is closed (since $T_4T_8 = T_0$, etc.), and every element has an opposite in the set (T_0 is its own opposite, and T_4, T_8 are opposites of each other. Since it's a group, it forms a subgroup of the larger group that contains it.

Table:

	T_0	T_8	T_4
T_0	T_0	T_8	T_4
T_8	T_8	T_4	T_4
T_4	T_4	T_0	T_8

c. Find all of the coset(s) of $\{T_0, T_8, T_4\}$ in this group.
Answer: Since the original group has 12 elements, and the subgroup has 3 elements, there must be $12 \div 3 = 4$ cosets. They are:

- $\{T_0, T_8, T_4\}$, the original subgroup
- $\{T_1, T_9, T_5\}$, the subgroup generated by T_1
- $\{T_2, T_{10}, T_6\}$, the subgroup generated by T_2
- $\{T_3, T_{11}, T_7\}$, the subgroup generated by T_3

6. How many elements are there in each of the following groups? Briefly explain each of your answers. (Note: to receive full credit, your explanation must consist of more than “I remember this from class”…)

a. The cyclic subgroup $\langle T_3 I \rangle$

Answer: Since $T_3 I \; T_3 I$ simplifies to T_0, the cyclic subgroup $\langle T_3 I \rangle$ consists of only two elements: $T_3 I$ and T_0

b. The group of *all* possible variations involving transpositions, inversions, and/or retrogrades

Answer: There are 48 such variations in all. After simplifying any combination of variations, the result is always of one of the four forms: $T_n, T_n I, T_n R$, or $T_n IR$. Since n can take on any value from 0 to 11 (12 possibilities), there is a total of $4 \times 12 = 48$ distinct variations.

c. The group of all actions on a square (that is, the number of different ways a square can be moved) generated by 90-degree right-rotations and/or horizontal flips

There are 8 such actions. (Recall that this is the “dihedral group of order 8” discussed in class.) This is because there are four distinct rotations; also, if we flip the square first, we get four more distinct rotations. As was demonstrated in class, every possible combination of right-rotations and horizontal flips is equivalent to one of these 8 actions.