Math 201, Test #1A
Solutions

1. For this problem,

$$f(x) = \frac{2x^2 - 5x + 2}{x^2 - 4}.$$

Find each limit, if it exists. If a limit does not exist, write DNE.

(a) $\lim_{x \to 2} f(x)$

Solution: First, note that $f(x)$ is not continuous at $x = 2$, since $f(2)$ is undefined (zero denominator). However, both its numerator *and* its denominator are zero at $x = 2$, which implies the limit as $x \to 2$ may exist.

Since the numerator and denominator are both polynomials with $x = 2$ as a root, it must be the case that $x - 2$ is a common factor. Thus, we can factor both polynomials, which allows us to simplify the expression for $f(x)$:

$$\begin{aligned} f(x) &= \frac{2x^2 - 5x + 2}{x^2 - 4} \\ &= \frac{(2x-1)(x-2)}{(x+2)(x-2)} \\ &= \frac{2x-1}{x+2}, x \neq 2 \end{aligned}$$

Therefore,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x-1}{x+2} = \frac{2(2)-1}{(2)+2} = \frac{3}{4}.$$

(b) $\lim_{x \to -2} f(x)$

Solution: This limit does not exist (DNE). This is because the denominator approaches zero as $x \to -2$, but the numerator does *not* approach zero as $x \to -2$. Thus, there is no real number that $f(x)$ approaches as $x \to -2$.

Note: It would not be correct to say that the limit is ∞, since the left- and right-hand limits are different. To see this, look at the simplified expression for $f(x)$, $\frac{2x-1}{x+2}$.

As $x \to -2$, the numerator approaches the value -5, which is negative. If $x \to -2$ from the left, we have $x + 2 < 0$, which means the fraction's value is positive (negative divided negative); thus, the left-hand limit of $f(x)$, as $x \to -2^-$, is $+\infty$.

On the other hand, as $x \to -2$ from the right, $x + 2 > 0$, which (by similar reasoning to that in the preceding paragraph) implies the right-hand limit of $f(x)$, as $x \to -2^+$, is $-\infty$.

Since the left- and right-hand limits are not the same (one is positive, the other negative), our conclusion must be that the limit does not exist.

(c) $\lim_{x \to \infty} f(x)$

Solution: The short-cut solution (which I accepted for this problem) is to find the ratio of the leading terms on the top and bottom. In this case, we'd have $2x^2/x^2 = 2$, which implies a horizontal asymptote of $y = 2$. Therefore, $\lim_{x \to \infty} f(x) = 2$.

More formally, we can prove $\lim_{x \to \infty} f(x) = 2$ algebraically:

$$\begin{aligned}
\lim_{x \to \infty} \frac{2x^2 - 5x + 2}{x^2 - 4} &= \lim_{x \to \infty} \frac{x^2 \left(2 - \frac{5}{x} + \frac{2}{x^2}\right)}{x^2 \left(1 - \frac{4}{x^2}\right)} \\
&= \lim_{x \to \infty} \frac{2 - \frac{5}{x} + \frac{2}{x^2}}{1 - \frac{4}{x^2}} \\
&= \frac{\lim_{x \to \infty} \left(2 - \frac{5}{x} + \frac{2}{x^2}\right)}{\lim_{x \to \infty} \left(1 - \frac{4}{x^2}\right)} \\
&= \frac{2 - 0 + 0}{1 - 0} \\
&= 2
\end{aligned}$$

(d) $\lim_{x \to \infty} 3^{f(x)}$

Solution: We know that all compositions of "elementary" functions (including exponential functions) are continuous on their domains. Further, we know that, for continuous functions f and g, $\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right)$.

Since (from part (c)) $\lim_{x \to \infty} f(x) = 2$, it follows that the limit of $3^{f(x)}$ as $x \to \infty$ must be 3^2. That is,

$$\lim_{x \to \infty} 3^{f(x)} = 3^{\lim_{x \to \infty} f(x)} = 3^2 = 9$$

2. If an object is dropped from the top of a 100-meter-tall building, its height after t seconds is given by the function $f(t) = 100 - 5t^2$ (meters).

   (a) Find the object's average velocity for the time period from $t = 0$ to $t = 4$.

   Solution: Average velocity is displacement divided by time elapsed. Over the time span from $t = 0$ to $t = 4$, 4 seconds have elapsed, and the displacement is $f(4) - f(0) = 20 - 100 = -80$. Therefore, the average velocity is $v = \dfrac{-80}{4} = -20$ meters per second.

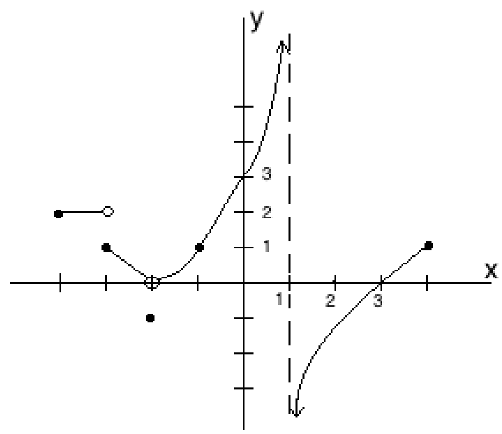
   (b) Find the object's instantaneous velocity at time $t = 4$.

   Solution: Instantaneous velocity is given by the derivative of the position function:

$$\begin{aligned}
f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\
&= \lim_{x\to 4} \frac{(100 - 5x^2) - 20}{x - 4} \\
&= \lim_{x\to 4} \frac{80 - 5x^2}{x - 4} \\
&= \lim_{x\to 4} \frac{5(16 - x^2)}{x - 4} \\
&= \lim_{x\to 4} \frac{5(4 - x)(4 + x)}{x - 4} \\
&= \lim_{x\to 4} -5(4 + x) \\
&= -5(4 + 4) = -40.
\end{aligned}$$

   Therefore, the object's instantaneous velocity at time $t = 4$ is -40 meters per second.

3. The following diagram shows the graph of $y = f(x)$. The dashed line at $x = 1$ is a vertical asymptote. Each point represented by a dot has integer coordinates.



Note: For part (a), no explanations are required. For (b) and (c), write your answers in the space provided; use the back of this page if you need more room.

(a) Find each of the following limits. If an answer does not exist, write "DNE."

$\lim_{x \to -3} f(x) DNE$ $\lim_{x \to -2} f(x) = 0$ $\lim_{x \to -1} f(x) = 1$

$\lim_{x \to 0} f(x) = 3$ $\lim_{x \to 1} f(x) DNE$ $\lim_{x \to 3} f(x) = 0$

(b) List all of the values of x at which $f(x)$ is not continuous. Provide a brief (one sentence) explanation for each of your answers.

Solution: f is not continuous at -3, -2, and 1.

- At $x = -3$, there is a jump discontinuity; this is because the left- and right-hand limits of f are unequal at $x = -3$. In particular, $\lim_{x \to -3^-} f(x) = 2$ while $\lim_{x \to -3^+} f(x) = 1$.
- At $x = -2$, there is a removable discontinuity, since $\lim_{x \to -2} f(x) = 0$ but $f(-2) = -1$.
- Since $f(1)$ is undefined, f is discontinuous at $x = 1$.

(c) Find the interval(s) on which $f'(x)$ is positive, the interval(s) on which $f'(x)$ is negative, and the interval(s) on which $f'(x)$ is zero. Briefly (one sentence) explain each of your answers.

Solution:

- $f'(x)$ is positive on the intervals (-2, 1) and (1, 4), since f is increasing on those intervals.
- $f'(x)$ is negative on (-3, -2), since f is decreasing on that interval.
- $f'(x)$ is zero on the interval (-4, -3), since f is constant on that interval.

4. Find an equation for the line tangent to the parabola $y = x^2 - 4x + 6$ at the point (4, 6).

   Solution: To find an equation for the tangent line, we must find its slope; this will be the derivative of the function $x^2 - 4x + 6$ (with respect to x) evaluated at $x = 4$:

$$\begin{aligned}\lim_{x\to 4} \frac{(x^2 - 4x + 6) - ((4)^2 - 4(4) + 6)}{x - 4} &= \lim_{x\to 4} \frac{x^2 - 4x + 6 - 6}{x - 4} \\ &= \lim_{x\to 4} \frac{x^2 - 4x}{x - 4} \\ &= \lim_{x\to 4} \frac{x(x - 4)}{x - 4} \\ &= \lim_{x\to 4} x \\ &= 4\end{aligned}$$

   Thus, the tangent line has slope $m = 4$ and passes through the point (4,6). Using the point-slope form, we find the equation

$$y - 6 = 4(x - 4),$$

   or equivalently

$$y = 4x - 10.$$

5. Use the precise definition of the limit to prove that $\lim_{x\to 2}(8 - 3x) = 2$.

   Solution: Let $\varepsilon > 0$ be any positive number. We want to restrict x in such a way that $|(8 - 3x) - 2| < \varepsilon$.

   Algebraically, the following are equivalent:

$$\begin{aligned}|(8 - 3x) - 2| &< \varepsilon \\ |6 - 3x| &< \varepsilon \\ 3|2 - x| &< \varepsilon \\ |2 - x| &< \frac{\varepsilon}{3}\end{aligned}$$

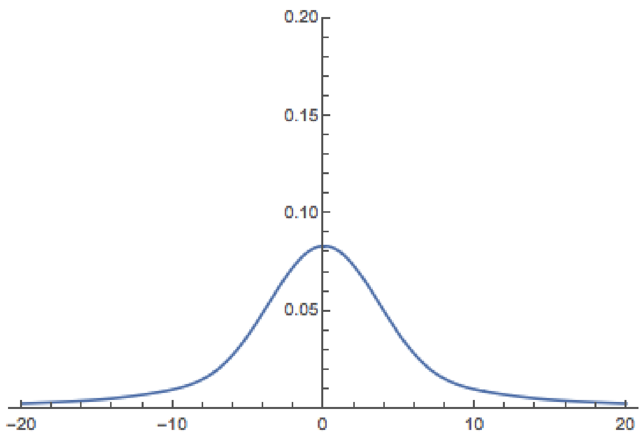
   We've shown that $|(8 - 3x) - 2| < \varepsilon$ whenever $|x - 2| < \frac{\varepsilon}{3}$. (Notice that $|2 - x|$, from the last line above, is the same thing as $|x - 2|$.) In other words, if we choose x to be withn $\frac{\varepsilon}{3}$ units of 2, it will follow that $8 - 3x$ is within ε units of 2, which is what we wanted.

   So, if we select $\delta = \frac{\varepsilon}{3}$, then $|(8 - 3x) - 2| < \varepsilon$ whenever $|x - 2| < \delta$. Thus, by the precise definition of the limit, we've shown $\lim_{x\to 2}(8 - 3x) = 2$.

   Note: if we were to choose δ to be any positive number less than or equal to $\frac{\varepsilon}{3}$, the above conclusion would still follow. The objective is always to find a positive number δ that is *small enough* to make the argument work.

6. The diagram below is a Mathematica graph of the function

$$f(x) = \frac{2\cos(x) + x^2 - 2}{x^4}.$$



(a) From the graph, estimate $\lim_{x \to 0} f(x)$, correct to the nearest hundredth.

Solution: From the graph, we see that $\lim_{x\to 0} f(x)$ is approximately 0.08. (Comment: The *exact* value, which can't be inferred directly from a graph but can be found using methods we'll learn later in the semester, turns out to be 1/12, or 0.083333....)

(b) Is f continuous at $x = 0$? Why, or why not? Write at least one sentence of explanation.

Solution: No, f is not continuous at $x = 0$. In order for f to be continuous at 0, $f(0)$ must exist and be equal to $\lim_{x \to 0} f(x)$. In this case, $f(0)$ clearly does not exist, since we can't divide by zero.

Comment: As we've seen multiple times in class (and in the textbook, and on Mathematica assignments), a computer graph can be misleading when there is a removable discontinuity, as is the case here.

(c) (Extra credit – optional): Based on your answer to part (a), evaluate

$$\lim_{x \to 0} \frac{2\cos(x) - 2}{x^2}.$$

Explain your answer. (No credit will be given without a valid explanation!)

Solution: As demonstrated in class with a Mathematica graph, the value of this limit turns out to be -1. Since no one solved this on the day of the test, I'm leaving it open - the first person who can show me a valid proof that this limit's value is -1 will have *five* extra credit points added to his/her score for this test. Note: for your proof, you may assume (without proof) that the limit shown in part (a) is correct.