Trade and the Topography of the Spatial Economy: Online Appendix*

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Abstract

This online appendix presents a number of additional theoretical and empirical results regarding: (1) allowing workers to commute across locations; (2) the conditions necessary for geographic trade costs to satisfy the triangle inequality; (3) the efficiency of the competitive equilibrium; (4) approximating the spatial equilibrium using Fourier series; and (5) incorporating multiple sectors.

1 Commuting

In this section, we extend our framework to allow workers to commute across locations. We then show how commuting affects the estimated topography of amenities and productivities.

Suppose that an individual living in location $i \in S$ derives utility from the amenity in location i and from the goods consumed in location i but can potentially commute to other locations to work. Because a worker consumes her goods at home, she chooses her work location to maximize her nominal wage net of commuting costs:

$$\max_{s \in S(i)} \frac{w(s)}{C(i,s)} \varepsilon_v(i,s),$$

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where we assume $\varepsilon_v(i,s)$ is a worker specific idiosyncratic term that is extreme value distributed. The density of workers living in i commuting to s can then be written as:

$$\pi(i,s) = \frac{w(s)^{\theta} C(i,s)^{-\theta}}{\int_S w(k)^{\theta} C(i,k)^{-\theta} dk}$$

so that the total number of workers working in location i, $\tilde{L}(i)$, can be written as

$$\tilde{L}(i) = \int_S \pi(s,i) L(s) ds = w(i)^{\theta} \int_S \frac{C(s,i)^{-\theta}}{\int_S w(k)^{\theta} C(s,k)^{-\theta} dk} L(s) ds. \tag{1}$$

We assume that workers choose where they live before they know where exactly they will be commuting. Hence welfare equalization implies:

$$\bar{W} = \frac{\tilde{w}(i)}{P(i)} u(i),$$

where $\tilde{w}(i)$ is the expected wage (or, equivalently, the average wage of residents) in each location:

$$\tilde{w}(i) \equiv E\left[\frac{w(s)}{C(i,s)} \varepsilon_v(i,s)\right] = \gamma \left(\int_S w(s)^{\theta} C(i,s)^{-\theta} ds\right)^{\frac{1}{\theta}} \tag{2}$$

and where the price index is:

$$P(i)^{1-\sigma} \equiv \int_S T(s,i)^{1-\sigma} A(s)^{\sigma-1} w(s)^{1-\sigma} ds. \tag{3}$$

Note that the gravity equation with commuting implies the value of trade from i to s is:

$$X(i,s) = T(i,s)^{1-\sigma} w(i)^{1-\sigma} A(i)^{\sigma-1} P(s)^{\sigma-1} \tilde{w}(s) L(s). \tag{4}$$

In equilibrium, the total income earned in a location (which is equal to the product of the wage in the location and the number of workers working in that location) is equal to the sales of the good produced in that location:

$$
\begin{aligned}
w(i)\,\tilde{L}(i) = \int_S X(i,s)\,ds = \int_S T(i,s)^{1-\sigma} w(i)^{1-\sigma} A(i)^{\sigma-1} P(s)^{\sigma-1} \tilde{w}(s) L(s)\,ds \iff \\
w(i)^{\sigma} \tilde{L}(i) A(i)^{1-\sigma} = \int_S T(i,s)^{1-\sigma} P(s)^{\sigma-1} \tilde{w}(s) L(s)\,ds \iff \\
w(i)^{\sigma} \tilde{L}(i) A(i)^{1-\sigma} = \int_S T(i,s)^{1-\sigma} W(s)^{1-\sigma} u(s)^{\sigma-1} \tilde{w}(s)^{\sigma} L(s)\,ds \iff \\
w(i)^{\sigma} \tilde{L}(i) A(i)^{1-\sigma} = \bar{W}^{1-\sigma} \int_S T(i,s)^{1-\sigma} u(s)^{\sigma-1} \tilde{w}(s)^{\sigma} L(s)\,ds, \quad (5)
\end{aligned}
$$

where the last line used welfare equalization.

Also in equilibrium, the total income spent in a region is equal to the amount purchased from all locations

$$
\begin{aligned}
\tilde{w}(i) L(i) = \int_S X(s,i)\,ds = \int_S T(s,i)^{1-\sigma} w(s)^{1-\sigma} A(s)^{\sigma-1} P(i)^{\sigma-1} \tilde{w}(i) L(i)\,ds \iff \\
P(i)^{1-\sigma} = \int_S T(s,i)^{1-\sigma} w(s)^{1-\sigma} A(s)^{\sigma-1}\,ds \iff \\
\tilde{w}(i)^{1-\sigma} u(i)^{1-\sigma} = \bar{W}^{1-\sigma} \int_S T(s,i)^{1-\sigma} w(s)^{1-\sigma} A(s)^{\sigma-1}\,ds \quad (6)
\end{aligned}
$$

where the last line used welfare equalization. Note that the integral on the right hand side is equal to $P(i)^{1-\sigma}$, so that equation (6) is equivalent to welfare equalization.

For a given trade cost function T, commuting cost function C, elasticity of substitution σ, shape parameter governing the idiosyncratic commuting costs θ, productivities A and amenities u, equilibrium is defined by functions L (determining the distribution of where people reside), $\tilde{L}$ (determining the distribution of where people work), w (determining the wages earned when producing in each location), and $\tilde{w}$ (determining the average wage earned by a person living in each location) such that equations (1), (2), (5) and (6) hold.

Suppose we observe T, C, σ, θ, w, and L; can we infer what the distribution of amenities and productivities are? The answer is yes: given w, L, C, and θ, $\tilde{L}$ can be determined from equation (1). Similarly, given w, C, and θ, $\tilde{w}$ can be determined from equation (2). Then note that equations (5) and (6) can be re-written as:

$$
A(i) = \left(\bar{W}^{1-\sigma} \int_S T(i,s)^{1-\sigma} \frac{\tilde{w}(s)^{\sigma} L(s)}{w(i)^{\sigma} \tilde{L}(i)} u(s)^{\sigma-1}\,ds \right)^{\frac{1}{1-\sigma}} \quad (7)
$$

$$u(i)=\left(\bar{W}^{1-\sigma}\int_S T(s,i)^{1-\sigma}\left(\frac{w(s)}{\tilde{w}(i)}\right)^{1-\sigma}A(s)^{\sigma-1}ds\right)^{\frac{1}{1-\sigma}}. \tag{8}$$

Equations (7) and (8) can be solved jointly to determine the distribution of amenities and productivities.

Figure 1 below presents the estimated composite productivities and amenities when the commuting costs are assumed to be ten times larger than the trade costs estimated in the paper, i.e for all $i,j\in S$, $C(i,j)=10\times(T(i,j)-1)+1$. While the estimated productivities are very similar to those estimated in the paper (see the top panel of Figure 13), the estimated topography of amenities with commuting are substantially smoother across space, reflecting workers abilities to work in nearby locations with higher observed wages.

2 Geographic trade costs and the triangle inequality

In this section we show that geographic trade costs satisfy the triangle inequality if and only if the function f such that $T(i,j)=f(t(i,j))$, where $t(i,j)$ is defined in equation (17) in the paper, is log-concave.

The triangle inequality is said to be satisfied if and only if for all $i,j,k\in S$:

$$T(i,k)\leq T(i,j)\,T(j,k) \tag{9}$$

Since $t(i,j)$ is defined as the least cost route from i to j, for all $i,j,k\in S$ we have:

$$t(i,k)\leq t(i,j)+t(j,k), \tag{10}$$

with the inequality strict when j is on the least cost route from i to k. Consider any function $f(t)$ that is monotonically increasing such that $f(0)=1$. What conditions do we requite on f such that the triangle inequality holds?

Proposition 1 *The triangle inequality holds for monotonically increasing f such that $f(0)=1$ if and only if f is log subadditive, i.e. $\ln f(x+y)\leq\ln f(x)+\ln f(y)$ for all $x,y\in\left[0,\frac{D}{2}\right]$,where D is the maximum distance between any two points in S.*

Proof. We first prove that if f is log subadditive, then it satisfies the triangle inequality. Suppose f is log subadditive. Then for all $i,j,k\in S$ we have:

$$\ln f(t(i,j)+t(j,k))\leq\ln f(t(i,j))+\ln f(t(j,k)).$$

Because f is monotonically increasing, so too is $\ln f$. Then from equation (10) we have:

$$\ln f\left(t\left(i,k\right)\right)\leq\ln f\left(t\left(i,j\right)+t\left(j,k\right)\right)$$

Combining the two equations yields:

$$\ln f\left(t\left(i,k\right)\right)\leq\ln f\left(t\left(i,j\right)\right)+\ln f\left(t\left(j,k\right)\right)\iff T\left(i,k\right)\leq T\left(i,j\right)T\left(j,k\right),$$

i.e. the triangle equality is satisfied.

We now prove that if f satisfies the triangle inequality, then it is log subadditive. Suppose f satisfies the triangle inequality. Then for all $i,j,k\in S$ we have:

$$T\left(i,k\right)\leq T\left(i,j\right)T\left(j,k\right)\iff\ln f\left(t\left(i,k\right)\right)\leq\ln f\left(t\left(i,j\right)\right)+\ln f\left(t\left(j,k\right)\right).$$

Note that for any $x,y\in\left[0,\frac{D}{2}\right]$, we can find and $i,k\in S$ such that $t\left(i,k\right)=x+y\leq D$. We can then find a $j\in S$ such that $t\left(i,j\right)=x$ and $t\left(j,k\right)=y$ and by simply moving from i along the least cost route for x units. We then have for all $x,y\in\mathbb{R}_{+}$:

$$\ln f\left(x+y\right)\leq\ln f\left(x\right)+\ln f\left(y\right),$$

i.e. f is log subadditive. ■

3 Efficiency of the competitive equilibrium

In this section, we derive the equilibrium conditions for the social planners problem and show that the competitive equilibrium presented in the paper is efficient if and only if $\alpha+\beta=0$.

Consider the following social planner's problem. The social planner chooses trade flows and the population in each location to maximize total utility:

$$\max_{q(s,i),\ L(s)}\int Q\left(i\right)L\left(i\right)^{\beta}di,$$

where

$$Q\left(i\right)\equiv\left(\int q\left(s,i\right)^{\frac{\sigma-1}{\sigma}}ds\right)^{\frac{\sigma}{\sigma-1}}\bar{u}\left(i\right),$$

subject to the following constraints:

1. First, the total population is equal to $\bar{L}$:

$$\int L(i)\, di \leq \bar{L} \tag{11}$$

2. Second, the production in each location is constrained by the total effective units of labor in that location:

$$\int q(i,s)\, T(i,s)\, ds \leq \bar{A}(i)\, L(i)^{1+\alpha} \tag{12}$$

3. Third, the per capita welfare is constrained to be equal to $\bar{W}$ in every location:

$$\left(\int q(s,i)^{(\sigma-1)/\sigma}\, ds\right)^{\sigma/(\sigma-1)} \bar{u}(i) \geq \bar{W} L(i)^{1-\beta} \tag{13}$$

The social planner's Lagrangian can be written as:

$$\begin{aligned}\mathcal{L} = \quad & \int \left(\int q(s,i)^{\frac{\sigma-1}{\sigma}}\, ds\right)^{\frac{\sigma}{\sigma-1}} \bar{u}(i)\, L(i)^{\beta}\, di - \lambda\left(\int L(i)\, di - \bar{L}\right) \\ & - \int \gamma(i)\, L(i)^{-\beta}\left(\bar{W} L(i) - \left(\int q(s,i)^{(\sigma-1)/\sigma}\, ds\right)^{\sigma/(\sigma-1)} \bar{u}(i)\, L(i)^{\beta}\right) di \\ & - \int \mu(i)\left(\int q(i,s)\, T(i,s)\, ds - \bar{A}(i)\, L(i)^{1+\alpha}\right) di\end{aligned}$$

Notice that with this formulation all the Lagrange multiplies are non-negative. The problem yields the following first order conditions:

1. The FOC with respect to $L(i)$ yields:

$$\beta L(i)^{\beta-1}\left(\int q(s,i)^{\frac{\sigma-1}{\sigma}}\, ds\right)^{\frac{\sigma}{\sigma-1}} \bar{u}(i) + (1+\alpha)\, \mu(i)\, \bar{A}(i)\, L(i)^{\alpha} = \lambda + (1-\beta)\, \gamma(i)\, \bar{W} L(i)^{-\beta} \implies$$

and using the fact that in equilibrium, the constraint 13, binds we have

$$\beta \bar{W} + (1+\alpha)\, \mu(i)\, \bar{A}(i)\, L(i)^{\alpha} = \lambda + (1-\beta)\, \gamma(i)\, \bar{W} L(i)^{-\beta} \tag{14}$$

2. The FOC with respect to $q(i,s)$ yields:

$$\left(L(s)^{\beta} + \gamma(s)\right)\left(\int q(i,s)^{(\sigma-1)/\sigma}\, di\right)^{\frac{1}{\sigma-1}} q(i,s)^{-\frac{1}{\sigma}}\, \bar{u}(s) = \mu(i)\, T(i,s) \tag{15}$$

and of course the two constraints 13 and 12.

Combining the two first order conditions and using the definition of $Q(s)$ gives us the consumption of each worker in location i:

$$q(i,s) = Q(s)\,\bar{u}(s)^{\sigma-1}\left(\frac{T(i,s)}{\bar{A}(i)}\right)^{-\sigma}(1+\alpha)^{\sigma}\left(L(s)^{\beta}+\gamma(s)\right)^{\sigma}L(i)^{\alpha\sigma}\,\Gamma_1(i)\,, \tag{16}$$

where $\Gamma_1(i) \equiv \left(\lambda + (1-\beta)\gamma(i)\bar{W}L(i)^{-\beta} - \beta L(i)^{\beta-1}Q(i)\right)^{-\sigma}$. Substituting equation (16) into the production constraint (12) yields our first equilibrium condition:

$$L(i)^{1-\alpha(\sigma-1)}\,\Gamma_2(i) = \bar{W}^{1-\sigma}\int T(i,s)^{1-\sigma}\,\bar{A}(i)^{\sigma-1}\,\bar{u}(s)^{\sigma-1}\,L(s)^{1+\beta(\sigma-1)}\left(1+\gamma(s)\,L(s)^{-\beta}\right)^{\sigma}ds \tag{17}$$

where $\Gamma_2(i) \equiv \left(\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{\frac{\lambda}{\bar{W}}-\beta}{1-\beta}+\gamma(i)\,L(i)^{-\beta}\right)\right)^{\sigma}$ and we used the constraint (13) per capita welfare is equalized everywhere (so that $Q(i)\,L(i)^{\beta-1}=\bar{W}$) and the definition of $p(i,s)$.

Substituting equation (16) into the definition of $Q(s)$ yields our second equilibrium condition:

$$\left(1+\gamma(s)\,L(s)^{-\beta}\right)^{1-\sigma}L(s)^{\beta(1-\sigma)} = \bar{W}^{1-\sigma}\int T(i,s)^{1-\sigma}\,\bar{A}(i)^{\sigma-1}\,\bar{u}(s)^{\sigma-1}\,L(i)^{\alpha(\sigma-1)}\,\Gamma_2(i)^{\frac{1-\sigma}{\sigma}}\,di \tag{18}$$

We further need to derive one more equation to characterize the ratio of $\lambda/\bar{W}$

To proceed multiply both sides of equation 15 by $\bar{u}(s)^{1-\sigma}\left(L(s)^{\beta}+\gamma(s)\right)^{1-\sigma}Q(s)$, we get

$$\int \mu(i)^{1-\sigma}\,\bar{u}(s)^{\sigma}\left(L(s)^{\beta}+\gamma(s)\right)^{\sigma}Q(s)T(i,s)^{1-\sigma}\,di = \bar{u}(s)\,Q(s)\left(L(s)^{\beta}+\gamma(s)\right)$$

Integrate the above equation and equation (12), by changing the order of integrals it is easy to show that $\int \bar{u}(s)\,Q(s)\left(L(s)^{\beta}+\gamma(s)\right)ds = \int \mu(i)\,\bar{A}(i)\,L(i)^{1+\alpha}\,di$.

Multiply both sides of equation 14 by $L(i)$, and implement integration, we get

$$\lambda \bar{L} = \int \beta L(i)^{\beta} \bar{u}(i) Q(i) di + \int (\beta - 1) \gamma(i) Q(i) \bar{u}(i) di + \int (1 + \alpha) \mu(i) \bar{A}(i) L(i)^{\alpha+1} di$$
$$= (1 + \alpha + \beta) \int L(i)^{\beta} \bar{u}(i) Q(i) di + (\alpha + \beta) \int \gamma(i) Q(i) \bar{u}(i) di$$

where $\int L(i)^{\beta} \bar{u}(i) Q(i) di = W = \bar{L}\bar{W}$. Hence:

$$\lambda = (1 + \alpha + \beta)\bar{W} + (\alpha + \beta)\bar{W} \frac{\int \gamma(i) L(i)^{1-\beta} di}{\bar{L}}, \tag{19}$$

which immediately implies $\lambda = \bar{W}$ if and only if $\alpha + \beta = 0$. Indeed, because $\gamma(i) \geq 0$, if $\alpha + \beta > 0$ then $\frac{\lambda}{\bar{W}} > 1$ and vice versa.

Thus, the solution of the social planner's problem, constituting of functions for $\gamma(i)$, $L(i)$, and variables λ,$\bar{W}$ such that (17), (18), the aggregate labor constraint (11) and labor market clearing constraint (19) hold. This allows to to prove the following proposition:

Proposition 2 *The distribution of the population in the competitive equilibrium is a solution to the social planner problem (i.e. is efficient) if and only if $\alpha + \beta = 0$.*

Proof. Prior to beginning the proof, it is helpful to re-write the equations characterizing the social planner problem:

$$L(i)^{1-\alpha(\sigma-1)} \Gamma_2(i) = \bar{W}^{1-\sigma} \int \left(\frac{T(i,s)}{\bar{A}(i) \bar{u}(s)} \right)^{1-\sigma} L(s)^{1+\beta(\sigma-1)} \left(1 + \gamma(s) L(s)^{-\beta}\right)^{\sigma} ds$$
$$\left(1 + \gamma(i) L(i)^{-\beta}\right)^{1-\sigma} L(i)^{\beta(1-\sigma)} = \bar{W}^{1-\sigma} \int \left(\frac{T(s,i)}{\bar{A}(s) \bar{u}(i)} \right)^{1-\sigma} L(s)^{\alpha(\sigma-1)} \Gamma_2(s)^{\frac{1-\sigma}{\sigma}} ds$$

as well as the equations characterizing the competitive equilibrium solution:

$$L(i)^{1-\alpha(\sigma-1)} w(i)^{\sigma} = \bar{W}^{1-\sigma} \int_S T(i,s)^{1-\sigma} \bar{A}(i)^{\sigma-1} \bar{u}(s)^{\sigma-1} L(s)^{1+\beta(\sigma-1)} w(s)^{\sigma} ds$$
$$w(i)^{1-\sigma} L(i)^{\beta(1-\sigma)} = \bar{W}^{1-\sigma} \int_S T(s,i)^{1-\sigma} \bar{A}(s)^{\sigma-1} \bar{u}(i)^{\sigma-1} w(s)^{1-\sigma} L(s)^{\alpha(\sigma-1)} ds$$

For the first "if" part, it is straightforward to show that $\alpha + \beta = 0$ implies that SP is the same as CE. When $\alpha + \beta = 0$, from above we know that $\lambda = \bar{W}$, which implies that $\left(\frac{1-\beta}{1-\alpha}\right) \left(\frac{\frac{\lambda}{\bar{W}} - \beta}{1-\beta} + \gamma(i) L(i)^{-\beta} \right) = 1 + \gamma(s) L(s)^{-\beta}$. In this case, the social planner problem equations are identical to the competitive equilibrium equations when $w(i) =$

$C\left(1+\gamma\left(s\right)L\left(s\right)^{-\beta}\right)$, where C is some scalar. As a result, any distribution of population satisfying the competitive equilibrium will also satisfy the social planner conditions.

We prove the second "only if" part of the proof by contradiction. Suppose that $\alpha+\beta\neq 0$ but the competitive equilibrium distribution of labor satisfies the social planner conditions. Note that we can write the first social planner equation as:

$$\tilde{w}\left(i\right)^{\sigma}=\bar{W}^{1-\sigma}\int T\left(i,s\right)^{1-\sigma}\bar{A}\left(i\right)^{\sigma-1}\bar{u}\left(s\right)^{\sigma-1}L\left(s\right)^{1+\beta(\sigma-1)}\kappa\left(i\right)^{-\sigma}L\left(i\right)^{\alpha(\sigma-1)-1}\tilde{w}\left(s\right)^{\sigma}ds$$

where $\tilde{w}\left(i\right)\equiv\left(1+\gamma\left(s\right)L\left(s\right)^{-\beta}\right)$ and $\kappa\left(i\right)\equiv\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{\frac{\frac{\lambda}{W}-\beta}{1-\beta}+\gamma(i)L(i)^{-\beta}}{1+\gamma(s)L(s)^{-\beta}}\right)$. Because this equation is a (linear) eigenequation, we know from the generalized Jentzsch theorem that there exists a unique $\tilde{w}$ that solves the equation for any given $\kappa\left(i\right)$ and labor distribution $L\left(i\right)$. Similarly, we can write the first competitive equilibrium condition as:

$$w\left(i\right)^{\sigma}=\bar{W}^{1-\sigma}\int T\left(i,s\right)^{1-\sigma}\bar{A}\left(i\right)^{\sigma-1}\bar{u}\left(s\right)^{\sigma-1}L\left(s\right)^{1+\beta(\sigma-1)}L\left(i\right)^{\alpha(\sigma-1)-1}w\left(s\right)^{\sigma}ds,$$

which again is a linear eigenequation, so there exists a unique w that satisfies each equation given a labor distribution $L\left(i\right)$. Since the distribution of population (by assumption) is the same in both eigenequations, this immediately implies that $\tilde{w}\left(i\right)=\phi w\left(i\right)$ if and only if $\kappa\left(i\right)=1$, where ϕ is an arbitrary scalar. Note that $\kappa\left(i\right)=1$ in turn if and only if:

$$\begin{aligned}
\kappa\left(i\right)=1 &\iff\\
\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{\frac{\lambda}{W}-\beta}{1-\beta}+\gamma\left(i\right)L\left(i\right)^{-\beta}\right)=1+\gamma\left(s\right)L\left(s\right)^{-\beta} &\iff\\
\frac{\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{\frac{\lambda}{W}-\beta}{1-\beta}\right)-\frac{1-\beta}{1-\alpha}}{1-\frac{1-\beta}{1-\alpha}}=\left(1+\gamma\left(s\right)L\left(s\right)^{-\beta}\right) &\iff\\
\frac{\frac{\lambda}{W}-1}{\alpha+\beta}=\tilde{w}\left(i\right)&
\end{aligned}$$

Hence, the wages coincide if and only if the wages are constant across all locations (which is generically not the case when trade costs are positive). A similar argument also shows that $\tilde{w}\left(i\right)\kappa\left(i\right)\neq\phi w\left(i\right)$. To complete the proof, we finally show that if $\tilde{w}\neq w$, then it cannot be the case that the competitive equilibrium distribution of the population coincides with the social planner's problem. Note that if distribution of population is the same in both the social planner problem and the competitive equilibrium problem, it has to be the case that the following equation (derived by equating the second equiliibrum equations for the social planner and the competitive equilibrium) holds for all i:

$$\int \left(\frac{\bar{A}(s)\,\bar{u}(i)\,L(s)^{\alpha}\,\tilde{w}(i)}{T(s,i)\,\kappa(s)\,\tilde{w}(s)} \right)^{\sigma-1} ds = \int_S \left(\frac{\bar{A}(s)\,\bar{u}(i)\,L(s)^{\alpha}\,w(i)}{T(s,i)\,w(s)} \right)^{\sigma-1} ds$$

Since $\frac{\kappa(s)\tilde{w}(s)}{\tilde{w}(i)} \neq \frac{w(s)}{w(i)}$, the equation cannot hold. Hence we have a contradiction. ■

4 Characterizing the spatial equilibrium using Fourier series

In this section, we show how the population function in the model without spillovers can be approximated with the help of a Fourier series when the geography of the world is a circle, there exist two symmetric borders, and the trade cost function is the exponential functon. We formulate the original problem with the eigenequation

$$\lambda y(x) = \int T(x,x')y\left(x'\right)dx' \tag{20}$$

where λ is the eigenvalue of the system. We will largely follow Fabinger (2011) and adapt the methods suggested there to provide an approximation for our model. Notice that the resulting solution is only an approximation because there is a nonlinear relationship between parameters (such as trade costs) and the population distribution and, in addition, the eigenvalue of the system changes as we change the parameters of the problem.

Step 0: Perturbation

Consider a small perturbation of the trade costs so that the Kernel becomes $T(x,x') \to T(x,x') + \kappa t\left(x,x'\right)$ so that equation (20) becomes

$$\lambda y(x) = \int \left(T(x,x') + \kappa t\left(x,x'\right)\right) y\left(x'\right)dx' \tag{21}$$

We consider the perturbations $\lambda = \lambda_0 + \kappa\lambda_1 + O(\kappa^2)$

$$y(x) = y_0(x) + \kappa y_1(x) + O(\kappa^2) \tag{22}$$

Substituting back in equation (21), completing the derivations, setting second order terms to zero and dividing by κ we obtain

$$\lambda_1 y_0(x) + \lambda_0 y_1(x) = \int T(x,x')y_1(x')dx' + \int t(x,x')y_0(x')dx'$$

We solve for the first order approximation of $y(x)$. Notice that $y_1(x)$ can be written as the following operator

$$y_1(x) = (Gy_1)(x) + (By_0)(x) \tag{23}$$

where $G = \frac{T}{\lambda_0}$, $B = \frac{t}{\lambda_0} - \frac{\lambda_1}{\lambda_0}G$. It is easy to verify that if $y_1(x)$ is a solution, $y_1(x)+ay_0(x)$ is also a solution and thus, we focus in the space that is orthogonal to y_0. This combined with the knowledge that 1 is the largest eigenvalue of operator G that corresponds to y_0 since $y_0 = Gy_0$, so that all the other eigenvalues are strictly lower than one implies $\lim G^n y_1 = 0$, and we can iterate y_1 in the above equation,

$$y_1 = \sum_{m=0}^{\infty} G^m B y_0.$$

The corresponding Fourier coefficient are $y_{1,n} = \sum_{m=0}^{\infty}(G^m B y_0)_n$.

Step 1: Specifying space and trade costs

We study the example of the *circle* with domain $(-\pi,\pi)$ with country A in $(-\pi/2,\pi/2)$ and country B in $(-\pi,-\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi)$, the kernel is $T(\theta,\theta') = e^{(1-\sigma)\tau d(\theta-\theta')}$ between place θ and θ' where $d(\theta,\theta') = \min(|\theta-\theta'|, 2\pi - |\theta-\theta'|)$. We consider a perturbation of the trade costs in the border of the two countries, so that after the perturbation the kernel is $(T(\theta,\theta') + \kappa t(\theta,\theta')) = T(\theta,\theta') - \kappa b(\theta,\theta')T(\theta,\theta')$, i.e. $t(\theta,\theta') = b(\theta,\theta')T(\theta,\theta')$ where $b(\theta,\theta') = 1_A(\theta)1_B(\theta') + 1_A(\theta')1_B(\theta)$, and $1_A, 1_B$ are country indicator functions so that $b(\theta,\theta')$ is 1 if θ and θ' are in different countries.

Step 2: Consider the solution for a constant y_0

First, notice that $y_0 = 1$ is a solution to 20. Replacing the solution the corresponding eigenvalue must be

$$\begin{aligned}
\lambda_0 &= \int_{-\pi}^{\pi} T(x,x')dx' \\
&= \int_{-\pi}^{\pi} e^{(1-\sigma)\tau \min(|\theta-\theta'|, 2\pi-|\theta-\theta'|)} d\theta' \\
&= \int_{-\pi+\theta}^{\pi+\theta} e^{(1-\sigma)\tau \min(|t|, 2\pi-|t|)} dt,
\end{aligned}$$

where the last equation used a change of variables setting $t = \theta' - \theta$. This function is periodic in the interval $[-2\pi, 2\pi]$ with the period being 2π, so that we can evaluate it at any θ and thus we do so at $\theta = 0$ where $\min(|t|, 2\pi - |t|) = |t|$. We finally have

$$\lambda_0 = \frac{2(e^{(1-\sigma)\tau\pi} - 1)}{(1-\sigma)\tau}. \tag{24}$$

Step 3: Specifying a Fourier series expansion for y_1

We focus on solutions that y_0 is constant and we restrict y_1 to be orthogonal to y_0. We will compute an expansion of $y_1(x)$ with the help of Fourier expansion (that can be computed for any square integrable function in the circle) written as

$$y_1(x) = \sum_{n=-\infty}^{n=\infty} y_{1,n} e^{inx} \tag{25}$$

where $i = \sqrt{-1}$ is the imaginary unity and the Fourier coefficients are given by

$$y_{1,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} y_1(x) \, e^{-inx} dx.$$

Step 4: Computing Fourier series coefficients of y_1

We now compute the Fourier coefficients of y_1.

i) For $n = 0$: From step 2 we have $y_0 = 1$ and we restrict y_1 to be orthogonal to y_0, the first Fourier coefficient for $n = 0$, $y_{1,0} = 0$

ii) For $n \neq 0$: The operator $G(\theta, \theta')$ could be represented by $G(\theta - \theta')$. We can make use of the convolution theorem and $(G * F)_n = 2\pi G_n F_n$, we thus have

$$y_{1,n} = \sum_{m=0}^{\infty} (2\pi G_n)^m (By_0)_n = \sum_{m=0}^{\infty} (2\pi G_n)^m (\frac{t}{\lambda_0} y_0)_n - \frac{\lambda_1}{\lambda_0} \sum_{m=0}^{\infty} (2\pi G_n)^{m+1} (y_0)_n.$$

Notice however that the Fourier coefficients of y_0 for $n \neq 0$ are all 0 by assumption (since it considered to be a constant). Thus,

$$y_{1,n} = \sum_{m=0}^{\infty} (2\pi G_n)^m (By_0)_n = \sum_{m=0}^{\infty} (2\pi G_n)^m (\frac{t}{\lambda_0} y_0)_n = \frac{1}{1 - 2\pi G_n} (\frac{t}{\lambda_0} y_0)_n, \tag{26}$$

and we do not need to consider λ_1. In addition, by the definition of t,

$$(\frac{t}{\lambda_0}y_0)_n = -(bGy_0)_n,$$

so that

$$y_{1,n} = -\frac{1}{1-2\pi G_n}(bGy_0)_n.$$

We can now focus on computing 1) $2\pi G_n$ and 2) $(bGy_0)_n$.

For the first term, we substitute for the operator for G and compute the Fourier coefficient

$$\begin{aligned}
2\pi G_n &= \frac{2\pi}{2\pi}\int_{-\pi}^{\pi} G(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{\lambda_0}\int_{-\pi}^{\pi} T(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{\lambda_0}[\int_{0}^{\pi} e^{((1-\sigma)\tau-in)\theta}d\theta + \int_{-\pi}^{0} e^{((\sigma-1)\tau-in)\theta}d\theta].
\end{aligned} \tag{27}$$

In the appendix, we show that completing the integration and using the value for λ obtained in (24) we finally have

$$2\pi G_n = \frac{(1-\sigma)^2\tau^2[(-1)^n e^{(1-\sigma)\tau\pi}-1]}{[(1-\sigma)^2\tau^2+n^2](e^{(1-\sigma)\tau\pi}-1)}. \tag{28}$$

For the second term, notice

$$\begin{aligned}
(bGy_0)(\theta) &= \int b(\theta,\theta')G(\theta,\theta')y_0(\theta')d\theta' \\
&= \int 1_A(\theta)1_B(\theta')G(\theta,\theta')d\theta' + \int 1_A(\theta')1_B(\theta)G(\theta,\theta')d\theta'
\end{aligned}$$

Using Fourier series expansion of the last two terms, also provided in the appendix we finally obtain

$$(bGy_0)_n = \begin{cases} 0 & n\,odd \\ -\frac{4}{\pi^2}(-1)^{\frac{n}{2}}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2-n^2}2\pi G_{2m+1} & n\,even \end{cases}. \tag{29}$$

Step 5: Combining the terms

We now combine all the terms of the epxansion. Notice that for even and nonzero n, using (28) we have

$$\frac{1}{1-2\pi G_n} = \frac{(1-\sigma)^2\tau^2 + n^2}{n^2}. \tag{30}$$

Additionally, using (28) we can compute $(bGy_0)_n$ for n even using

$$-(bGy_0)_n = -\frac{4(1-\sigma)^2\tau^2[e^{(1-\sigma)\tau\pi}+1](-1)^{\frac{n}{2}}}{\pi^2(e^{(1-\sigma)\tau\pi}-1)} \sum_{m=0}^{\infty} \frac{[(1-\sigma)^2\tau^2+n^2]^{-1}}{[(2m+1)^2-n^2][(1-\sigma)^2\tau^2+(2m+1)^2]}$$

$$\begin{aligned} &= -\frac{4(1-\sigma)^2\tau^2[e^{(1-\sigma)\tau\pi}+1](-1)^{\frac{n}{2}}}{\pi^2[(1-\sigma)^2\tau^2+n^2](e^{(1-\sigma)\tau\pi}-1)} \sum_{m=0}^{\infty} \left\{ \frac{1}{[(2m+1)^2-n^2]} - \frac{1}{[(1-\sigma)^2\tau^2+(2m+1)^2]} \right\} \\ &= \frac{4(1-\sigma)^2\tau^2[e^{(1-\sigma)\tau\pi}+1](-1)^{\frac{n}{2}}}{\pi^2[(1-\sigma)^2\tau^2+n^2](e^{(1-\sigma)\tau\pi}-1)} \sum_{m=0}^{\infty} \left\{ \frac{1}{[(1-\sigma)^2\tau^2+(2m+1)^2]} - \frac{1}{[(2m+1)^2-n^2]} \right\} \end{aligned} \tag{31}$$

Combining these two equations $y_{1,n}=0$ when n is odd or zero and

$$y_{1,n} = \frac{4(1-\sigma)^2\tau^2[e^{(1-\sigma)\tau\pi}+1]}{\pi^2 n^2(e^{(1-\sigma)\tau\pi}-1)}(-1)^{\frac{n}{2}} \sum_{m=0}^{\infty} \left\{ \frac{1}{[(1-\sigma)^2\tau^2+(2m+1)^2]} - \frac{1}{[(2m+1)^2-n^2]} \right\}.$$

and based on the mathematical identity 1.421-1 found in Jeffrey and Zwillinger (2007):

$$\tan\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2},$$

whereby letting $k=m+1$, $x=2k$, and using the fact that $\tan\pi k = 0$ for $k=1,2,...,$

$$y_{1,n} = \frac{4(1-\sigma)^2\tau^2[e^{(1-\sigma)\tau\pi}+1]}{\pi^2 n^2(e^{(1-\sigma)\tau\pi}-1)}(-1)^{\frac{n}{2}} \sum_{m=0}^{\infty} \frac{1}{[(1-\sigma)^2\tau^2+(2m+1)^2]}.$$

Finally, using these coefficients in (25)

$$
\begin{aligned}
y_1(x) &= \sum_{n=-\infty}^{\infty} y_{1,n} e^{inx} \\
&= \sum_{k=1}^{\infty} y_{1,2k}(e^{2ikx} + e^{-2ikx}) \\
&= \sum_{k=1}^{\infty} 2y_{1,2k} cos(2kx) \\
&= \sum_{k=1}^{\infty} \frac{2\left((1-\sigma)\tau\right)^2 \left[(e^{a\pi}+1\right]}{k^2\pi^2(e^{a\pi}-1)} (-1)^k \sum_{m=0}^{\infty} \frac{1}{\left(((1-\sigma)\tau\right)^2 + (2m+1)^2} cos(2kx) \\
&= \frac{(1-\sigma)\tau}{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} cos(2kx) \qquad (32)
\end{aligned}
$$

and the last equality by using identity 1.421-2 from Jeffrey and Zwillinger (2007)

$$\tanh \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{\left(2k-1\right)^2 + x^2}.$$

Letting again $k = m + 1$, $x = (1-\sigma)\tau$ and noting that by definition, $\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$.

$$\frac{4(1-\sigma)\tau}{\pi} \sum_{k=1}^{\infty} \frac{1}{\left(2m+1\right)^2 + \left((1-\sigma)\tau\right)^2} = \tanh \frac{\pi(1-\sigma)\tau}{2} = \frac{e^{\frac{\pi(1-\sigma)\tau}{2}} - e^{-\frac{\pi(1-\sigma)\tau}{2}}}{e^{\frac{\pi(1-\sigma)\tau}{2}} - e^{-\frac{\pi(1-\sigma)\tau}{2}}} = \frac{e^{\pi(1-\sigma)\tau} - 1}{e^{\pi(1-\sigma)\tau} + 1}$$

Although the form for $y_1(x)$ given in equation 32 looks quite complex, the first six Fourier coefficients $y_{1,2k}$ are 47.1239 -11.7810 5.2360 -2.9452 1.8850 -1.3090, so that it is rather safe to say that the first component dominates, i.e y_1 will be related to the $cos(2x)$ function, similarly to what happens in the line example of our paper. Taken together, our approximation of the population function using $y(x) \approx y_0(x) + \kappa y_1(x)$ is the sum of a constant function and a function that is roughly approximated by the $cos(2x)$ function.

Additionally, note that the expression for $y_1(x)$ can be further simplified with one more transformation, using identity 1.443-4 from Jeffrey and Zwillinger (2007)

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \qquad \text{for } [-\pi \leq x \leq \pi],$$

and setting $x = 2x$ or $x = -2x$ and substituting out in equation 32 we finally obtain

$$y_1(x) = \frac{(\sigma - 1)\tau}{2\pi}\left(\frac{\pi^2}{12} - x^2\right) \text{ for } [-\pi/2 \leq x \leq \pi/2],$$

or

$$y_1(x) = \frac{(\sigma - 1)\tau}{2\pi}\left(\frac{\pi^2}{12} + x^2\right) \text{ for } [x \leq -\tfrac{\pi}{2} \text{ and } \tfrac{\pi}{2} \leq x].$$

5 Multiple sectors

In this section, we extend the model in the paper to a world in which there are two sectors. We first prove that an equilibrium (without spillovers) exists and is unique when there are two sectors. We then examine how trade costs affect the distribution of economic activity on the line.

5.1 Existence and Uniqueness

Consider two sectors $l = 1, 2$ in the setup of the main paper. The price index of the consumer is a Cobb-Douglas aggregate of the price indices of each sector, with weights equal to the consumption shares, $P(i) = (P_1(i))^{\gamma_1}(P_2(i))^{\gamma_2}$ where $\gamma_1 + \gamma_2 = 1$ and

$$P_l(i)^{1-\sigma} = \int_S T_l(s,i)^{1-\sigma} A_l(s)^{\sigma-1} w(s)^{1-\sigma} ds.$$

The CES assumption implies that the welfare of living in a particular location can be written as an indirect function of the real wage and the amenity value:

$$W(i) = \frac{w(i)}{(P_1(i))^{\gamma_1}(P_2(i))^{\gamma_2}} u(i), \tag{33}$$

where welfare is said to be *equalized* if for all $i \in S$ there exists a $W > 0$ such that $W(i) \leq W$, with the equality strict if $L(i) > 0$. That is, welfare is equalized if the welfare of living in every inhabited location is the same and the welfare of living in every uninhabited location is no greater than the welfare of the inhabited locations. In addition, markets are said to *clear* if for all $i \in S$:

$$w(i) L(i) = \sum_l \int_S X_l(i,s) ds. \tag{34}$$

Given a regular geography with parameters σ, α, and β, we define a *spatial equilibrium* as a distribution of economic activity such that (i) trade is balanced; (ii) welfare is equalized; and (iii) the aggregate labor market clears:

$$\int_S (L_1(s) + L_2(s))\, ds = \int_S L(s)\, ds = \bar{L}. \tag{35}$$

We now discuss sufficient conditions for the existence and uniqueness of regular spatial equilibria in our equilibrium system, i.e. equilibria where $L(i) > 0$ and $W(i) = W$ for all i. We start by using the market clearing equation 35. Substituting out for trade flows and the indirect utility function (33), we can write the balanced trade condition (34) for all $i \in S$ as:

$$L(i) = \sum_l \gamma_l \int_S \left(\frac{T_l(i,s)}{A(i)\, P_l(s)} \right)^{1-\sigma} w(i)^{-\sigma}\, w(s)\, L(s)\, ds \tag{36}$$

where notice that the price index is a function of wages alone.

It is straightforward to notice that from the two equilibrium conditions, equation 33 only involves wages whereas 36 involves wages and labor. Now, notice that the latter one, given wages, forms an equilibrium system with a positive kernel that is linear on $L(i)$ so that following the arguments in Allen and Arkolakis (2013) given wages there exists a unique function $L(i)$ that satisfies it. Thus, it suffices to prove that welfare equalization, given 33 is satisfied by a unique function $w(i)$. We provide the proof of the following result below which completes the argument.[1]

Proposition 3 *Consider the system of equations formed by 33 and $W(i) = \bar{W}$ for all i. There exists a unique function $w(i)$ and a unique $\bar{W}$ that satisfies this system.*

Proof: Consider the welfare equalization equation 33 and substitute the definition of the price index and consider welfare equalization $W(i) = \bar{W}$, so that

$$w(i)^{1-\sigma} = \bar{W}^{1-\sigma} \prod_{l=1}^{2} \left(\int_S T_l(s,i)^{1-\sigma}\, u(i)^{\sigma-1} A_l(s)^{\sigma-1}\, w(s)^{1-\sigma}\, ds \right)^{\gamma_j}$$

It can be represented with the following simpler form

$$\lambda f(i) = (K_1 f(i))^{\gamma_1} (K_2 f(i))^{\gamma_2} \tag{37}$$

[1] We thank Xiangliang Li for suggesting the main outline of this proof.

where $\lambda = \bar{W}^{\sigma-1}$, $f(i) = w(i)^{1-\sigma}$,$K_l f(i) = \int K_l(s,i)f(s)ds$ $(l = 1,2)$ in which $K_l(s,i) = T_l\left(s,i\right)^{1-\sigma} u(i)^{\sigma-1} A_l\left(s\right)^{\sigma-1} f(s)$. Our task is to solve equation 37. In the following, we will successively prove the solution existence, uniqueness. The results are similar to the linear integral equation case.

Before we get into the details, we introduce two operators, T and T^* where

$$Tf(i) = \left(K_1 f(i)\right)^{\gamma_1} \left(K_2 f(i)\right)^{\gamma_2},$$

and

$$T^* f(i) = \frac{\left(K_1 f(i)\right)^{\gamma_1} \left(K_2 f(i)\right)^{\gamma_2}}{\int \left(K_1 f(i)\right)^{\gamma_1} \left(K_2 f(i)\right)^{\gamma_2} di}.$$

The proof proceeds in three steps. We first illustrate the existence of a solution for the operator T^*, then argue that this solution implies existence for the operator T, and finally prove uniqueness for the operator T.

Step 1: Existence of a solution for the operator T^* In this section, we employ the Shauder's Fixed Point Theorem (see, e.g. Agarwal, Meehan, and O'Regan (2001), p.38) to prove the existence of a solution for T^*, i.e $T^*f = f$ has a solution. First, we introduce the space of functions C= $\{f \in C:\ f \geq 0,\ f$is bounded, $f$is continuous, $\int f(i)di = 1\}$, where we solve the system and show that is closed, and convex; One may think that we could avoid considering the function to be continuous and bounded as there may be discountinous or unbounded solutions. However, in the proof, this is not possible under the current model setting and thus we restrict attention to such functions. Second, we show that $T*$ a) maps C to C, b) continuous, and c) compact. After proving those, we can use Schauder Fixed Point Theorem - which requires a continuous compact operator that maps a space closed convex space C into itself - to prove existence of a fixed point.

a) T^* maps C to C.

Let C be the space of functions that are nonnegative, continuous, bounded and $\int f(i)di = 1$. Define the norm $||f|| = \max\limits_i |f(i)|$. It is straightforward that C is closed (by Lebesgue's Dominated Convergence Theorem) and convex.

From the assumption of the model, it is easy to show that $K_1(s,i)$ and $K_2(s,i)$ are positive and bounded, say $0 < N \leq K_1(s,i), K_2(s,i) \leq M$. Then $N \leq K_1 f(i), K_2 f(i) \leq M$, so T^*f is positive and bounded. Furthermore, T^*f is also continuous. At first, it is easy to show that $K_1 f$ is continuous. In closed set S, $K(s,i)$ is uniformly continuous, i.e there exists δ so that $||a - b|| < \delta$ means $|K(s,a) - K(s,b)| < \epsilon$ for any $s \in S$ and $\epsilon > 0$.

Thus $|K_1f(a) - K_1f(b)| = |\int(K_1(s,a) - K_1(s,b))f(s)ds| < \epsilon$, K_1f is continuous. It is the same with K_2f. As a result, $T^*f = (K_1f(i))^{\gamma_1}(K_2f(i))^{\gamma_2}/P$ is also continuous, where $P = \int(K_1f(i))^{\gamma_1}(K_2f(i))^{\gamma_2}di$ (for a specific f , P is constant). The above derivation does not use the condition that f is continuous or bounded, which also means the restriction in last section that we only search the solution in continuous and bounded function space C is reasonable.

b) $T^*$is continuous

To prove $T*$is continuous, it is equivalent to show that for any given $f_a \in C$ and arbitrary $\epsilon > 0$, there exists δ such that for any f_b satisfying $||f_a - f_b|| < \delta$, $||T^*f_a - T^*f_b|| < \epsilon$.

Then, $|K_1f_a(i) - K_1f_b(i)| < \delta M * mes(S)$. Repeatedly using this inequality combined with some basic algebral transformation, one can make $|T^*f_a(i) - T^*f_b(i)| < \epsilon$ by changing the value of δ.

c) $T^*$is compact

By definition, to show $T*$ is compact, we only need to confirm $T^*(C)$ is compact. To prove this, we employ Ascoli-Arzela Theorem (see e.g. Aliprantis and Burkinshaw (1998), p.75). Thus, we need to show $T*(C)$ is closed, bounded, and equicontinuous.

First, it is bounded. From the above, $N * mes(S) \le P \le M * mes(S)$, so $T^*f(i) \le M/P \le M * (N * mes(S))^{-1}$. For any given $f \in C$, $||T^*f|| \le M * (N * mes(S))^{-1}$, which means $T^*(C)$ is bounded.

Second, it is equicontinuous.

$$
\begin{aligned}
|T^*f(a) - T^*f(b)| &= |(K_1f(a))^{\gamma_1}(K_2f(a))^{\gamma_2} - (K_1f(b))^{\gamma_1}(K_2f(b))^{\gamma_2}|/P \\
&\le [M^{\gamma_1}|(K_2f(a))^{\gamma_2} - (K_2f(b))^{\gamma_2}| + M^{\gamma_2}|(K_1f(a))^{\gamma_1} - (K_1f(b))^{\gamma_1}|]/P \\
&= \left[M^{\gamma_1}\gamma_2(m_2)^{\gamma_2-1}|K_2f(a) - K_2f(b)| + M^{\gamma_2}\gamma_1(m_1)^{\gamma_1-1}|K_1f(a) - K_1f(b)|\right]/P \\
&\le \epsilon * G
\end{aligned}
$$

where $G = [\gamma_2M^{\gamma_1}N^{\gamma_2-2} + \gamma_1M^{\gamma_2}N^{\gamma_1-2}]/mes(S)$ is constant unrelated with f, and m_1 is some value between $K_1f(a)$ and $K_1f(b)$(the result of using the mean value theorem), $m_2$similarly . Thus, the function set $T^*(C)$ is equicontinuous.

Third, as C is closed and $T*$ is continuous , so $T*(C)$ must be closed.

Having proved a)-c) for T^* we can thus apply the Schauder Fixed Point Theorem. Thus, there is a fixed point f_0 satisfying $T*f = f$.

Step 2: Existence for the operator T Step 1, also means that f_0 is also the solution of $Tf = \lambda f$ and the corresponding λ is $\lambda_0 = \int(K_1f_0(i))^{\gamma_1}(K_2f_0(i))^{\gamma_2}di$. Having proven the existence for T we can now turn to show that the solution f_0 is the unique one.

Step 3: Uniqueness for the operator T

In this section, we show the uniqueness of the solution for the operator T. First, λ_0 is the only eigenvalue with positive eigenfunction. Suppose there is another $\lambda_1$and corresponding eigenvector $f_1$satisfying $Tf_1 = \lambda_1 f_1$. Considering $\lim ||T^n f_0||^{1/n} = \lim ||\lambda_0^n f_0||^{1/n} = \lambda_0$, similarly we have $\lim ||T^n f_1||^{1/n} = \lambda_1$. From the above section, $f_0$and f_1 are positive continuous function. As S is compact, both f_0 and $f_1$must have positive lower and upper bounds. Thus there must be $\mu_0 > 0$ so that $f_1(i) > \mu_0 f_0(i)$ for every $i \in S$. Then, $T^n f_1(i) > \mu_0 T^n f_0(i)$, furthermore $\lim ||T^n f_1||^{1/n} \geq \lim ||T^n f_0||^{1/n}$. Similarly, we have $\lim ||T^n f_1||^{1/n} \leq \lim ||T^n f_0||^{1/n}$. The above means that $\lambda_0 = \lambda_1$.

Second, f_0 is the unique positive eigenvector(up to scale) corresponding λ_0. Suppose not, f_2 is another positive eigenvector. Set a proper value ofμ (it can be found) so that $f_0(s) - \mu f_2(s) \geq 0$ for all $s \in S$, and there is some$i \in S$ $f_0(i) - \mu f_2(i) = 0$.

$$\begin{aligned}
\lambda_0(f_0(i) - \mu f_2(i)) &= Tf_0(i) - \mu T f_2(i) \\
&= \left([K_1(f_0 - \mu f_2 + \mu f_2)](i)\right)^{\gamma_1} \left([K_2(f_0 - \mu f_2 + \mu f_2)](i)\right)^{\gamma_2} \\
&\quad - \left(K_1 \mu f_2(i)\right)^{\gamma_1} \left(K_2 \mu f_2(i)\right)^{\gamma_2}
\end{aligned}$$

As $f_2 \neq f_0$, for a nonzero measure $f_0 - \mu f_2 > 0$, thus it can be shown that $K_1(f_0 - \mu f_2)(i) > 0$, $[K_1(f_0 - \mu f_2 + \mu f_2)](i) > K_1 \mu f_2(i)$, similarly $[K_2(f_0 - \mu f_2 + \mu f_2)](i) > K_2 \mu f_2(i)$. In all, $\lambda_0(f_0(i) - \mu f_2(i)) > 0$ which contradicts $f_0(i) - \mu f_2(i) = 0$. QED.

5.2 Distribution of economic activity on the line with two sectors

In this subsection, we calculate the equilibrium distribution of economic activity on the line with two sectors. As far as we are aware, there is no analytical characterization of the solution, so we rely on computational methods.

Suppose that our space is a finite interval, i.e. $S = [-1, 1]$ and trade costs for each sector are constant across the line, so that $T_j(i, s) = \exp(\tau_j |i - s|)$ for sectors $j \in \{1, 2\}$. Suppose too that there are no spillovers, and let the distribution of exogenous amenities and productivities be uniform. Finally, set $\sigma = 4$ and $\gamma_1 = \gamma_2 = \frac{1}{2}$. Then we can solve for the equilibrium distribution of wages and population using equations (33) and (36) above.

We are interested in how changes in the level of trade costs affect the concentration of economic activity. The top panel of Figure 2 shows that increasing trade costs in either or both sector leads to increased concentration of the population near the center of the line when trade costs in both sectors are positive. In contrast, the bottom panel of Figure 2 shows that when trade in the second sector is costless, increasing trade costs in the first sector actually reduces economic concentration. Intuitively, there are two effects: on the one hand, as the trade costs in either sector increase, the sector's price index will become more unequal between the center of the line and its edges, which will provide a force for concentration. On the other hand, as the trade costs in the first sector increase relative to the second sector, a greater fraction of workers will be employed in the second sector, where the price index is relatively equal across space, providing a force of dispersion. If the trade costs in the second sector are sufficiently low, the latter effect will dominate the former.

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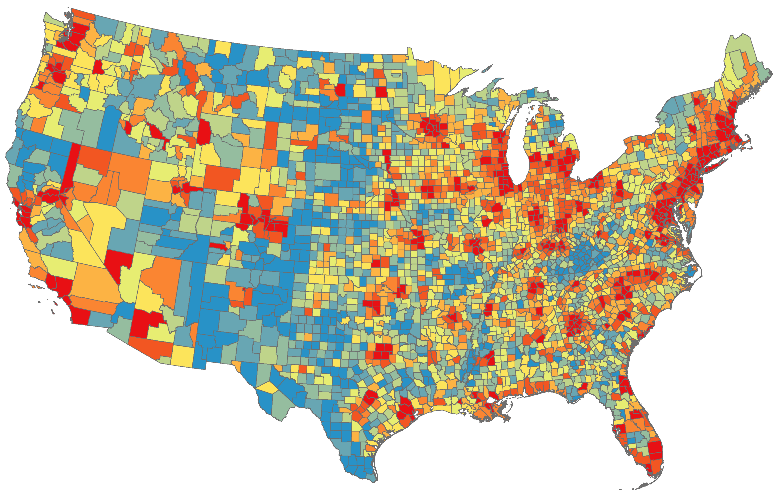
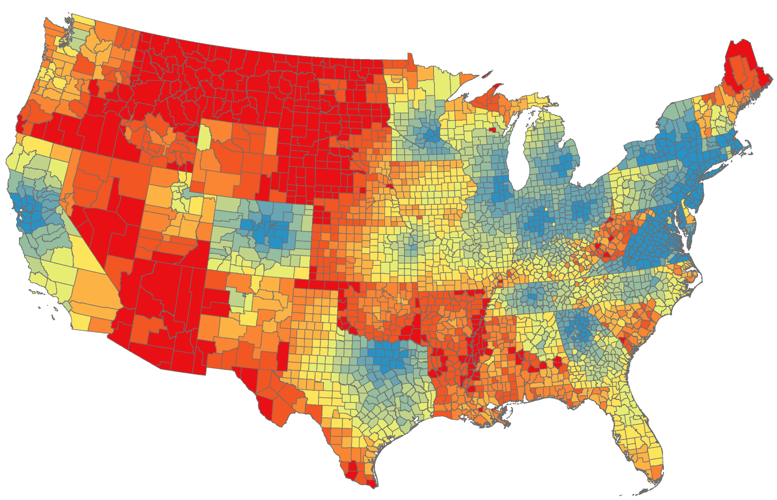
Figures

Figure 1: Estimated composite amenities and productivities allowing for commuting when commuting costs are ten times larger than the trade costs



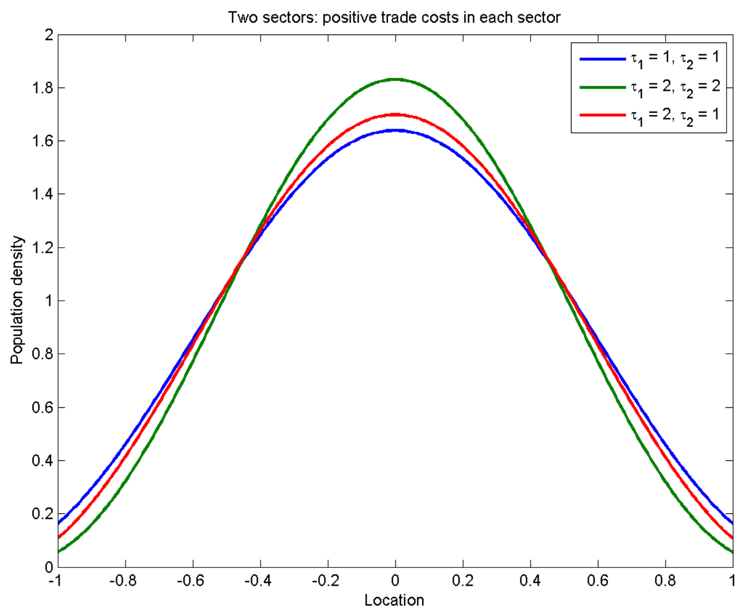
Composite productivities



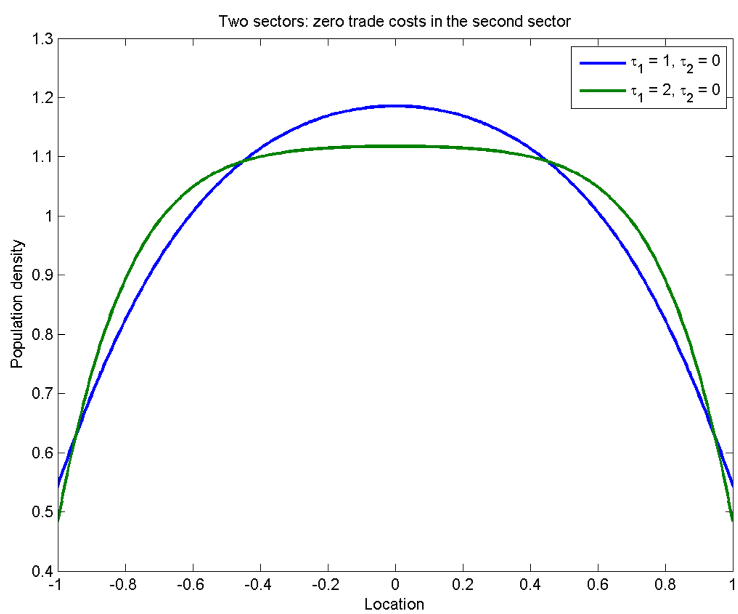
Composite amenities

Notes: This figure shows the spatial distribution of estimated composite productivities A and amenities u when we allow workers to commute. The commuting costs are assumed to be ten times larger (in ad valorem terms) than the trade costs. The color of the county indicates the decile of the productivity/amenity distribution: red indicates higher deciles, blue indicates lower deciles.

Figure 2: Two sectors and the effect of trade



Positive trade costs in the second sector



Zero trade costs in the second sector

Notes: This figure shows the equilibrium distribution of economic activity on the line when there are two sectors. In the top figure, the second sector is assumed to have positive trade costs. In this case, increasing the trade costs in either sector is associated with greater economic concentration. In the bottom figure, the second sector is assumed to have zero trade costs. In this case, increasing trade costs in the first sector reduces economic concentration.

Appendix

In this Appendix we provide the details of the derivations of expressions 28 and 29 in the main text.

For the first one, notice that we have

$$\int_0^{\pi} e^{((1-\sigma)\tau - in)\theta} d\theta = \frac{e^{(1-\sigma)\tau\pi} e^{-in\pi} - 1}{(1-\sigma)\tau - in} = \frac{e^{(1-\sigma)\tau\pi} cos(n\pi) - 1}{(1-\sigma)\tau - in} = \frac{(-1)^n e^{(1-\sigma)\tau\pi} - 1}{(1-\sigma)\tau - in}.$$

Similarly,

$$\int_{-\pi}^{0} e^{((\sigma-1)\tau - in)\theta} d\theta = \frac{1 - e^{(1-\sigma)\tau\pi} e^{in\pi}}{(\sigma-1)\tau - in} = \frac{1 - e^{(1-\sigma)\tau\pi} cos(n\pi)}{(\sigma-1)\tau - in} = \frac{1 - e^{(1-\sigma)\tau\pi}(-1)^n}{(\sigma-1)\tau - in}.$$

Below we denote $(1-\sigma)\tau - in$ by c, then $(\sigma-1)\tau - in = -((1-\sigma)\tau + in) = -\bar{c}$ where $\bar{c}$ stands for the complex conjugate of c it can be. The conjugate pairs have the following properties $c + \bar{c} = 2Re(c) = 2(1-\sigma)\tau$, $c\bar{c} = Re(c)^2 + Img(c)^2 = (1-\sigma)^2\tau^2 + n^2$, where $Re(c)$ stands for the real part of a complex conjugate and $Img(c)$ for the imaginary part. Adding the two terms we obtain

$$\begin{aligned}
\frac{(-1)^n e^{(1-\sigma)\tau\pi} - 1}{(1-\sigma)\tau - in} + \frac{1 - e^{(1-\sigma)\tau\pi}(-1)^n}{(\sigma-1)\tau - in} &= \frac{(-1)^n e^{(1-\sigma)\tau\pi} - 1}{c} - \frac{1 - (-1)^n e^{(1-\sigma)\tau\pi}}{\bar{c}} \\
&= \frac{\bar{c}(-1)^n e^{(1-\sigma)\tau\pi} - \bar{c} - c + c(-1)^n e^{(1-\sigma)\tau\pi}}{c\bar{c}} \\
&= \frac{2(1-\sigma)\tau[(-1)^n e^{(1-\sigma)\tau\pi} - 1]}{(1-\sigma)^2\tau^2 + n^2}.
\end{aligned}$$

Substitutingλ_0 from equation 24 and the expression for the two terms in equation to (27) using the above derivations, we finally obtain equation (28) in the main text.

Now we want to compute

$$\begin{aligned}
(bGy_0)(\theta) &= \int b(\theta,\theta')G(\theta,\theta')y_0(\theta')d\theta' \\
&= \int 1_A(\theta)1_B(\theta')G(\theta,\theta')d\theta' + \int 1_A(\theta')1_B(\theta)G(\theta,\theta')d\theta'.
\end{aligned}$$

For the first term

$$\begin{aligned}\int 1_A(\theta)1_B(\theta')G(\theta,\theta')d\theta' &= 1_A(\theta)\int 1_B(\theta')G(\theta,\theta')d\theta' \\ &= 1_A(\theta)(G*1_B)(\theta) \\ &= (\sum_{(n-m)=-\infty}^{\infty} 1_{A,n-m}e^{i(n-m)\theta})(\sum_{m=-\infty}^{\infty} 2\pi G_m 1_{B,m}e^{im\theta}) \\ &= \sum_{n=-\infty}^{\infty}(\sum_{m=-\infty}^{\infty} 1_{A,n-m}1_{B,m}2\pi G_m)e^{in\theta},\end{aligned}$$

with

$$1_{A,n} = \begin{cases} \frac{1}{2} & n=0 \\ 0 & n \text{ is even and nonzero} \\ \frac{(-1)^{\frac{n-1}{2}}}{\pi n} & n \text{ odd,} \end{cases} \tag{38}$$

and

$$1_{B,n} = \begin{cases} \frac{1}{2} & n=0 \\ 0 & n \text{ is even and nonzero} \\ \frac{(-1)^{\frac{n+1}{2}}}{\pi n} & n \text{ odd,} \end{cases} \tag{39}$$

are the Fourier coefficients of the indicator function and where third equality is simply the Fourier series expansion of the indicator functions, and the fourth equation cross-multiplies all the terms. Combining the results for the two integrals, we obtain the Fourier coefficients for all n :

$$(bGy_0)_n = \sum_{m=-\infty}^{\infty} 2\pi G_m(1_{A,n-m}1_{B,m} + 1_{B,n-m}1_{A,m}),$$

and using expressions 38 and 39 we finally obtain

$$(bGy_0)_n = \begin{cases} 0 & n \text{ odd} \\ \pi G_0\delta_{0n} - \frac{4}{\pi^2}(-1)^{\frac{n}{2}}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2-n^2}2\pi G_{2m+1} & n \text{ even} \end{cases}$$

where for our example G_0 which finally implies equation 29.