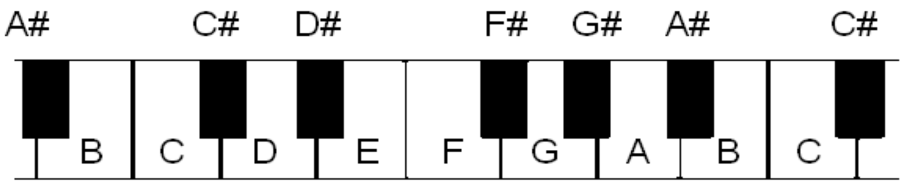
Math 105 – Music & Mathematics
Tuning Systems – Practice Exercises
Solutions (please try each exercise yourself before reading its solution)

1.



Make sure you're familiar with the layout of the standard piano keyboard!

2. Suppose we set out to devise a **just intonation** system based on "A" rather than "C". That is, the base frequency will be identified with one of the "A" tones on a keyboard, and other tones' frequencies will be found accordingly. Under this system, what should be the frequency, in Hz, of each of the following tones? (Where necessary, round answers to the nearest hundredth.)

Answers:

a) E above A:220.
Since the E is seven semitones above A:220, the A-E interval should be a perfect fifth under just intonation, with a frequency ratio of 3/2. Therefore, the frequency of the E above A:220 should be $220\times\frac{3}{2} = 330\ Hz$.

b) D above A:220.
The A-D interval consists of 5 semitones, so it would be tuned as a perfect fourth, with frequency ratio 4/3. So, the frequency of the D above A:220 under this system would be $220\times\frac{4}{3} = \frac{880}{3} \approx \mathbf{293.33\ Hz}$.

c) F above A:220
The A-F interval consists of 8 semitones, so it would be tuned as a minor sixth, with frequency ratio 8/5. So, the frequency of the F above A:220 under this system would be $220\times\frac{8}{5} = \mathbf{352\ Hz}$.

d) F below A:220.
Note that there are two ways to find the frequency for this note…

Solution: Since the F above A:220 is tuned to 352 Hz, the frequency of the F an octave lower would be one half of 352 Hz: $352\times\frac{1}{2} = \mathbf{176\ Hz}$.

Alternate solution: the F-A interval consists of 4 semitones, so it would be tuned as a major third, with frequency ratio 5/4. Since we'd have to lower A:220 by a major third in this case, we must divide by the corresponding frequency ratio: $220 \div \frac{5}{4} = 220\times\frac{4}{5} = \frac{880}{5} = \mathbf{176\ Hz}$.

e) G above A:220
The A-G interval consists of 10 semitones, so it would be tuned as a minor seventh, with frequency ratio 9/5. So, the frequency of the F above A:220 under this system would be $220\times\frac{9}{5} = \mathbf{396\ Hz}$.

f) A three octaves below A:220.

To lower a pitch by an octave, we multiply its frequency by 1/2. So, we'll have to multiply 220 by 1/2 three times, once for each time we lower by an octave. This gives us $220\times\frac{1}{2}=110$, then $110\times\frac{1}{2}=55$, then $55\times\frac{1}{2}=\frac{\mathbf{55}}{\mathbf{2}}$ or $\mathbf{27.5}\ \boldsymbol{Hz}$.

Note: An easier way to find this answer would be to multiply by 1/2 three times all at once, rather than through three separate steps: $220\times\underbrace{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}_{\substack{multiply\\ these\ first!}}=220\times\frac{1}{8}=\frac{220}{8}$, which reduces to $55/2$, or 27.5 Hz.

3. Find the frequency of each of the tones from #2 if we use **Pythagorean tuning,** rather than just intonation, to devise a tuning system based on A:220. (This means the frequency of each tone is found by raising or lowering by perfect fifths starting from A:220.) Hint: you may wish to use the "Circle of Fifths" diagram attached at the end of this document to help you figure out how many times to raise/lower by fifths.

Answers:

a) E above A:220.

Since the E is seven semitones above A:220, the A-E interval should be a perfect fifth, with a frequency ratio of 3/2. Therefore, the frequency of the E above A:220 should be $220\times\frac{3}{2}=330\ Hz$.

b) D above A:220.

Starting from A, we must lower by a perfect fifth once (then raise by an octave) to tune the next higher D. (You can find this by counting seven semitones to the left starting from A on a keyboard; or, on the "circle of fifths," count counterclockwise one place from A to find D.) So, we'll need to divide by 3/2 (to lower by a fifth), then multiply by 2 (to raise by an octave)…

$\underbrace{220\div\frac{3}{2}=220\times\frac{2}{3}=\frac{440}{3}}_{lower\ by\ a\ perfect\ fifth};\ \underbrace{\frac{440}{3}\times\frac{2}{1}=\frac{880}{3}}_{raise\ by\ an\ octave}$, or approximately 293.33 Hz.

c) F above A:220

Starting from A, we must lower by fifths four times to tune an F. (Again, this can be found by counting semitones on a keyboard, or by reading the "circle of fifths" diagram.) So, we'll have to divide by 3/2 four times – in other words, multiply by 2/3 four times – to tune an F; after this, we'll raise by octaves to get a result in the 220-440 Hz range. (Note: the exercise asks for "F above A:220," meaning the next higher F – this must have a frequency somewhere between 220 Hz and the next higher A, whose frequency is 440 Hz.)

First, we'll lower by fifths four times:

$$220\times\underbrace{\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}}_{\substack{it's\ best\ to\ deal\\ with\ these\ first\ldots}}=220\times\frac{2^4}{3^4}=220\times\frac{16}{81}=\frac{3520}{81}\text{ , or about }43.46Hz$$

Now, we'll need to raise this by octaves a few times to get a result in the 220-440 Hz range. Note that I'm sticking with fractions, because they are more precise than (rounded off) decimals…

Raise by octaves: first, we get $\frac{3520}{81}\times\frac{2}{1}=\frac{7040}{81}\approx 86.91$ (still too low…)

Another octave: $\frac{7040}{81}\times\frac{2}{1}=\frac{14080}{81}\approx 173.83$ (still under 220, but one more octave should do it…)

One more octave: $\frac{14080}{81}\times\frac{2}{1}=\frac{28160}{81}\approx\mathbf{347.65}\ \boldsymbol{Hz}$, which is our final answer!

Comment: This is one instance where just intonation and Pythagorean tuning diverge. Under just intonation, this F had a frequency of 352 Hz; under Pythagorean tuning, its frequency is about 347.65 Hz. This is due to the different frequency ratios – under just intonation, the A-F interval has frequency ratio 8/5; under Pythagorean tuning, its effective frequency ratio (combining all of the above multiplications into one step) is $\underbrace{\left(\frac{2}{3}\right)^4}_{\substack{lowering \\ by\ fifths}} \times \underbrace{\left(\frac{2}{1}\right)^3}_{\substack{raising \\ by\ octaves}} = \frac{16}{81}\times\frac{8}{1} = \frac{128}{81}$!

d) F below A:220.
We actually answered this in the work for the preceding example (F above A:220). Just before tuning the final F, we found that the frequency of the next lower F was approximately **173.83 Hz.**

e) G above A:220
Starting from A, we must lower by fifths twice to tune a G. (Again, this can be found by counting semitones on a keyboard, or by reading the "circle of fifths" diagram.) So, we'll have to divide by 3/2 twice– in other words, multiply by 2/3 twice – to tune a G; after this, we'll raise by octaves to get a result in the 220-440 Hz range. (This is very similar to the "F above A:220" example above)

First, we'll lower by a fifth twice:

$$220\times\frac{2}{3}\times\frac{2}{3} = 220\times\frac{4}{9} = \frac{880}{9} \text{ , or about } 97.78 Hz$$

Now, we'll need to raise this by octaves a couple of times to get a result in the 220-440 Hz range….
Raise by octaves: first, we get $\frac{880}{9}\times\frac{2}{1} = \frac{1760}{9} \approx 194.44\ Hz$
One more octave: $\frac{1760}{9}\times\frac{2}{1} = \frac{3520}{9} \approx \mathbf{388.89\ Hz}$. Since this is in the 220-440 Hz range, as requested, this is our answer.

Comment: Again, notice the difference between this result and the result under just intonation (396 Hz).

f) A three octaves below A:220.
Solution: 27.5 Hz. This is found in exactly the same way under Pythagorean tuning as it is found under just intonation, since octaves have a 2/1 frequency ratio under both tuning systems.

4. Suppose we tune C to a base frequency of 648 Hz. Consider the 12-tone scale from C:648 up to the C one octave higher (C:1296). Tune the 12-tone scale in each of the following ways. When necessary, round answers to the nearest hundredth of a Hertz.

(a) Find the correct frequency for each tone in the scale using Pythagorean tuning.
(b) Find the correct frequency for each tone in the scale using Just Intonation.
(c) Find the correct frequency for each tone in the scale using equal temperament.

Answers:
Part (a) – Pythagorean Tuning. (Similar to Collected HW #1)
Recall that to tune a scale by Pythagorean tuning, we raise or lower by perfect fifths, adjusting by octaves when necessary. Specifically, raise by fifths six times, and lower by fifths five times…
Base: C:648 Hz. We'll start by finding R1, R2, etc…
R1. Raising by a fifth from C gives us a G. Frequency: $648\times\frac{3}{2}=\frac{1944}{2}=972\ Hz$. So, we have $\boldsymbol{G}\text{: }\mathbf{972\ \boldsymbol{Hz}}$.
R2. Raising by a fifth from G gives us D. (Recall – you can find these by counting 7 semitones on a keyboard, or by following the attached "circle of fifths" diagram in the clockwise direction.) So, we'll multiply our previous result of 972 Hz by 3/2… $972\times\frac{3}{2}=\frac{2916}{2}=1458\ Hz$. But, note that 1458 Hz falls outside of the range of our scale (648-1296 Hz), so we'll need to lower it by an octave: $1458\times\frac{1}{2}=729$. So, we have $\boldsymbol{D}\text{: }\mathbf{729\ \boldsymbol{Hz}}$.

Proceeding in a similar way, we get the following results for R3, R4, R5 and R6. (I'm just including the answers below; make sure that you understand where they come from and can work them out for yourself!...) For each I'm giving an exact answer as a fraction, as well as a decimal (rounded to the nearest hundredth if necessary).
R3. Tone: **A.** Frequency: **2187/2 Hz**, or $\mathbf{1093.5\ \boldsymbol{Hz}}$
R4. Tone: **E.** Frequency: **6561/8 Hz**, which is approximately $\mathbf{820.13\ \boldsymbol{Hz}}$
R5. Tone: **B.** Frequency: **19683/32 Hz**, which is approximately $\mathbf{1230.19\ \boldsymbol{Hz}}$
R6. Tone: **F#.** Frequency: **59049/64 Hz**, which is approximately $\mathbf{1845.28\ \boldsymbol{Hz}}$

Next, we'll find L1, L2, etc…
L1. Lowering by a fifth from C gives us an F, whose frequency will be $648\div\frac{3}{2}=648\times\frac{2}{3}=\frac{1296}{3}=432$ Hz. However, we'll need to raise this by an octave, giving us $432\times 2=864\ Hz$. So, we have $\boldsymbol{F}\text{: }\mathbf{864\ \boldsymbol{Hz}}$.

L2. Lowering by a fifth from F gives us A# (or Bb). As before, we multiply by 2/3 to lower a frequency by a perfect fifth: $864\times\frac{2}{3}=\frac{1728}{3}=576$ Hz. Again, we'll need to raise by an octave, giving us $576\times 2=1152\ Hz$. So, we have $\boldsymbol{A}\#\text{: }\mathbf{1152\ \boldsymbol{Hz}}$.

As before, I'll list the correct answers for L3, L4 and L5 below; verify for yourself that you understand and are able to duplicate these answers…

L3. **D#:** $\mathbf{768\ \boldsymbol{Hz}}$
L4. **G#:** $\mathbf{1024\ \boldsymbol{Hz}}$
L5: **C#:** $\mathbf{2048/3\ \approx 682.67\ \boldsymbol{Hz}}$.

#4, cont.

Part (b) – Just intonation (similar to Collected HW #2)

To tune each tone in the octave, just multiply 648 Hz by the corresponding frequency ratio of each interval based at C:648. We'll find the answers in chromatic order (that is, keyboard order) from C# up to B… (Note: the just intonation frequency ratios that we use are given in the diagram for exercise #12.)

C# (one semitone): $648\times\frac{16}{15}=\frac{10368}{15}=\frac{3456}{5}=691.2\ Hz$

D (two semitones): $648\times\frac{9}{8}=\frac{5832}{8}=729\ Hz$

(Note: the above calculation becomes a bit easier if you notice that 8 is a factor of 648:

$$\frac{\overset{81}{\cancel{648}}}{1}\times\frac{9}{\underset{1}{\cancel{8}}} = 81\times 9 = 729$$

This cross-cancellation is not strictly necessary, but it helps! We'll use it below where applicable…)

D# (three semitones): $648\times\frac{6}{5}=\frac{3888}{5}=777.6\ Hz$

E (four semitones): $648\times\frac{5}{4}=\frac{648}{1}\times\frac{5}{4}=\frac{162}{1}\times\frac{5}{1}=810\ Hz$

F (five semitones): $648\times\frac{4}{3}=\frac{648}{1}\times\frac{4}{3}=\frac{216}{1}\times\frac{4}{1}=864\ Hz$

F# (six semitones): $648\times\frac{45}{32}=\frac{81}{1}\times\frac{45}{4}=\frac{3645}{4}=911.25\ Hz$

G (seven semitones): $648\times\frac{3}{2}=\frac{648}{1}\times\frac{3}{2}=324\times3=972\ Hz$

G# (eight semitones): $648\times\frac{8}{5}=\frac{5184}{5}=1036.8\ Hz$

A (nine semitones): $648\times\frac{5}{3}=\frac{648}{1}\times\frac{5}{3}=\frac{216}{1}\times\frac{5}{1}=1080\ Hz$

A# (ten semitones): $648\times\frac{9}{5}=\frac{5832}{5}=1166.4\ Hz$

B (eleven semitones): $648\times\frac{15}{8}=\frac{648}{1}\times\frac{15}{8}=\frac{81}{1}\times\frac{15}{1}=1215\ Hz$

C (twelve semitones): $648\times2=1296\ Hz$

Part (c) – Equal Temperament: Under 12-TET, we raise the base tone by n semitones by multiplying its frequency by $2^{n/12}$.

C# (one semitone above C): $648\times2^{1/12}\approx686.53\ Hz$

D (two semitones above C): $648\times2^{2/12}\approx727.36\ Hz$

D# (3 semitones): $648\times2^{3/12}\approx770.61\ Hz$

E: (4 semitones): $648\times2^{4/12}\approx816.43\ Hz$

F: (5 semitones): $648\times2^{5/12}\approx864.98\ Hz$

F#:(6 semitones): $648\times2^{6/12}\approx916.41\ Hz$

G: (7 semitones): $648\times2^{7/12}\approx970.90\ Hz$

G#: (8 semitones): $648\times2^{8/12}\approx1028.64\ Hz$

A: (9 semitones): $648\times2^{9/12}\approx1089.80\ Hz$

A#: (10 semitones): $648\times2^{10/12}\approx1154.60\ Hz$

B: (11 semitones): $648\times2^{11/12}\approx1223.26\ Hz$

C: (12 semitones): $648\times2=1296\ Hz$

5. For the following, use a just intonation frequency ratio for each interval:
Suppose, from a starting pitch of 360 Hz, the pitch is raised by an octave, then lowered by a perfect fifth, then lowered by a major sixth, then raised by a perfect fourth, then lowered by an octave, and finally raised by a minor third.
a) At what frequency do we end up? What are the other frequencies we hit on the way there?
b) What is the frequency ratio of the interval formed by the ending pitch and the starting pitch (360 Hz)?
c) Can you think of a way we could have found the answer to part (b) more quickly (without necessarily finding the answers to (a))?

Solution:
a) The final pitch is **230.4 Hz.** The intermediate pitches are, in order:

- "raised by an octave:" $360\times2 = \mathbf{720}$ **Hz**
- "lowered by a perfect fifth:" $720\div\frac{3}{2} = 720\times\frac{2}{3} = \mathbf{480}$ **Hz**
- "lowered by a major sixth:" $480\div\frac{5}{3} = 480\times\frac{3}{5} = \mathbf{288}$ **Hz**
- "raised by a perfect fourth:" $288\times\frac{4}{3} = \mathbf{384}$ **Hz**
- "lowered by an octave:" $384\times\frac{1}{2} = \mathbf{192}$ **Hz**
- "raised by a minor third:" $192\times\frac{6}{5} = \mathbf{230.4}$ **Hz**

b) We started at 360 Hz and ended up at 230.4 Hz. To compute a frequency ratio we typically divide the higher frequency by the lower frequency; in this case, this gives us a frequency ratio of $\frac{360}{230.4} = 1.5625$.

c) We could have found the answer more quickly by combining all of the operations of part (a) into one computation. Starting from 360 Hz, we knew (from the instructions) that we'd have to multiply by 2, then 2/3, then 3/5, then 4/3, then ½, then 6/5. Since multiplication is commutative (order doesn't matter) and associative (groupings are arbitrary), we could have proceeded as follows:

$$360\times\frac{2}{1}\times\frac{2}{3}\times\frac{3}{5}\times\frac{4}{3}\times\frac{1}{2}\times\frac{6}{5} = 360\times\frac{2}{1}\times\frac{2}{\cancel{3}}\times\frac{\cancel{3}}{5}\times\cancel{\frac{4}{3}\times\frac{1}{2}}\times\frac{\cancel{6}}{5} \quad (3\times2=6)$$

(Look for common factors to "cancel" from top and bottom)

$$= 360\times\frac{2\times2\times4}{5\times5} = 360\times\frac{16}{25}$$

… and $360\times\frac{16}{25} = 230.4\ Hz$, as expected (since that was the answer to part (a).) Again, the point here is that we could have multiplied all of the frequency ratios first, to find that the ratio of the ending frequency to the beginning frequency is 16/25, then multiplied 360 by that number to get our result.

Note: The reason our ratio from (b), 1.5625, and our ratio from (c), 16/25 (or 0.64) are unequal is that they work in different "directions," since in (c) we determined how to lower (rather than raise) from the beginning frequency to the ending frequency. Another way to look at this: since we're actually lowering 360 Hz down to 230.4 Hz, we can look at the above step of multiplying by 16/25 as actually dividing by 25/16 – which, if you calculate its decimal form, is equal to 1.5625, corresponding to our result for part (b).

6. Redo #5, but use equal temperament rather than just intonation frequency ratios.

Answers:
Remember that under 12-TET, all frequency ratios are of the form $2^{x/12}$, where x is the number of semitones between the upper and lower tones in the interval. So, we get the following results…

a) The final pitch is **230.4 Hz.** The intermediate pitches are given in order below. Note that I've rounded each individual result off to the nearest hundredth; however, you should leave it as-is on your calculator to go from one step to the next (otherwise you'll get some errors, due to rounding at intermediate steps)…

* "raised by an octave:" $360 \times 2 = \mathbf{720}$ **Hz**
* "lowered by a perfect fifth:" $720 \div 2^{7/12} \approx \mathbf{480.54}$ **Hz**
* "lowered by a major sixth:" previous result $\div\, 2^{9/12} \approx \mathbf{285.73}$ **Hz**
* "raised by a perfect fourth:" previous result $\times 2^{\frac{5}{12}} \approx \mathbf{381.41}$ **Hz**
* "lowered by an octave:" previous result $\times \frac{1}{2} \approx \mathbf{190.70}$ **Hz**
* "raised by a minor third:" previous result $\times 2^{\frac{3}{12}} \approx \mathbf{226.79}$ **Hz**

b) As in #5, divide the higher by the lower: $\frac{360}{226.79} \approx 1.587 \ldots$

c) This problem is actually MUCH easier to do if we combine steps. In fact, we can make use of the fact that we're in equal temperament – in which every semitone is consistent – to observe that all we really have to do here is count semitones! Step by step: first, "raise by an octave" means we raise by 12 semitones. Next, "lower by a perfect fifth" means lower by 7 semitones; "lower by a major sixth" means lower by 9 semitones; and so on. Proceeding in this way, we can find the number of semitones' difference between the starting and ending tones… $\underbrace{+12}_{\substack{raise\\octave}} \underbrace{-7}_{\substack{lower\\fifth}} \underbrace{-9}_{\substack{lower\\maj.sixth}} \underbrace{+5}_{\substack{raise\\fourth}} \underbrace{-12}_{\substack{lower\\octave}} \underbrace{+3}_{\substack{raise\\min.third}} = -8$. That is, the ending tone is eight semitones below the starting tone. In 12-TET, in order to lower a pitch by 8 semitones, we divide by the corresponding frequency ratio of $2^{8/12}$. So, our result is $360 \div 2^{8/12} \approx \mathbf{226.79}\ \boldsymbol{Hz}$

(Comment: Note that $2^{8/12} \approx 1.587$, corresponding to our result from part (b) above.)

7. Convert each of the following frequency ratios into “cents.” Round each answer to the nearest whole number.

Solutions: For each, we may use the formula $c = 1200\times\frac{\log(r)}{\log(2)}$ to convert from a frequency ratio into cents:

a) $1200\times\frac{\log(7/6)}{\log(2)} \approx 267$ cents

b) $1200\times\frac{\log(12/7)}{\log(2)} \approx 933$ cents

c) $1200\times\frac{\log(11/6)}{\log(2)} \approx 1049$ cents

d) $1200\times\frac{\log(7/5)}{\log(2)} \approx 583$ cents

e) $1200\times\frac{\log(14/5)}{\log(2)} \approx 1783$ cents

f) $1200\times\frac{\log(3)}{\log(2)} \approx 1902$ cents

8. Convert each of the following to frequency ratios; round your answers to the nearest hundredth.

Solutions: For each, use the formula $r = 2^{c/1200}$ (that's 2 to the *power* $c/1200$) to convert from cents into a frequency ratio:

a) $2^{351/1200} \approx 1.22$

b) $2^{580/1200} \approx 1.40$

c) $2^{885/1200} \approx 1.67$

d) $2^{949/1200} \approx 1.73$

e) $2^{1404/1200} \approx 2.25$

f) $2^{1586/1200} \approx 2.50$

#8.5 – Optional exercise - The answers to #8 could be considered "close," respectively, to the following:

- $2^{350/1200} \approx 1.22$, *which could reasonably be considered "close" to either 6/5 (exactly 1.2) or 11/9 (1.222…)*
- $2^{580/1200} \approx 1.40$, *which is 7/5*
- $2^{885/1200} \approx 1.67$, *which is very close to 5/3 (1.6666…)*
- $2^{949/1200} \approx 1.73$, *which is pretty close to 7/4 (1.75) or 19/11 (1.727272…)*
- $2^{1404/1200} \approx 2.25$, *which is 9/4*
- $2^{1586/1200} \approx 2.50$, *which is 5/2*

9. Simplify each of the following, using properties of exponents. You should be able to write each answer as a whole number.

a) $3^7 \times 3^{-5}$ b) $\left(5^{1/4}\right)^8$ c) $9^{3/2}$ d) $8^{2/3}$ e) $\frac{(12^{20} \times 12^{30})}{12^{49}}$

Answers:
a) By the product rule, we can add exponents: $3^7 \times 3^{-5} = 3^{7-5} = 3^2$ = 9.
b) By the power-of-a-power rule, we can multiply exponents: $\left(5^{1/4}\right)^8 = 5^{\left(\frac{1}{4}\right) \times 8} = 5^2 = 25.$
c) Recall that a fraction exponent means that we will first find a root, and then raise that root to a power. In this case, an exponent of 3/2 means the second root of 9 will be raised to the third power. The second, or "square," root of 9 is 3; 3 to the third power is $3 \times 3 \times 3 = 27$. Thus, $9^{3/2} = \left(9^{1/2}\right)^3 = 3^3 = 27.$
d) Similar to part (c) – this time we're finding the third root of 8, then raising that to the second power. The third root of 8 is 2 (since $2^3 = 8$), so $8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4.$
e) By the product rule, the numerator of this expression is $12^{20+30} = 12^{50}$.
By the quotient rule (which allows us to subtract one exponent from another), then,

$$\frac{12^{20} \times 12^{30}}{12^{49}} = \frac{12^{50}}{12^{49}} = 12^{50-49} = 12^1 = 12.$$

10. Simplify each of the following, using properties of logarithms. You should be able to write each answer as a whole number.

a) $\log_{10}(100)$ b) $\log_2(32)$ c) $\log_4(32)$ d) $\log_3(9^{1000})$

For each of these, remember that the expression "$\log_b(n)$" stands for an exponent – specifically, it stands for the unique real number, x, such that b^x will be equal to n.

Answers:
a) We'd raise 10 to the 2nd power to get 100; therefore, by definition of logarithm, $\log_{10}(100) = 2$

b) What power of 2 gives us 32? It turns out to be a whole number – if you multiply by 2 five times, the result is 32. Therefore, $2^5 = 32$, which means the value of $\log_2(32)$ is 5.

c) What power of 4 gives us 32? This one turns out not to be a whole number. This is because $4^2 = 16$, and $4^3 = 64$, so $\log_4(32)$ must be somewhere between 2 and 3. To answer this without a calculator, we need the following observations: first, note that we already figured out (part (b)) that $2^5 = 32$. Next, notice that 2 is the square root of 4 – that is, $2 = 4^{1/2}$. Put this together, and we have the following: if we take the second ("square") root of 2, then raise that number to the 5th power, the result is 32 – that is,

$$2^5 = \underbrace{\left(4^{1/2}\right)}_{\substack{\text{this is} \\ \text{equal to } 2}}{}^5 = 4^{5/2} = 32.$$

This idea of raising a root to a power is exactly what leads us to fractional exponents – in particular, the second root gives us a denominator (in the exponent) of 2, and the fifth power gives us a numerator of 5. So, what we're describing here is raising 4 to the 5/2 power. Therefore, $\log_4(32) = 5/2$.

(Short answer: $\log_4(32) = \log_4(2^5) = \log_4\left(\left(4^{1/2}\right)^5\right) = \log_4(4^{5/2}) = 5/2.$)

c) (cont.) Another way to figure out the value of $\log_4(32)$ is to observe that $32 = 2^5$. This allows us to use the "log-of-a-power" property of logarithms as follows:

$$\log_4(32) = \log_4(2^5) = 5 \cdot \log_4(2).$$

At this point, note that 2 is the square root of 4 – that is, $2 = 4^{1/2}$- which implies $\log_2(4) = 1/2$.
Therefore, we can substitute ½ for $\log_2(4)$ in the above calculation, giving us:

$$\log_4(32) = \log_4(2^5) = 5 \cdot \log_4(2) = 5 \cdot (1/2) = 5/2.$$

d) To find the value of $\log_3(9^{1000})$, we must answer the question: what power of 3 gives us 9^{1000}. (Resist the urge to turn on your calculator – it probably won't be able to help anyway!) The key to this problem is to observe that $9 = 3^2$; therefore, anywhere we see a 9 we can substitute 3^2 in its place. In particular:

$$9^{1000} = \underbrace{(3^2)}_{\substack{\text{this is}\\ \text{equal}\\ \text{to 9}}}{}^{1000} \underbrace{= 3^{2\times 1000}}_{\substack{\text{Here we use the}\\ \text{power of a power}\\ \text{rule...}}} = 3^{2000}.$$

Thus, 9^{1000} is equal to 3^{2000}; that is, we'd have to raise 3 to the 2000th power to get 9^{1000}.
In other words: $\log_3(9^{1000}) = 2000$.

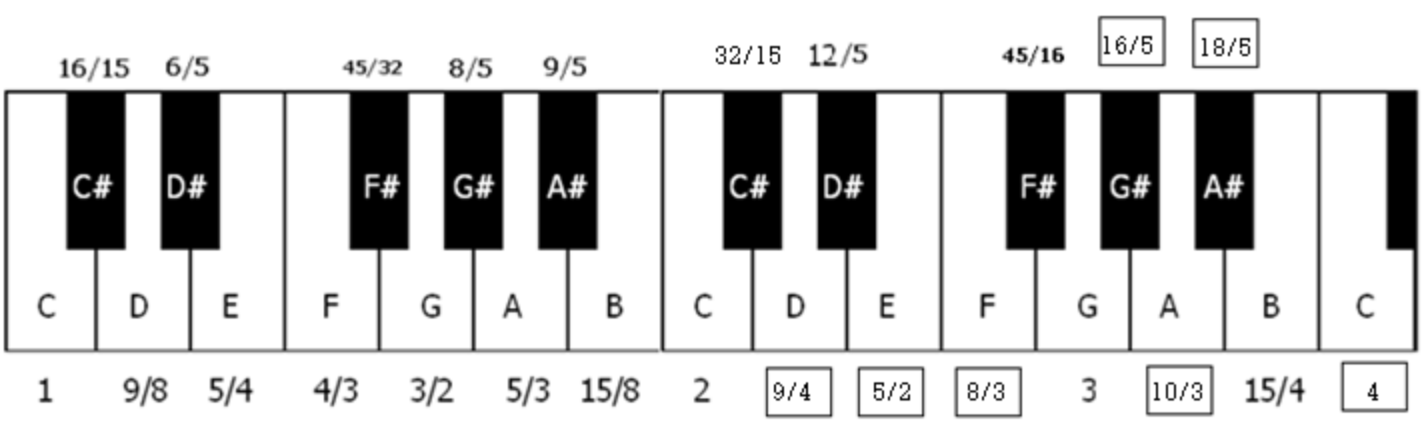
(Short answer: $\log_3(9^{1000}) = \log_3((3^2)^{1000}) = \log_3(3^{2000}) = 2000$.)

Alternative solution – again, using the "log-of-a-power" rule, we could compute $\log_3(9^{1000})$ as follows:

$$\log_3(9^{1000}) = 1000 \cdot \underbrace{\log_3(9)}_{\substack{9=3^2,\ \text{so}\\ \log_3(9)=2}} = 1000 \cdot 2 = 2000.$$

11. Consider the following diagram, which represents our version of just intonation based on "C":

Answers for part (a) shown in the diagram below:



Solutions:
a) Each frequency ratio is found based on the fact that the frequency ratio of an octave is 2/1. So, for example: the G# in the second octave is one octave above the G# in the first octave, whose frequency (as shown) is 8/5 of the base frequency. Therefore, to raise by an octave, we multiply the frequency ratio by two, giving us $\frac{2}{1} \times \frac{8}{5} = \frac{16}{5}$ of the base frequency.

#11, cont.

b) Here are a few examples of "broken fifths:"

- D – A: This one is explained in the instructions (it was given as an example of a "broken fifth").

- F# - C#: These two tones are separated by seven semitones. However, the frequency ratio of this interval – that is, the higher frequency divided by the lower frequency – is
$$\frac{32}{15} \div \frac{45}{32} = \frac{32}{15} \times \frac{32}{45} = \frac{1024}{675} \approx 1.52.$$

  This is not equal to 1.5, so F# - C# is a "broken fifth" under this version of just intonation.

- A# - F: These are also separated by seven semitones; however, the frequency ratio of this interval is
$$\frac{8}{3} \div \frac{9}{5} = \frac{8}{3} \times \frac{5}{9} = \frac{40}{27} \approx 1.48,$$

  which is not equal to 1.5. Therefore, A#-F is yet another "broken fifth" under this intonation.

These are all the examples of "broken fifths" that I found. (Did I miss any?)

c) A few examples of "broken major thirds" (see if you can find others)…

- E – G#: These two notes are four semitones apart; however, the frequency ratio of this interval is
$$\frac{8}{5} \div \frac{5}{4} = \frac{8}{5} \times \frac{4}{5} = \frac{32}{25} = 1.28,$$

  which is not equal to 1.25. Thus, this is a "broken major third."

- F# - A#: These two notes are four semitones apart; however, the frequency ratio of this interval is
$$\frac{9}{5} \div \frac{45}{32} = \frac{9}{5} \times \frac{32}{45} = \frac{288}{225} = 1.28,$$

  which is not equal to 1.25. Thus, this is another "broken major third."

- A – C#: These two notes are four semitones apart; however, the frequency ratio of this interval is
$$\frac{32}{15} \div \frac{5}{3} - \frac{32}{15} \times \frac{3}{5} = \frac{96}{75} = 1.28,$$

  which is not equal to 1.25. Thus, this is another "broken major third."

  (Note that E-G#, F#-A#, and A-C# all have exactly the same frequency ratio: 36/25, or 1.28. This isn't necessarily significant, but it is interesting!)

These are all the "broken major thirds" that I found. (Did I miss any?)

12. Suppose a musician decides to construct a keyboard that divides the octave into 17 tones, rather than the usual 12, using equal temperament. (In other words, consider "17-TET.") Note – just for the following questions, we'll refer to the interval between two consecutive tones of the 17-tone scale as a "semitone."

a) What would be the frequency interval of each "semitone" (rather than the usual $2^{1/12}$, or $\sqrt[12]{2}$)?

Answer: Under 17-TET, each semitone's frequency ratio would be $2^{1/17}$ rather than $2^{1/12}$, so that raising by a "semitone" 17 times would have a cumulative effect of doubling the original frequency, resulting in an octave.

b) Find the frequency of each tone in an octave (as a multiple of the base frequency). Give an exact answer for each, as well as a decimal rounded to the nearest thousandth (three places).

Answer: These would be: $2^{1/17}, 2^{2/17}, 2^{3/17}$, etc… all the way up to $2^{16/17}$ (the 16th note of the scale, just before the octave is reached), and then 2 (for the octave). The decimal approximations should be as follows:

$$2^{1/17} \approx 1.042;\ 2^{2/17} \approx 1.085; 2^{3/17} \approx 1.130;\ 2^{4/17} \approx 1.177; 2^{5/17} \approx 1.226;\ 2^{6/17} \approx 1.277;$$
$$2^{7/17} \approx 1.330; 2^{8/17} \approx 1.386;\ 2^{9/17} \approx 1.443;\ 2^{10/17} \approx 1.503; 2^{11/17} \approx 1.566; 2^{12/17} \approx 1.631;$$
$$2^{13/17} \approx 1.699; 2^{14/17} \approx 1.770;\ 2^{15/17} \approx 1.843;\ 2^{16/17} \approx 1.920; 2^{17/17} = 2.$$

c) Under a 17-TET tuning system, an interval of how many "semitones" comes the closest to approximating a perfect fifth? (Hint: because we're currently in 17-TET rather than 12-TET, the answer will *not* be seven "semitones.")

Solutions: A perfect fifth has a frequency ratio of 1.5, so we'd want to find the frequency ratio from part (b) that is closest to this value. Of these ratios, the closest fit is $2^{10/17} \approx 1.503$; therefore, under 17-TET, a ten-"semitone" interval would be the closest approximation to a perfect fifth. (The next closest fit would be a nine-"semitone" interval, whose frequency ratio is $2^{9/17} \approx 1.443$. These frequency ratios are "off" by 0.003 and 0.057 respectively.)

Comment: Recall that under 12-TET, the seven-semitone interval has frequency ratio $2^{\frac{7}{12}} \approx 1.498$, which is closer to the desired 3/2 ratio than we can achieve in a 17-TET scale. This serves to illustrate why 12-TET is generally used in preference to other equal temperament systems such as 17-TET. (In fact, no ET tuning system gives us a better fit to the perfect fifth unless we are willing to consider scales with more than 40 tones per octave!)

d) How many 17-TET "semitones" would most closely approximate a perfect fourth? …a major third? …a major sixth?

Solution: We'll just look at the major third here as an example. The frequency ratio for a major third is 1.25. Looking at the frequency ratios we found in part (b) above, we see that none of the notes in the 17-tone scale comes very close to this. The closest ones are 1.226 (for a 5-tone interval) or 1.277 (for a 6-tone interval). The former is off by 0.024 (that is, 1.250-1.226=0.024), while the latter is off by 0.027, so in that sense the 5-step interval is slightly closer.

Another good way to compare intervals is by using "cents" measurement. We'll use "cents" here, just for a little extra practice with the formula…

A “pure” major third has a width of $1200 \times \frac{\log(1.25)}{\log(2)} \approx 386.3$ cents

The 5-step interval, whose frequency ratio is $2^{5/17}$, would have a width of $1200 \times \frac{\log(2^{5/17})}{\log(2)} \approx 352.9$ cents

The 6-step interval, whose frequency ratio is $2^{6/17}$, would have a width of $1200 \times \frac{\log(2^{6/17})}{\log(2)} \approx 423.5$ cents

The 5-step interval is off by 386.3-352.9=33.4 cents; the 6-step interval is off by 423.5-386.3=37.2 cents. So, the “winner,” by about 3.8 cents, is the 5-“semintone” interval in 17-TET. (Note: with a 33-cent “error,” this is *not* a very good approximation of a pure major third.)