UNIT FOUR – COUNTING

In this unit, we explore problems in which the objective is to count the number of objects in a collection, or the number of ways to complete a given task, without having to actually write out a complete list to be counted one by one. Such an example, from the preceding chapter, would be counting the number of changes (orderings) required to complete an extent on a given number of bells. We saw that this number grew very quickly relative to the number of bells – 5 bells required 120 changes, 7 bells required 5040 changes, and so on. Fortunately we discovered that "factorials" allow us to compute these totals without the need to write out the entire list of changes. (Note: factorials will be re-introduced in this chapter, for the benefit of those who did not read/cover Chapter 3.)

In this chapter, we will develop new techniques – many of which follow closely from the factorials method from the previous chapter – to solve more complex counting problems.

Before we proceed, let's introduce some useful vocabulary:

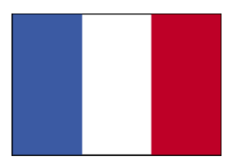
Set

The word "set" indicates an unordered collection of "objects," or "elements." We'll use curly brackets { } to indicate a set. For example, {red , white, blue} indicates a set of colors including red, yellow and blue; no ordering is implied. This simply indicates that our set includes these three colors, while other colors (say, purple, yellow, maroon, etc) are not included in our set.

{red, white, blue}

For example, it might make sense to consider the *set* {red, white, blue} when discussing the colors that appear on the U.S. flag. This is because {red, white, blue} includes all of the colors on the flag, but does not imply any particular ordering of these colors. (Question: although these colors are usually expressed verbally as "red, white and blue," in that order, does the ordering have any real significance in this context? Why or why not? This is the basic question we use to distinguish between "sets" and "lists.")

List



[blue, white, red]

The word "list" indicates an ordered collection of objects. We'll use square brackets [ ] to indicate a list. For example, [blue, white, red] indicates that we are considering these three colors in this specific order. This distinction is drawn whenever ordering *is* relevant – for example, if we were discussing national flags, there are examples for which it *would* make sense to discuss the order in which the colors appear – for instance, from left to right in the flag's design. (Do you know which national flag's colors are typically described as [blue, white, red], in that order?)

Factorials

Definition: "n factorial," which is also written as $n!$, is defined as follows:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

Factorials arise naturally in solutions to many problems involving counting.

Example: Given a set of n distinct notes, it turns out that are $n!$ different "melodies" that use each note exactly once. (Here we are not taking rhythm into account – just the sequence of the notes.)

For example, there are 3!, or 6, ways to arrange the notes C,D,E:

If we add a fourth note – say, G – then the number of arrangements is multiplied by 4, as illustrated by the following diagram:

Note that in this diagram, each row consists of the original row of 3-note melodies. However, in each row, the new note, G, is inserted in a different location. The idea is that we obtain six 4-note melodies by putting the G in the first position, six more 4-note melodies by putting G in the second position, six more by putting G third, and six more by putting G fourth. This is why we end up with four times as many 4-note arrangements as 3-note arrangements; hence, there are $4 \times 3!$, or $4!$, ways to rearrange four notes. This gives us a total of 24 "melodies."

Similarly, from each of the 4! 4-note melodies shown above, we can get five new 5-note melodies by placing a fifth note in one of its five possible positions; hence, there would be 5!

distinct 5-note melodies. Proceeding similarly, we see that there are, in general, $n!$ ways to arrange n different notes.

But… what if we allow repetition of notes? And, what if we don't require every possible note to be used in our melody? This modifies the count somewhat.

Example: If we restrict ourselves to the six notes {C, D, E, F, G, A}, how many distinct 4-note melodies are possible, assuming that no note is used more than once?

Answer: There are 6 possibilities for the first note, 5 possibilities for the second note, 4 for the third note, and 3 for the fourth note. (Note that, in this example, the number of options at each step decreases by 1, since repetition of a previously used note is not allowed. This is typical of counting problems which do not allow repetition.) So, the total number of possible melodies of this type is $6 \times 5 \times 4 \times 3$, or 360.

<u>Permutations</u>

A "permutation" is defined as an ordered selection (or "list") of objects in which no object may be repeated.

For example, permutations of the letters in the word "MATH" include MATH, MHTA, THAM, ATHM, HAMT, etc. As another example: the set of numbers {1, 3, 5} has six permutations: [1,3,5]; [1,5,3]; [3,1,5]; [3,5,1]; [5,1,3]; [5,3,1].

Question: How many different permutations are there of a set of n objects – for example, the first n counting numbers: {1, 2, … , n}?

Answer: This is easily answered for smaller sets.

- n=2: The permutations of {1, 2} are [1,2] and [2,1].

- n=3: The permutations of {1, 2, 3} are :
  [1,2,3] [2,1,3] [3,1,2]
  [1,3,2] [2,3,1] [3,2,1]
  The above set of permutations is arranged to show that the number of permutations starting with 1 is that same as the number starting with 2, or with 3. (Hopefully this makes sense intuitively – there's no reason why more ordered lists would start with one number than with another.) So, there are three groups of two permutations, and $3 \times 2 = 6$.

- n=4: The permutations of {1, 2, 3, 4} are:

[1,2,3,4]	[2,1,3,4]	[3,1,2,4]	[4,1,2,3]
[1,2,4,3]	[2,1,4,3]	[3,1,4,2]	[4,1,3,2]
[1,3,2,4]	[2,3,1,4]	[3,2,1,4]	[4,2,1,3]
[1,3,4,2]	[2,3,4,1]	[3,2,4,1]	[4,2,3,1]
[1,4,2,3]	[2,4,1,3]	[3,4,1,2]	[4,3,1,2]
[1,4,3,2]	[2,4,3,1]	[3,4,2,1]	[4,3,2,1]

Note that there are four groups of six permutations – specifically, six that start with 1, six that start with 2, and so on. There are six in each group because, once we've selected the first number in the permutation, there are three remaining numbers to select; and, as we've established earlier (see "n=3" above), there are six ways to rearrange three objects. Therefore, there are $4 \times 6 = 24$ permutations of any four objects, as listed above.

This should seem pretty familiar – our results (and reasoning) so far are very similar to what we saw in the "melody-counting" examples at the beginning of this chapter. This is no coincidence; in fact, melodies without repeated notes are perfect examples of permutations. What we are considering now is just a generalization of that idea to ordered, non-repeating selections in general, regardless of the context.

We can summarize our observations thus far with the following formulas:

PERMUTATION FORMULA #1:
Given a set of n objects (where n is any counting number), the number of permutations involving all of the objects in the set is n factorial:

$$n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1.$$

For example, $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$, which is the number of different permutations of any set of 10 objects.

There's a more general counting principle implied by the above development of Permutation Formula #1. This more general principle is called the "Multiplication Principle" for counting problems:

MULTIPLICATION PRINCIPLE:
Suppose a certain process, or task, can be completed in a certain sequence of steps, and at each step we have a specific, fixed, number of options for completing that step. ("Fixed," meaning that previous choices will not affect the number of options available.) Then, the number of ways to complete the process or task can be found by multiplication: if we determine the number of options at each step and list these numbers, then the number of ways to complete the process or task will simply be the *product* of these numbers.

Refer back to the example in which we counted the number of non-repeating four-note melodies selected from a set of six notes. Our result, $6 \times 5 \times 4 \times 3$, was an application of this "Multiplication Principle." Here is another example that applies the same sort of reasoning:

Example: Suppose we want to write a three-note melody consisting of notes from the C-major scale (that is, "white keys" on the keyboard), with the restriction that no note may be used twice. (Disregard octaves for this example – that is, if we use "C" as the first note, then none of the other notes can be a "C," even if it's in a different octave.) How many such melodies are possible?

First, observe that this "task" (writing a melody) can be thought of as a three-step process: choose the first note, then the second, then the third. This suggests that the Multiplication Principle can be used to answer our question:

- Step 1 (first note): There are 7 options: C, D, E, F, G, A, or B
- Step 2 (second note): Since repetition is disallowed, there are 6 options at this step. (Note that this is true no matter which note was selected as the first note.)
- Step 3 (third note): Again, since repetition is disallowed and 2 notes have been used already, there are 5 options remaining at this step.

Therefore, by the Multiplication Principle, there are $7 \times 6 \times 5 = 210$ possible ways to write the melody.

Example: Find the number of permutations of ten objects taken four at a time. (Equivalent problem: find the number of sequences of four digits, with the condition that no digit may be repeated.)

Answer: Again, we will think this through using the Multiplication Principle. There are 10 options for the first digit in our permutation. After this selection is made – no matter which digit is selected – there are 9 remaining options for our second digit, then 8 options for the third digit, then 7 options for the fourth digit. Therefore, the number of ways to complete the selection – in other words, the number of permutations of ten objects taken four at a time – is $10 \times 9 \times 8 \times 7 = 5040$.

The two preceding examples suggest a general strategy, or formula, for computing the number of permutations of, say, n objects taken r at a time (where n and r can be any two counting numbers, with the restriction that r is between 0 and n.) That formula – along with mathematical notation commonly used as shorthand for counting permutations – is given below:

NOTATION:

The symbol $P(n,r)$ is shorthand for "the number of possible permutations of n objects, taken r at a time. For instance, the preceding example could be summarized by writing "$P(10,4) = 10 \times 9 \times 8 \times 7 = 5040$.

PERMUTATION FORMULA #2:
$P(n,r)$ is a "falling product" of the form $n \times (n-1) \times \ldots$, where the product ends after we've included r distinct factors. The last number to be multiplied will always be $n-(r-1) = n-r+1$; so, the formula is as follows:

$$P(n,r) = \underbrace{n \times (n-1) \times \ldots \times (n-r+1)}_{\substack{r\ distinct\ factors: \\ one\ for\ each\ object\ selected}}$$

Alternate form (which is sometimes more convenient):

$$P(n,r) = \frac{n!}{(n-r)!}$$

It's not immediately obvious that the two above formulas are equivalent, but in fact they are, due to how factorials behave when one is divided by the other. For example, consider $P(10,4)$, which (from the previous example) is known to be $10 \times 9 \times 8 \times 7$, or 5040. According to the second version of the above formula – with $n = 10$ and $r = 4$ — this formula would give us

$$P(10,4) = \frac{10!}{6!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}\ .$$

While this may appear intimidating at first glance, it helps to recall that fractions (such as the one above) c can be found in both the numerator and in the denominator, and so each of these six factors can be cancelled out. (This cancellation is the basis of the formula; it's what makes the two versions of Permutation Formula #2 equivalent.) What's left over after the cancellation is complete? Exactly what we expected: $10 \times 9 \times 8 \times 7$, as before.

In general, $\frac{n!}{(n-r)!}$ will always simplify to $n \times (n-1) \times \ldots \times (n-r+1)$; thus, the two formulas are equivalent, so either one may be used when counting permutations of n objects taken r at a time.

COMBINATIONS

A "combination" is defined as an unordered selection without repetition. This is different from "permutation" in that we are not considering the order of selection, but simply the (unordered) *set* of objects being selected. The difference between "combination" and "permutation" depends on whether the order of selection matters.

For example, suppose we wish to select a "*chord*" – that is, a set of two or more notes played *simultaneously* – from among the seven notes of the C-major scale. For example, the *set* {C, E, G} would make up one possible chord of this type. Before developing notation, formulas, etc., it's important to point out that the difference between "chord" and "melody" is analogous to the difference between "combination" and "permutation" – since all three notes will be played at the same time, it doesn't matter which is selected first. That is, the lists (or melodies) [C,E,G], [C,G,E], [G,C,E], [G,E,C], [E,C,G] and [E,G,C] would all correspond to the same set (or chord) of {C, E, G}. Notice that six melodies all correspond to the same chord; this is true in general, and it implies that there are six times as many melodies as there are chords. (Based on that observation – how many chords should there be? Why?)

Before proceeding, let's introduce notation for combinations (which will be similar to what we use for permutations):

> NOTATION: The symbol $C(n,r)$ is shorthand for "the number of possible combinations of n objects, taken r at a time. (For instance, the preceding example could be summarized by writing "$C(7,3) = \frac{210}{6} = 35.$)

To further explore this connection between permutations and combinations, let us consider non-repeating melodies and chords selected from the 5-note set {C, D, E, F, G}:

- <u>Five notes</u>:
  We know that there are $5! = 120$ non-repeating 5-note melodies we can select from a set of five notes. (In other words, $P(5,5) = 5! = 120$.)
  On the other hand, the only way to select a 5-note *chord* from this set of five notes would be to play all five notes at once! This means that there is exactly one 5-note chord that can be selected from a set of 5 notes; in other "words," $C(5,5) = 1$.
  (Comment: this implies a general combinations formula: $C(n,n) = 1$. Good news!)

- <u>Four notes</u>:
  First, we'll note that $P(5,4) = 5 \times 4 \times 3 \times 2 = 120$, which means that there are 120 distinct 4-note non-repeating *melodies*. (Question: Does it make sense that the number of 4-note non-repeating melodies is the same as the number of 5-note non-repeating melodies? Why?)

  To select a four-note *chord* from {C,D,E,F,G}, on the other hand, all we have to decide is which of these five notes NOT to select. For example, if we leave out C, then we'll keep the other four, giving us the chord {D, E, F, G}. The other four chords are found similarly; therefore, there are 5 distinct 4-note chords from {C,D,E,F,G}. In other words, $C(5,4) = 5$. (By similar reasoning, $C(n, n-1) = n$ for all positive integers n.)

- <u>Three notes</u>:
  There are $P(5,3) = 5 \times 4 \times 3 = 120$ three-note non-repeating melodies possible.

  Counting chords is a little trickier once you start "leaving out" more than one note, so we'll proceed differently. Consider any sample 3-note non-repeating melody – for instance, [F, D, G]. We know that there are $3! = 6$ different arrangements of these three notes, so these three notes can be arranged into six distinct *melodies*. However, these same three notes only give us *one chord*: {D, F, G}. This implies that, if we were to list all 120 three-note melodies, we could break this list into sections of six melodies each, each of which corresponds to only one chord. How many such “sections” would there be? Same question: how many sixes go into 120? The answer is $120 \div 6$, which is 20. Therefore, there are 20 possible 3-note chords.

Comment: To go from counting permutations to counting combinations, we can always divide by the number of rearrangements of each individual permutation, as demonstrated above.

We’ve seen that to go from counting permutations (melodies) to counting combinations (chords), we must adjust for multiple-counting of combinations. For each combination of r objects, there are $r!$ rearrangements of those r objects. Therefore, to adjust for multiple-counting, we must divide by $r!$. We can summarize this by the formula:

COMBINATIONS FORMULA:

$$C(n,r) = \frac{\{\#\ of\ permutations\}}{\{\#\ ways\ each\ combination\ can\ be\ rearranged\}} = \frac{P(n,r)}{r!}$$

<u>Example</u>: How many different 4-note chords are possible if we select from the 12-tone scale?

<u>Answer</u>: Since chords are combinations (unordered and without repetition), so the answer to this question is $C(12,4)$:

$$C(12,4) = \frac{P(12,4)}{4!} = \frac{12\times 11\times 10\times 9}{4\times 3\times 2\times 1} = \frac{\overset{\overset{1}{\cancel{3}}}{\cancel{12}}\times 11\times \overset{5}{\cancel{10}}\times 9}{\underset{1}{\cancel{4}}\times \underset{1}{\cancel{3}}\times \underset{1}{\cancel{2}}\times 1} = 11\times 5\times 9 = 495$$

Next, we’ll consider an example in which *some* repetition is allowed. Note that when repetition is allowed, we’re no longer dealing with simple permutations; this is a slightly different type of counting problem. (What the following example has in common with permutations is that the order of selection matters.)

<u>Example</u>: Suppose we are (still) selecting notes from the set {C, D, E, F, G}. How many different five-note melodies are possible, with the condition that each melody consists of exactly four different notes, one of which will be used twice?

A few examples of such melodies would be: CDEGG, FDGAD, CECFD, etc.

Solution: First, let's consider the special case where the repeated note will appear first and second in our melody – for example, CCDEF, FFECG, etc. The number of such melodies can be found by multiplication: there are 6 options for the first two notes, then 5 options remaining for the third note, 4 for the fourth note, and 3 for the fifth note. So, there are $6 \times 5 \times 4 \times 3$, or 360, such arrangements. (Note that this "case" is equivalent to the previous example.)

But 360 isn't our final answer – we've considered only one possible case. The repeated note could come in the first and third positions (e.g., CDCEFG), or the second and fourth positions (e.g., DCECFG), etc. How many different possibilities are there? We need to figure out the number of ways in which we can select two locations, out of six, in which to place the repeated notes.

There are ten ways to do this, as illustrated by the following arrangements of AABCD. (In each, the repeated notes – the A's – are highlighted in **boldface font**.)

AABCD	**A**B**A**CD	**A**BC**A**D	**A**BCD**A**	B**AA**CD
B**A**C**A**D	B**A**CD**A**	BC**AA**D	BC**A**D**A**	BCD**AA**

So, we would say that there are ten "*combinations*" of two locations out of five locations. (We'll formally define the word "combination" in a moment.)

For each of these ten arrangements, there are $6 \times 5 \times 4 \times 3$, or 360 possible melodies. This is because there are always 6 ways to choose the repeated note, then 5 ways to choose the first non-repeating note, and so on, just as in the "special case" we considered earlier. Therefore, if we were to list all of the possible melodies, we would end up with ten lists of 360 melodies each. Thus, the total number of melodies would be 10 *times* 360, which is **<u>3600</u>**.

Let's consider another example of selection without regard to order: playing "chords."

Definition: a "chord" consists of two or more notes played simultaneously. Any combination of two or more notes can be considered to be a "chord." (Of course, some chords will sound better than others!)

Example: Restricting ourselves to the notes {A, B, C, D, E, F, G}, how many two-note chords are possible? How many three-note chords?

<u>Two-note chords:</u>
For two notes: first, let's list all ordered selections ("melodies") of two notes:

A,B   A,C   A,D   A,E   A,F   A,G

b,a		B,C	B,D	B,E	B,F	B,G
c,a	c,b		C,D	C,E	C,F	C,G
d,a	d,b	d,c		D,E	D,F	D,G
e,a	e,b	e,c	e,d		E,F	E,G
f,a	f,b	f,c	f,d	f,e		F,G
g,a	g,b	g,c	g,d	g,e	g,f	

Note that each combination appears twice – once in upper-case, once in lower-case. That is, for any upper-case pair of notes, you'll find the same pair of notes in lower-case, in the reverse order. The point of this arrangement is to illustrate that in the list of ordered selections of two notes, each combination of notes occurs exactly twice. In other words, the two-note chords are "double-counted" by the above list.

We know that there are $7 \times 6 = 42$ ways to select two notes in order from the 7 notes {A,B,C,D,E,F,G}. Since this list "double-counts" chords, though, we need to divide by 2 to find the number of two-note chords. Therefore, there are 42/2 ,or 21, different two-note chords.

<u>Example</u>: Consider melodies consisting entirely of notes from the C-major scale: {C, D, E, F, G, A, B}. How many ways are there to write a six-note melody such that includes exactly 2 C's, exactly 2 G's, and no other repeated notes?

For example, a few allowable melodies would be: CGCGAB, CDCGGE, FCGAGC.
(Note that FCGFCG, for instance, would not be allowed, since the two F's would break the "no other repeated notes" condition.)

If you like, you can start by trying to list all of the melodies satisfying these conditions. (This is never a bad idea, at least to help you start thinking about how to approach the problem.) However, it turns out that there are well over one thousand such melodies, so we recommend against completing this list unless you have a lot of free time on your hands.

Note that this example is similar to the previous example [NOTE TO SELF – eventually number exercises; refer back to correct number here] in which we allowed one note out of five to be repeated. We'll follow a similar strategy here.

In this example, we'll introduce the idea of devising a "selection process," by which we mean a process in which our selection is broken into a sequence of steps such that the number of ways to complete each step doesn't depend on what was done in previous steps. (This system allows us to apply the Multiplication Principle to solve the problem.)

<u>**Selection Process**</u>:

1. <u>Decide which two notes in our melody will be the C's.</u>

   For example, we could have C C _ _ _ _ (where blanks indicate the other four notes in the melody), or C _ C _ _ _, etc. We could list all of the ways to make this selection – again, see the preceding similar example. However, there is a way around that: we can observe that we are actually counting combinations. In particular, if we consider the six note locations in our melody – {first, second, third, fourth, fifth, sixth} – then what we are really doing at this step is selecting two out of these six locations to be the C's in our melody. This is a selection *without repetition* (we can't put both C's in the same place) and it is an *unordered* selection (since the C's are not distinct from one another).

So, we can use the combinations formula to count the number of ways to select two locations from six locations to be C's – there are $C(6,2) =$ **<u>15 ways</u> to make this selection.**

(Example: one of our 15 possible selections is _ _ C _ _ C.)

2. <u>Decide which two notes (of the four remaining notes to pick from) will be the G's.</u>

   This step is similar to the preceding step – we are selecting a combination of two locations from our melody to be the G's. The only difference is that, since we already filled two locations in our melody in the previous step of this process, we now have only four locations left to choose from. So, by similar reasoning to that used in Step 1, there are $C(4,2) =$ **<u>6 ways</u> to make this selection.**

   (Example: if we had selected _ _ C _ _ C in step 1, then one of the 6 available options here in step 2 based on this would be G G C _ _ C.)

3. <u>Select our last two notes.</u>

   Recall that the final condition for the melodies being counted was "no other repeated notes." This means that our last two notes must be selected from the set {D, E, F, A, B}, and we must select two *distinct* notes. Therefore, as opposed to what we did in steps 1 and 2, the order of selection *does* matter in this step. So, we can just select "left-to-right;" there are 5 ways to select the first note, and 4 ways to select the second note, for a total of $5 \times 4 =$ **<u>20 ways</u> to complete this step**.

   (Continuing from the running example in steps 1 and 2, one of the 20 ways to complete the G G C _ _ C melody would be G G C D A C.

   Optional exercise: find the other 19 ways to complete the G G C _ _ C melody so tha there are no repeated notes other than the 2 C's and the 2 G's we already have.)

To summarize: we've devised a selection process that counts each melody exactly once (no "double-counting," and nothing left out), and we've set it up in such a way that the number of options for completing each step does not depend on the choices we've made at previous steps. Therefore, we can solve the problem using the multiplication principle: since there are <u>15 ways</u> to complete the first step, <u>6 ways</u> to complete the second step, and <u>20 ways</u> to complete the third step, there is a total of $15 \times 6 \times 20 =$ <u>1800 ways</u> to complete the entire selection process.

Solution: There are 1800 distinct six-note melodies with notes selected from the C-major scale {C, D, E, F, G, A, B} with exactly two C's, two G's, and no other repeated notes.

Word Scrambles (a non-musical digression)

"JUMBLE" (www.jumble.com) is a popular puzzle that appears daily in hundreds of newspapers around the world. It consists of several "jumbled" words – words whose letters are rearranged – and the objective is to figure out what the original word must have been. For example, if the clue were "LIMCASU," the solution would be "MUSICAL."

Question: How many different ways are there to rearrange the word "MUSICAL?"

(Note: to simplify wording, we'll use the phrase "rearrange the word MUSICAL" as a shorthand for "rearrange the order of the letters in the word MUSICAL.")

Answer: Since "MUSICAL" contains 7 distinct letters, and the order of selection matters (it's the entire point of the question, in fact), this is pretty clearly a permutation problem. In particular, we can apply Permutations Formula #1 here: there are $7! = 5040$ ways to rearrange any 7 distinct letters, so "MUSICAL" has 5040 distinct rearrangements.

Comment (note to self – maybe move to end of chapter as an optional exercise, or some sort of "exploration" or something): This doesn't quite mean the word "MUSICAL" could appear in the "JUMBLE" game in 5040 different ways. For example, one of those 5040 rearrangements of "MUSICAL" is the word MUSICAL itself. A few other rearrangements, such as MUSICLA, MUSCIAL, or MUSILAC, would be far too easy to decode. An interesting question (which we won't go into here) might be to find a way to count the number of ways a word could be rearranged so that the original word isn't too obvious – that is, the number of rearrangements that would be useful *in a "JUMBLE"-type game.*

Question: How many ways are there to rearrange the word "SESAME?"

Before we proceed to answer this question, note that there is an important distinction between this example (SESAME) and the preceding one (MUSICAL): the word "SESAME" contains repeated letters. Since the letters are not all distinct, Permutation Formula #1 (the "factorials" formula) does not apply as it did for "MUSICAL." We must find a different counting strategy in order to deal with the repeated, non-distinct letters.

There are several ways to devise a selection process that will yield each rearrangement of "SESAME" exactly once. We will demonstrate one such process below.

Note: keep in mind that our word contains six letters: 2 S's, 2 E',s 1 A, and 1 M.

Selection Process:

1. Decide which two letters in our rearrangement will be the S's.

   For example, we could have S S _ _ _ _ (where blanks indicate the other four letters in the rearrangement), or S _ S _ _ _, etc. We could list all of the ways to make this selection – again, see the preceding similar example. However, there is a way around that: we can observe that we are actually counting combinations. In particular, if we consider the six letter locations in our rearrangement – {first, second, third, fourth, fifth, sixth} – then what we are really doing at this step is selecting two out of these six locations to be the S's in our rearrangement. This is a selection *without repetition* (we can't put both S's in the same place) and it is an *unordered* selection (since the S's are not distinct from one another).

   So, we can use the combinations formula to count the number of ways to select two letters from six letters to be S's – there are $C(6,2) =$ **15 ways to make this selection.**

   (Example: one of our 15 possible selections is _ _ S _ _ S.)

2. Decide which two letters (of the four remaining letters to pick from) will be the E's.

   This step is similar to the preceding step – we are selecting a combination of two letters from our rearrangement to be the E's. The only difference is that, since we already filled two locations in our melody – I mean, rearrangement - in the previous step of this process, we now have only four locations left to choose from. So, by similar reasoning to that used in Step 1, there are $C(4,2) =$ **6 ways to make this selection.**

   (Example: if we had selected _ _ S _ _ S in step 1, then one of the 6 available options here in step 2 based on this would be E E S _ _ S.)

   Comment: If you're reading through this chapter for the first time, you may have experienced a bit of déjà vu just now. This is because our selection processes for rearrangements of "SESAME" and (from the preceding section) six-note melodies containing two C's and two G's are, aside from a few words here and there, exactly the same. Compare and contrast the two examples to see why (virtually) the same process, up to this point, can be used for each. Also, make sure you understand why the third step in this example will be slightly different from the third step in the preceding example…

3. Select our last two letters.

   Once the two S's and the two E's have been placed, all that's left are the A and the M. These are two distinct letters, and there are only two locations left to put them in. It's clear that there are only **2 ways** to complete this step (either A,M or M,A). More formally,

   (Continuing from the running example: if we had selected E E S _ _ S in steps 1 and 2, then we would end up with either E E S A M S or E E S M A S as our final rearrangement, depending on which of the two available options we selected for step 3.)

   To summarize: we've devised a selection process that counts each rearrangement exactly once, and we've set it up in such a way that the number of options for completing each step does not depend on the choices we've made at previous steps. Therefore, we can solve the problem using the multiplication principle: since there are 15 ways to complete the first step, 6 ways to complete the second step, and 2 ways to complete the third step, there is a total of $15 \times 6 \times 2 =$ 180 ways to rearrange the letters of a word with 2 S's, 2 E's, 1 M, and 1 A.

   Solution: There are 180 distinct rearrangements of the word "SESAME."