MATH 210, TEST #1
SOLUTIONS

1. Assume that A, B and C are sets. Solve each of the following problems. No additional work or explanation is required for the problems on this page.

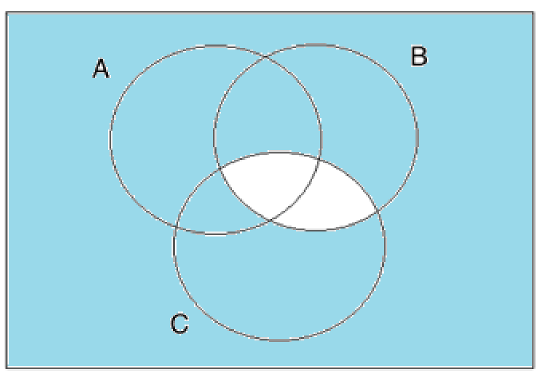
a) Describe, in words, the set $(A \cap B') \cup (B \cap C)$. Your written description may not include any of the words “union,” “intersection,” or “complement.”

Solution (wording may vary slightly):

The set of all elements that are in A and not in B, or that are in both B and C.

Note: something like “set A and not set B…” is in the ballpark, but doesn’t make sense as written. When describing a set, you are specifically describing a criterion that each element of the set must satisfy. If there’s no mention of “elements” (or something with that meaning), then you didn’t get full credit since your explanation was not clear.

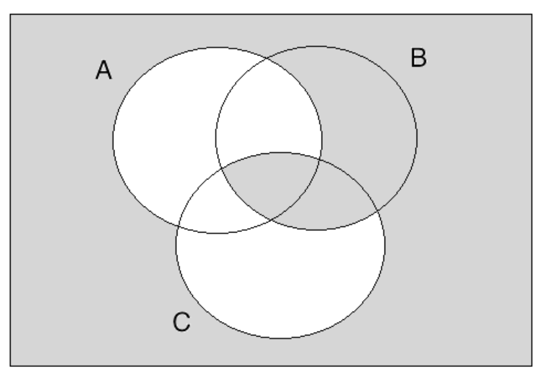
b) Shade in the region of the Venn diagram below which corresponds to the set $(B \cap C)'$.
Solution: Shade the entire diagram except for the region corresponding to $B \cap C$, as shown:



c) Use set operations (union, intersection, complement) to describe the set that corresponds to the shaded region in the Venn diagram below.

Solution: The simplest way (there are other correct ways) to describe the shaded area is as $(\boldsymbol{A} \cup \boldsymbol{C})' \cup (\boldsymbol{B} \cap \boldsymbol{C})$.

Another correct answer is
$(\boldsymbol{A}' \cap \boldsymbol{B}' \cap \boldsymbol{C}') \cup (\boldsymbol{B} \cap \boldsymbol{A}') \cup (\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C})$



2. Use a truth table to determine whether the proposition

$$\big((\boldsymbol{p} \to \boldsymbol{q}) \land (\boldsymbol{p} \lor \boldsymbol{q})\big) \leftrightarrow \boldsymbol{q}$$

is a theorem. Explain your conclusion (write at least one sentence of explanation).

Solution:

p	q	$p \to q$	$p \lor q$	$(p \to q) \land (p \lor q)$	$\big((p \to q) \land (p \lor q)\big) \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Since the $\big((p \to q) \land (p \lor q)\big) \leftrightarrow q$ column consists entirely of T's, we can see that this proposition is always true, regardless of the status of p or q. Therefore, this proposition is a theorem.

(Note: it would suffice to stop with the next-to-last column above, then write a sentence observing that $(p \to q) \land (p \lor q)$ is true if and only if q is true.)

3. Use an *element chasing* argument to prove: $(\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{B}' \cup \boldsymbol{C})' \subseteq \boldsymbol{B}.$

Solution: We want to prove that every element of $(A \cap B) \cup (B' \cup C)'$ must be an element of B.

Proof: Let $x \in (A \cap B) \cup (B' \cup C)'$. There are two cases to consider.

- Case 1: Suppose $x \in (A \cap B)$. Then, x is in both set A and set B, by definition of intersection. Thus, x is an element of set B.

- Case 2: Suppose $\in (B' \cup C)'$. Then, x is not an element of $B' \cup C$. This means x is not in the union of B' and C – that is, x can be and element of neither B' nor C. Since x is not an element B', it follows that x is an element of set B.

In both cases, we can conclude that $x \in B$. Thus, every element of $(A \cap B) \cup (B' \cup C)'$ must be an element of B. It follows that $(A \cap B) \cup (B' \cup C)' \subseteq B$. ■

Comment: If you did not read the instructions carefully, and tried to do a full element chasing proof of set equality, then you should have been unsuccessful, since $B \nsubseteq (A \cap B) \cup (B' \cup C)'$. For example, any element of A which is not also an element of B would be an element of $(A \cap B) \cup (B' \cup C)'$, but not of B.

4. Do **ONE** of the following; pick the **one** you feel more comfortable with. If you try both, clearly indicate which **one** you want to have graded. (You won't get extra credit for doing both!)

a. Use appropriate Boolean identity laws to prove the theorem:

$$\boldsymbol{\bar{x}y + (\overline{z+x}) = \bar{x}(y+\bar{z})}.$$

Show all necessary steps of your proof. (Hint: You should be able to complete this proof using only those laws you were instructed to memorize – commutative, associative, distributive, and/or De Morgan)

Solution: We'll start with the left-hand side, and show it's equal to the right-hand side…

$$\begin{aligned} \bar{x}y + (\overline{z+x}) &= \bar{x}y + \bar{z}\,\bar{x} \quad (De\ Morgan) \\ &= \bar{x}y + \bar{x}\,\bar{z} \quad (Commutative) \\ &= \bar{x}(y + \bar{z}) \quad (Distributive) \end{aligned}$$

b. Use appropriate set identity laws to prove the identity:

$$\boldsymbol{(A' \cap B) \cup (C \cup A)' = A' \cap (B \cup C')}$$

Solution: We'll start with the left-hand side, and show it's equal to the right-hand side…

$$\begin{aligned} (A' \cap B) \cup (C \cup A)' &= (A' \cap B) \cup (C' \cap A') \quad (De\ Morgan) \\ &= (A' \cap B) \cup (A' \cap C') \quad (Commutative) \\ &= A' \cap (B \cup C') \quad (Distributive) \end{aligned}$$

Comment: Aside from context (Boolean variables vs. sets), these proofs are identical!

5. Find the maxterm and minterm expansions for the Boolean function, f, which is defined by the following table:

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Solutions:

To find the minterm expansion, focus on the cases in which $f = 1$:
Row 1 minterm: $\bar{x}\,\bar{y}\,\bar{z}$
Row 2 minterm: $\bar{x}\,\bar{y}\,z$
Row 4 minterm: $\bar{x}\,y\,z$
Row 5 minterm: $x\,\bar{y}\,\bar{z}$
Row 7 minterm: $x\,y\,\bar{z}$

So, $f(x,y,z) = \bar{x}\,\bar{y}\,\bar{z} + \bar{x}\,\bar{y}\,z + \bar{x}\,y\,z + x\,\bar{y}\,\bar{z} + x\,y\,\bar{z}$.

To find the maxterm expansion, we'll first find the minterm expansion for $\bar{f}$:

$$\bar{f}(x,y,z) = \bar{x}\,y\,\bar{z} + x\,\bar{y}\,z + x\,y\,z$$

The maxterm expansion for f is the negation of the minterm expansion for $\bar{f}$:

$$\begin{aligned} f(x,y,z) &= \overline{\bar{x}\,y\,\bar{z} + x\,\bar{y}\,z + x\,y\,z} \\ &= \overline{\bar{x}\,y\,\bar{z}}\;\;\overline{x\,\bar{y}\,z}\;\;\overline{x\,y\,z} \\ &= (x + \bar{y} + z)(\bar{x} + y + \bar{z})\,(\bar{x} + \bar{y} + \bar{z}) \end{aligned}$$

6. Consider the following argument:

PREMISES
Everyone who can drive has a driver's license.
No toddler has a driver's license.
Ruby cannot drive.

CONCLUSION
Ruby is a toddler.

a) Use propositional calculus to write the premises and the conclusion of this argument symbolically. (Clearly define each of your "atoms" – don't make me guess at what your notation means.)

b) Determine whether the argument is valid. Show all necessary work, and write at least one sentence to explain your conclusion.

Solution:
Your atoms should be as follows (though your notation may be different):

D: It can drive.
L: It has a driver's license.
R: It is Ruby.
T: It is a toddler.

Then, the argument may be written as follows:

Premises: $D \to L\,; T \to \neg L\,; R \to \neg D$
Conclusion: $R \to T$

So, the argument is equivalent to the following proposition:

$$\big((D \to L) \wedge (T \to \neg L) \wedge (R \to \neg D)\big) \to (R \to T)$$

We can use the contrapositive and deduction laws to simplify this somewhat. By the contrapositive law, $T \to \neg L$ is equivalent to $L \to \neg T$. So, deduction gives us: $\big((D \to L) \wedge (L \to \neg T)\big) \to (D \to \neg T)$. So, we can deduce the following:

$$\begin{aligned}\big((D \to L) \wedge (T \to \neg L) \wedge (R \to \neg D)\big) &\leftrightarrow \big((D \to L) \wedge (L \to \neg T) \wedge (R \to \neg D)\big)\\ &\to \big((D \to \neg T) \wedge (R \to \neg D)\big)\end{aligned}$$

Unfortunately, we're stuck at this point – it's not possible to conclude from $(D \to \neg T)$ and $(R \to \neg D)$ that $R \to T$ is true. It is tempting to substitute $\neg D \to T$ for $D \to \neg T$ (since this would allow us to conclude $R \to \neg D \to T$ by deduction); however, this would be invalid, since an implication is not logically equivalent to its inverse. (We could, by the contrapositive law, substitute $T \to \neg D$ for $D \to \neg T$, but this would not help.)

In summary: If all of the premises are true, it is *possible* for R to be true while T is false; that is, it's possible that Ruby is not a toddler. More specifically: if atom R is true (it is Ruby) while D, L and T are all false (Ruby can’t drive, doesn’t have a license, and isn’t a toddler), then all three premises are true but the conclusion is false. Thus, the argument is rendered invalid.

Or, in other words: based on the premises, we don’t assume anything at all about people who can’t drive. So, the fact that Ruby can’t drive gives us no further information about her – in particular, we don’t know whether or not she’s a toddler. (Nor, for that matter, do we even know – based on the premises – whether she has a driver’s license!)

By the way: in reality, all of the premises *are* true, and…

…it just so happens that Ruby *is*, in fact, a toddler!

BUT – and this is an important point – *even though* the conclusion of the argument happens to be true, the fact remains that the *argument* is invalid! Make sure you understand the difference!