Math 105 – Sequences

A "sequence" is an ordered list of numbers. The numbers in a sequence are called the "terms" of the sequence. For example, in the sequence 6, 2, 5, 8, 7, 4, we would say that 6 is the first term of the sequence, 2 is the second term, 5 is the third term, and so on.

Many sequences are created using "recursion" – that is, each term in the sequence is determined based on one or more of the preceding terms. For example, consider the following sequences:

2, 5, 8, 11, 14, 17, 20, 23, … (each term is 3 more than the preceding term)
2, 10, 50, 250, 1250, 6250, … (each term is 5 times the preceding term)
8, 4, 2, 1, ½, ¼, … (each term is ½ of the preceding term)

In each case, each term depends entirely on the one term preceding it in the sequence. A sequence determined in this way is called a "one-term recursion," since the rule for continuing the sequence depends only on one term at a time.

Here are two types of one-term recursions that are seen frequently:

- A sequence in which the difference between consecutive terms is constant is called an arithmetic sequence. In other words: there is some constant number, call it n, such that from any given term in the sequence, the next term is always found by adding the same number, n.

  The above example 2, 5, 8, 11, … is an example of an arithmetic sequence, since the difference between consecutive terms (specifically, each term minus the term that came before it) is always equal to 3. Another example of an arithmetic sequence is, 1, 5, 9, 13, 17, … , in which the constant difference between consecutive terms is 4.

- A sequence in which the ratio between consecutive terms is constant is called a geometric sequence. In other words: there is some constant number, call it r, such that from any given term in the sequence, the next term is always found by multiplying by the same number, r.

  The above example 2, 10, 50, 250, … is an example of a geometric sequence, since the ratio between consecutive terms (specifically, each term divided by the term that came before it) is always 5. Another example of a geometric sequence (which we have seen in a different context earlier in the semester) is the following:

$$1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \frac{243}{32}, \ldots$$

  In this sequence, each term is 3/2 of the previous term, so the constant ratio between consecutive terms is 3/2.

For some sequences, the value of each term depends on the preceding *two* terms, rather than just one. For example, consider the "Fibonacci sequence:"

1, 2, 3, 5, 8, 13, 21, 34, …

In this sequence, each term (after the first two terms) is the sum of the preceding two terms. This is an example of a "two-term recursion." Here are a couple more examples of two-term recursions:

1, 2, 5, 13, 34, 89, … (here, each term is three times the preceding term, minus the term before that)
2, 3, 6, 18, 108, 1944, … (each term is the product of the preceding two terms)

Note: every one-term recursions *can* also be viewed as a two-term recursion. (Not that there's any advantage to doing so – it's just an interesting observation.) For example, in the sequence

2, 5, 8, 11, 14, 17, 20, 23…,

which we considered earlier (see above), we *could* say that each term is twice the preceding term, minus the term before that. (Try a few examples to see for yourself that this seems to work every time.)

When describing a sequence, or the rule used to continue a sequence, it is common practice to use a variable to stand for a term of the sequence, and a subscript to indicate the index (the location in the list) of each term. For example, in the Fibonacci sequence, we may choose to let "a_n" denote the nth term of the sequence. So, we'd have:

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13 \ldots$$

…and so on. This notation gives us a much easier way to write the two-term recursion rule: whenever n is a whole number greater than 2, the rule used to find the nth term of the Fibonacci sequence is:

$$a_n = a_{n-1} + a_{n-2},$$

where $a_1 = 1$ and $a_2 = 2$. (Note that a_1 and a_2 need to be defined separately, since the above equation only "works" if n is at least 3.) This indicates that any given term (the n^{th} term) of the sequence is the sum of the preceding term (the $(n-1)^{\text{th}}$ term of the sequence) and the term before that (the $(n-2)^{\text{th}}$ term of the sequence).

A few more examples:

- To describe the sequence 2, 5, 8, 11, 14, …, (the arithmetic sequence example from the beginning of this handout), we could choose to let x_n stand for the n^{th} term of the sequence. Then, we could accurately describe a recursive rule for this sequence in either of the following two ways:
  - $x_n = x_{n-1} + 3$, with $x_1 = 2$
    (this means each term is 3 more than the preceding term)
  - $x_n = 2x_{n-1} - x_{n-2}$, with $x_1 = 2$ and $x_2 = 5$
    (this means each term is twice the preceding term, minus the term before that)

- The sequence 1, 2, 5, 13, 34, 89, …, in which each term is three times the preceding term, minus the term before that, could be described by the equation $x_n = 3x_{n-1} - x_{n-2}$, with $x_1 = 1$ and $x_2 = 2$.

- 2, 3, 6, 18, 108, 1944, … , in which each term is the product of the preceding two terms, could be described by the equation $x_n = x_{n-1} \cdot x_{n-2}$ (with $x_1 = 2$ and $x_2 = 3$).