The Riemann Integral: Part 1

Introduction

The theory of integration forms an important part of mathematical analysis. Historically integration was used to find areas of plane figures. Archimedes used the very same process to find areas of parabola but he called it the *method of exhaustion*. The idea used by Archimedes was to divide the desired area in terms of smaller and smaller areas so that the sum of the areas of these smaller parts tended to a finite limit. It was the genius of Newton (and Leibniz too) to recognize that the process of integration could be viewed as the inverse process of differentiation. This greatly helped in finding areas of curves for which summing the areas of smaller parts was difficult. After Newton people started thinking of integration as the inverse of differentiation and the older approach based on summation was put at the back front.

At the beginning of nineteenth century Joseph Fourier started the new field of harmonic analysis using which a function could be expressed as a linear combination of sines and cosines as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

While analyzing such trigonometric series lot of questions arose about the idea of a function. While each term in the trigonometric series is continuous, differentiable (in fact differentiable as many times as we please) the sum of such series was not even guaranteed to be continuous. This shook the mathematical community and mathematicians were forced to rethink about the concept of a function. Before Fourier's time a function was thought of as a mathematical expression like $x^2, \sin x$ etc. where one could calculate the value of function for a given value of x using a formula. Fourier's trigonometric series forced mathematicians to take into account functions which were much weirder. And the question put forward was: what kind of functions does a trigonometric series represent?

Bernhard Riemann (pronounced *Ree-maann*) arrived on the mathematical scene in the mid-nineteenth century and he reasoned that a function should be viewed more as a correspondence so that to each value of the argument a unique value of the function could be associated. Inherent in the theory of Fourier's trigonometric series is the technique of integration of a function to calculate the coefficients of the series:

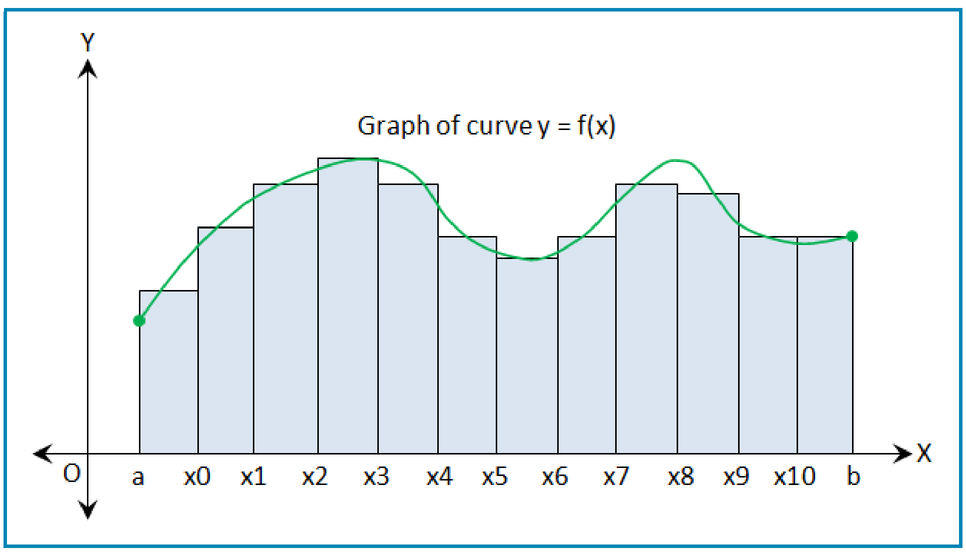
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Riemann thus realized that a proper investigation of the Fourier series could not be done unless there was a proper theory of integration. He chose to resort to the old age method of summation to define the integration. And in his approach he made only one assumption about the nature of a function: the function should be *bounded* in the range under consideration. Apart from this restriction the function could behave in weird ways. Riemann formalized the concept of integration as a process of summation and put it on a solid foundation.

In this series of posts we will discuss the approach used by Riemann and will focus mostly on the subtle points and not on the routine stuff (which is easily available in many textbooks). Our main focus would be the formal definition of integration provided by Riemann and the characterization of Riemann-integrable functions. The material presented in this regard would be somewhat abstract but it represents the true spirit of mathematical analysis and explores the themes of rigor and formalism in mathematics.

Riemann Integral: Definition

Riemann started with the basic geometrical notion of treating the integral as an area under a given curve. And like his predecessors he chose to calculate the area by splitting the region in smaller rectangles and then adding the areas of the rectangles. Following the path of Riemann, we begin with a function f defined on closed interval $[a, b]$ with the only restriction that f is *bounded* on $[a, b]$. The boundedness property seems to be natural to assume otherwise the graph of the function looks unbounded and seems to suggest that the area of the region under the graph would be infinite.



Area approximated by Riemann Sum

In the above graph we have shown that f is positive in $[a, b]$, but this is only for purposes of illustration. Riemann used a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of interval $[a, b]$ and formed the sum:

$$S(P, f) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(t_k)\Delta x_k$$

where t_k is any arbitrary point in $[x_{k-1}, x_k]$. Riemann argued that under a reasonable set of assumptions the sums of type $S(P, f)$ would converge to a limit when the *norm* of the partition P (i.e. $||P||$) gets smaller and smaller and finally tends to zero. Note that corresponding to any given partition there can be infinitely many Riemann sums based on the choice of points $t_k \in [x_{k-1}, x_k]$. This idea can be formalized into a definition as follows:

Let f be a bounded function in closed interval $[a, b]$. The number I is said to be the Riemann integral of f on $[a, b]$ if for any given number $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|S(P, f) - I| < \epsilon \text{ i.e. } \left|\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) - I\right| < \epsilon$$

for any Riemann sum $S(P, f)$ corresponding to any partition P with norm $\|P\| < \delta$. When such a number I exists we say that f is Riemann-integrable on interval $[a, b]$ and we write

$$I = \int_a^b f(x)\,dx$$

Riemann's Condition for Integrability

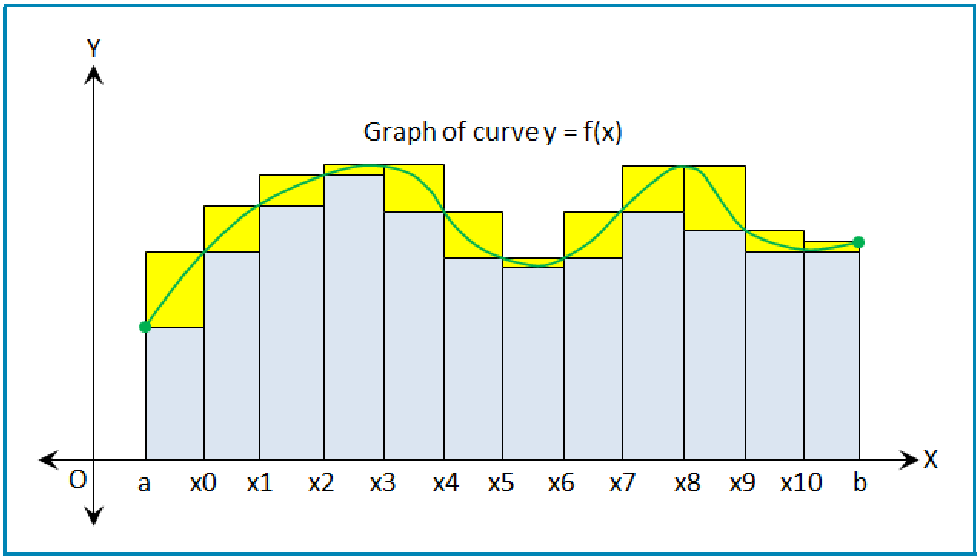
The definition above does not give any hint about deciding whether a function is Riemann-integrable or not because the test mentioned in the definition requires us to have prior knowledge of the number I with which the test of integrability can be performed. To understand the question of integrability further we need to examine the behavior of a function in more detail. It turns out that a very clear picture emerges when we replace the values $f(t_k)$ in the Riemann sum with the supremum and infimum of f in the interval $[x_{k-1}, x_k]$. This idea leads to the definition of Darboux sums.

As before let f be bounded in $[a, b]$ and let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$. Let M_k be the supremum and m_k be the infimum of f in interval $[x_{k-1}, x_k]$. M_k, m_k exist because the function f is bounded in $[a, b]$. We form the *Darboux Upper Sum*

$$U(P, f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} M_k \Delta x_k$$

and the *Darboux Lower Sum*

$$L(P, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} m_k \Delta x_k$$



Upper and Lower Darboux sums

In the above figure the light blue rectangles represent a lower Darboux sum and the light blue plus the yellow rectangles represent an upper Darboux sum.

Unlike the Riemann sums the Darboux sums are uniquely determined for a given partition P of $[a, b]$. It is quite obvious that any Riemann sum lies between its corresponding lower and upper Darboux sums i.e. $L(P, f) \leq S(P, f) \leq U(P, f)$. If we add an extra point say x' of the subinterval (x_{k-1}, x_k) in partition P to make it finer say $P' \supset P$, it is clear that

$$\sup\{f(x) \mid x \in [x_{k-1}, x']\} \leq M_k,\ \sup\{f(x) \mid x \in [x', x_k]\} \leq M_k$$

$$\inf\{f(x) \mid x \in [x_{k-1}, x']\} \geq m_k,\ \inf\{f(x) \mid x \in [x', x_k]\} \geq m_k$$

It therefore follows that $U(P', f) \leq U(P, f),\ L(P', f) \geq L(P, f)$ whenever $P \subseteq P'$. In plain English the Darboux upper sums decrease and the Darboux lower sums increase when the partition is made finer. Let M, m be the supremum and infimum of f in $[a, b]$ so that $M_k \geq m$ and $m_k \leq M$. Therefore we see that the upper sums are bounded below by $m \sum \Delta x_k = m(b-a)$ and the lower sums are bounded above by $M \sum \Delta x_k = M(b-a)$. It follows that there exist two numbers $\overline{J}, \underline{J}$ such that

$$\begin{aligned}\overline{J} &= \inf\{U(P, f) \mid P \text{ is a partition of } [a, b]\} \\ \underline{J} &= \sup\{L(P, f) \mid P \text{ is a partition of } [a, b]\}\end{aligned}$$

If P_1, P_2 are any partitions of $[a, b]$ then $P = P_1 \cup P_2$ is also a partition of $[a, b]$ which is finer than both P_1 and P_2 and hence

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f)$$

so that any lower sum cannot exceed any upper sum. It hence follows that

$$\underline{J} = \sup\{L(P, f) \mid P \text{ is a partition of } [a, b]\} \leq \overline{J} = \inf\{U(P, f) \mid P \text{ is a partition of } [a, b]\}$$

The number $\overline{J}$ is called the *upper Darboux integral* of f on $[a, b]$ and is denoted by

$$\overline{J} = \overline{\int_{a}^{b}} f(x)\,dx$$

Similarly the value $\underline{J}$ is called the lower Darboux integral of f on $[a, b]$ and is denoted by

$$\underline{J} = \underline{\int_{a}^{b}} f(x)\,dx$$

If $\underline{J} = \overline{J} = J$ then J is the unique number such that for any $\epsilon > 0$ there are partitions P, P' of $[a, b]$ such that

$$J - \epsilon < L(P, f) \leq U(P', f) < J + \epsilon$$

If we replace P, P' by their union $P_0 = P \cup P'$ then we see that for any partition $P \supseteq P_0$ (i.e. P finer than P_0) we have

$$J - \epsilon < L(P, f) \leq S(P, f) \leq U(P, f) < J + \epsilon$$

It follows that *for any given $\epsilon > 0$ there is a partition P_0 such that $|S(P, f) - J| < \epsilon$ for any partition $P \supseteq P_0$ of $[a, b]$*. We shall identify this number J with the Riemann integral of f in $[a, b]$. We need to ensure that under these conditions there is a $\delta > 0$ such that $|S(P, f) - J| < \epsilon$ whenever $||P|| < \delta$. The argument for this is bit abstract but highly instructive and reader should pay great attention here.

We replace ϵ by $\epsilon/2$ and let $K = \sup\{|f(x)| : x \in [a, b]\} + 1$ (this $+1$ is to guarantee $K > 0$). Then we have a partition P_0 such that

$$J - \epsilon/2 < L(P, f) \leq S(P, f) \leq U(P, f) < J + \epsilon/2$$

whenever $P \supseteq P_0$. Let N be the number of points in P_0 and we take $\delta = \epsilon/(2KN)$.

Let P be such that $||P|| < \delta$. Then we can write $U(P, f) = \sum M_k \Delta x_k = S_1 + S_2$ where the sum S_1 corresponds to those subintervals of P which contain no points of P_0 (and hence these subintervals are contained fully in subintervals of P_0) and S_2 corresponds to other subintervals made by P.

Clearly we have

$$S_1 \leq U(P_0, f) < J + \epsilon/2,\ \ S_2 \leq NK\,||P|| < NK\delta = \epsilon/2$$

so that $U(P, f) < J + \epsilon$. Similarly we can show that $J - \epsilon < L(P, f)$. It follows that $|S(P, f) - J| < \epsilon$ whenever $||P|| < \delta$. It follows that the number J is the Riemann integral of f over $[a, b]$.

What we have shown above is that:
A sufficient condition for the function f to be integrable on $[a, b]$ is that
$\underline{J} = \sup\{L(P, f) \mid P \text{ is a partition of } [a, b]\} = \overline{J} = \inf\{U(P, f) \mid P \text{ is a partition of } [a, b]\}$
and in this case $J = \underline{J} = \overline{J}$ is the Riemann integral of f on $[a, b]$.

In the course of establishing the above result we have also shown that the following can be taken as **an alternative and simpler definition of Riemann integral**:
Let f be bounded on $[a, b]$. If there is a number I such that for any $\epsilon > 0$ there is a partition P_ϵ of $[a, b]$ such that $|S(P, f) - I| < \epsilon$ whenever $P \supseteq P_\epsilon$ then the function f is said to be Riemann integrable on $[a, b]$ and I is called the Riemann integral of f on $[a, b]$.

The condition of integrability mentioned above in terms of upper and lower Darboux integrals is also necessary as we shall see below. Let us then suppose that f is Riemann integrable over $[a, b]$ and let I be its Riemann integral over $[a, b]$. Then for any $\epsilon > 0$ we have a partition P_ϵ such that $|S(P, f) - I| < \epsilon/3$ whenever $P \supseteq P_\epsilon$. Then we have

$$\left|\sum_{k=1}^{n} f(t_k)\Delta x_k - I\right| < \frac{\epsilon}{3}, \left|\sum_{k=1}^{n} f(t'_k)\Delta x_k - I\right| < \frac{\epsilon}{3}$$

so that

$$\left|\sum_{k=1}^{n}\{f(t_k) - f(t'_k)\}\Delta x_k\right| < \frac{2\epsilon}{3}$$

Since we know that $M_k - m_k = \sup\{f(x) - f(x') \mid x, x' \in [x_{k-1}, x_k]\}$, it follows that we can choose points t_k, t'_k such that

$$f(t_k) - f(t'_k) > M_k - m_k - \frac{\epsilon}{3(b-a)}$$

Therefore we can see that

$$\begin{aligned}U(P, f) - L(P, f) &= \sum_{k=1}^{n}\{M_k - m_k\}\Delta x_k\\ &< \sum_{k=1}^{n}\left\{f(t_k) - f(t'_k) + \frac{\epsilon}{3(b-a)}\right\}\Delta x_k\\ &< \frac{2\epsilon}{3} + \frac{\epsilon}{3(b-a)}\sum_{k=1}^{n}\Delta x_k\\ &= \epsilon\end{aligned}$$

It now follows that we must have $0 \leq \overline{J} - \underline{J} \leq \epsilon$ and thus $\underline{J} = \overline{J}$.

We have now established the **condition of integrability** as follows:
Let f be bounded on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if

$$\underline{\int_{a}^{b}} f(x)\,dx = \overline{\int_{a}^{b}} f(x)\,dx$$

or equivalently *if and only if corresponding to any $\epsilon > 0$ there is a partition P_ϵ of $[a, b]$ such that*
$U(P, f) - L(P, f) < \epsilon$ whenever $P \supseteq P_\epsilon$.

Since making a partition finer never increases $U(P, f) - L(P, f)$ finding only one partition P to meet above criteria is sufficient and therefor we have the following **condition for integrability** given by Riemann:
If f is bounded in $[a, b]$ then f is Riemann integrable on $[a, b]$ if and only if for a given $\epsilon > 0$ we can find a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) = \sum_{k=1}^{n} \{M_k - m_k\}(x_k - x_{k-1}) < \epsilon$$

The reader should observe that the sum $U(P, f) - L(P, f)$ is represented by the yellow rectangles in the last figure and should try to deduce the geometrical meaning of the above condition.

Introductory Definition of the Riemann Integral

We can now understand the rationale behind the definition of integral provided in introductory textbooks of calculus as a limit of sum. We take the partition such that every sub-interval is of equal length. Thus we divide the interval $[a, b]$ into n sub-intervals of equal length $(b - a)/n = h$ and the points t_k are chosen as $t_k = a + kh$. And the integral is then defined as:

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \frac{b - a}{n} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b - a}{n}\right)$$

Apart from being simple (in the sense that it can be used to evaluate integrals of functions like $x, x^2, \sin x, \cos x$) this definition is not at all useful for theoretical investigations. For complicated functions like x^p for a general p the above definition does not help. But if we change the partition in such a way that the points of partition themselves form a geometrical progression with $x_k = ar^k$ and $r^n = b/a$ then we can evaluate the integral for x^p. Clearly this works only when a, b are of same sign. Let's assume $0 < a < b$ so that $r = \sqrt[n]{b/a}$ is defined and as $n \to \infty$, $r \to 1+$. We then have for $x_k = ar^k$

$$\begin{aligned}\int_a^b x^p\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n}(x_k)^p(x_k - x_{k-1})\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}(ar^k)^p(ar^k - ar^{k-1})\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}a^{p+1}r^{k(p+1)-1}(r-1)\\ &= a^{p+1}\cdot\lim_{n\to\infty}(r-1)\sum_{k=1}^{n}r^{k(p+1)-1}\\ &= a^{p+1}\cdot\lim_{n\to\infty}(r-1)\cdot\frac{r^p(r^{n(p+1)}-1)}{r^{p+1}-1}\\ &= a^{p+1}\cdot\lim_{n\to\infty}(r-1)\cdot\frac{r^p((b/a)^{p+1}-1)}{r^{p+1}-1}\\ &= a^{p+1}\cdot\lim_{r\to 1+}(r-1)\cdot\frac{r^p((b/a)^{p+1}-1)}{r^{p+1}-1}\\ &= \frac{a^{p+1}((b/a)^{p+1}-1)}{p+1}\\ &= \frac{b^{p+1}-a^{p+1}}{p+1}\end{aligned}$$

This example alone shows the power of the general definition of Riemann even for purposes of calculation.

Cauchy's Condition for Integrability

If f is integrable over $[a, b]$ then for any $\epsilon > 0$ we have a partition P_ϵ such that $|S(P, f) - I| < \epsilon/2$ for all partitions $P \supseteq P_\epsilon$. Clearly this implies that $|S(P, f) - S(P', f)| < \epsilon$ whenever $P, P' \supseteq P_\epsilon$. It can be shown that this works in the reverse direction too. In other words if for any given $\epsilon > 0$ we have a partition P_ϵ of $[a, b]$ such that $|S(P, f) - S(P', f)| < \epsilon$ whenever $P, P' \supseteq P_\epsilon$ then the function f is integrable over $[a, b]$.

For any given positive integer n we have a partition P_n such that $|S(P, f) - S(P', f)| < 1/n$ whenever $P, P' \supseteq P_n$. If P_{n+1} is not finer than P_n then we can replace P_{n+1} by $P_{n+1} \cup P_n$ and therefore it is safe to assume that $P_{n+1} \supseteq P_n$. Now consider the sequence $\{a_n\}$ given by $a_n = S(P_n, f)$ for some choice of tags associated with P_n.

Clearly if $m > n$ then we have $|a_m - a_n| < 1/n$ and therefore $\{a_n\}$ is a Cauchy sequence and hence has a unique limit I. Letting $m \to \infty$ we see that the inequality $|a_m - a_n| < 1/n$ is transformed into $|a_n - I| \leq 1/n$. Next let $\epsilon > 0$ be given and we choose a positive integer $n > 2/\epsilon$. Let P be a partition of $[a, b]$ with $P \supseteq P_n$ and then we can see that

$$\begin{aligned}|S(P, f) - I| &= |S(P, f) - S(P_n, f) + S(P_n, f) - I|\\ &\leq |S(P, f) - S(P_n, f)| + |S(P_n, f) - I|\\ &< \frac{1}{n} + \frac{1}{n} = \epsilon\end{aligned}$$

It therefore follows that f is integrable on $[a, b]$. Thus we have established the Cauchy's condition of integrability which we can be stated as follows:

A function f *bounded on* $[a, b]$ *is Riemann integrable on* $[a, b]$ *if and only if for any given* $\epsilon > 0$ *there exists a partition* P_{ϵ} *of* $[a, b]$ *such that* $|S(P, f) - S(P', f)| < \epsilon$ *whenever* $P, P' \supseteq P_{\epsilon}$.

Note that using the standard definition of Riemann integral the above condition can also be expressed as:
A function f *bounded on* $[a, b]$ *is Riemann integrable on* $[a, b]$ *if and only if for any given* $\epsilon > 0$ *there exists a number* $\delta > 0$ *such that* $|S(P, f) - S(P', f)| < \epsilon$ *whenever* $||P|| < \delta, ||P'|| < \delta$.

We have provided the definitions of Riemann integral (limit of Riemann sums as the norm of partition gets smaller and smaller or as the partitions get finer and finer) and discussed the conditions of integrability in this post. In the next post we will focus on certain classes of integrable functions and study some of the important properties of Riemann integral.

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Monday, July 30, 2012

Labels: Mathematical Analysis