Math 105 - Practice Exercises: Variations and Groups (with solutions)
Solutions start on the second page of this document. As always, try to solve each problem on your own before reading the solution.

Note: Exercises #1-#5 are based on the musical variations introduced in Section 2.1 of the class notes.

1. For each of the following variations, find its opposite.

a) T_8 b) T_4R c) T_4I

2. Show that each of the following variations is its own opposite:

a) T_8I b) T_6R c) T_2IR

3. Find the cyclic subgroup (of the group of 48 variations) generated by each of the following.

a) T_4 b) T_4R c) T_3IR d) T_7

4. Which musical transpositions generate a cyclic subgroup consisting of exactly four variations? (Hint: there are *four* such variations. Can you find them all?)

5. For each of the following sets of variations (using the usual rules for combining variations) determine whether the set is a group. (Remember: to be a group, you need to have an identity, opposites, and closure.)
a) $\{T_0, R, T_4, T_4R, T_8, T_8R\}$
b) $\{T_0, I, T_4, T_4I\}$
c) $\{T_0, T_3, T_4, T_6, T_8, T_9\}$
d) $\{T_2, T_4, T_6, T_8, T_{10}\}$

Exericses #6 and #7 use "modular arithmetic" (as defined in Section 2.2 of the class notes).

6. Determine whether each of the following is a group.
   a) The set {0, 2, 4} under addition modulo 5
   b) The set {0, 2, 4} under addition modulo 6
   c) The set {1, 2, 3, 4} under multiplication modulo 7
   d) The set {1, 2, 4} under multiplication modulo 7

7. For each group, find the indicated cyclic subgroups.
   a) For the group {0, 1, 2, 3, 4, 5} under mod 6 addition, find the subgroups $\langle 2 \rangle, \langle 3 \rangle$, and $\langle 4 \rangle$.
   b) For the group {0, 1, 2, …, 14} under mod 15 addition, find the subgroups $\langle 3 \rangle$ and $\langle 5 \rangle$.
   c) For the group {1, 2, 3, 4, 5, 6} under mod 7 multiplication, find the subgroups $\langle 2 \rangle$ and $\langle 3 \rangle$.
   d) For the group {1, 2, 3, … , 12} under mod 13 multiplication, find the subgroups $\langle 3 \rangle$ and $\langle 4 \rangle$.

Solutions and Comments

1. Find the opposite of each of the following.

a) The opposite of T_8 is T_4. This is because $T_8T_4 = T_0$, which is the identity.

Comment: in general, the opposite of T_n is T_{12-n}.

b) The opposite of T_4R is T_8R. This is because $T_4\,\underbrace{R\,T_8}_{T_8R}\,R = \underbrace{T_4\,T_8}_{T_0}\underbrace{R\,\,R}_{T_0} = T_0$, which is the identity.

Comment: in general, the opposite of T_nR is $T_{12-n}R$.

c) The opposite of T_4I is T_4I. (Strangely enough, it is its *own* opposite!)

$$T_4\,\underbrace{I\;T_4}_{T_8I}\,I = \underbrace{T_4\,T_8}_{T_0}\,\underbrace{I\,I}_{T_0} = T_0$$

Comment: In fact, it turns out that T_nI is always its own opposite. This is an interesting “side-effect” of the rule for switching the order of inversions and transpositions. Actually, the underlying reason behind this property is that every variation of the form T_nI is actually another inversion – that is, an inversion centered somewhere other than C. (For example, T_2I is the inversion centered at F.)

Here’s how this works out in general: $T_n\,\underbrace{I\,T_n}_{T_{12-n}I}\,I = \underbrace{T_nT_{12-n}}_{T_0}\,\underbrace{II}_{T_0} = T_0$

2. Show that each of the following variations is its own opposite:

a) T_8I is its own opposite because combining T_8I with itself leaves us with the identity, T_0:

$$T_8I\,T_8I = T_8\,\underbrace{IT_8}_{T_{12-8}I}\,I = \underbrace{T_8T_4}_{T_0}\,\underbrace{II}_{T_0} = T_0$$

(As noted in the above comment on #1 part (c), T_nI is its own opposite for *all* values of n.)

b) T_6R is its own opposite: $T_6R\,T_6R = \underbrace{T_6T_6}_{T_0}\,\underbrace{RR}_{T_0} = T_0$

c) T_2IR is its own opposite:

$T_2I\,\underbrace{R\,T_2}_{T_2R}\,IR = T_2\,\underbrace{I\,T_2}_{T_{10}I}\,\underbrace{R\,I}_{IR}\,R = \underbrace{T_2T_{10}}_{T_0}\,\underbrace{I\,I}_{T_0}\,\underbrace{RR}_{T_0} = T_0$

(Note: similarly to T_nI, it turns out that T_nIR is its own opposite for *all* values of n.)

Comment on #2: It turns out that there are 28 variations (more than half of the set of all 48 variations) that are their own opposites. As noted above, all variations of the form T_nI or T_nIR are their own opposites. There are 12 variations of each of these forms (one for each value of n between 0 and 11, inclusive), for a total of 24 variations with ths property. The other four variations that are their own opposites are: T_0, T_6, R, and T_6R.

3. Find the cyclic subgroup (of the group of 48 variations) generated by each of the following.

a) T_4
Answer: the cyclic subgroup generated by T_4 is $\{T_4, T_8, T_0\}$. This is because combining T_4 with itself repeatedly gives us T_4, then T_8, then T_0.

b) T_4R
Answer: The cyclic subgroup generated by T_4R is $\{ T_4R , T_8 , R , T_4 , T_8R , T_0 \}$. See below for details:

One repetition: T_4R

Two repetitions:

$$T_4R\ T_4R = T_8$$

Three repetitions: Note that we know two repetitions give us T_8, so we don't need to do that again – just "add" another T_4R to the previous result, which was T_8:

$$\underbrace{T_8\ T_4}_{T_0}\ R = R$$

Four repetitions: As before, just "add" another T_4R to the preceding result:

$$\underbrace{R\ T_4}_{T_4R}\ R = T_4R\ R = \ T_4$$

Five repetitions: Proceed as before:

$$\underbrace{T_4\ \ T_4}_{T_8}\ R = T_8R$$

Six repetitions:

$$T_8\ \underbrace{R\ T_4}_{T_4R}\ R = \underbrace{T_8T_4}_{T_0}\ \underbrace{RR}_{T_0} = T_0$$

We see that six repetitions of T_4R result in the identity, and this is the smallest number of repetitions which give us this result.

c) T_3IR

Answer: As we noted above (in the solution for #1(d), and again in the solution for #2), any variation of the form T_nIR is its own opposite. Therefore, $T_3IR\ T_3IR = T_0$, so the cyclic subgroup generated by T_3IR only has two variations: $\{T_3IR,\ T_0\}$.

d) T_7

You would need to repeat T_7 12 times to end up with the identity, T_0. You should verify this for yourself. I won't show all the calculations here, but you should end up with – in order (relative to the number of times you've repeated T_7) – the following results:

$$T_7, T_2, T_9, T_4, T_{11}, T_6, T_1, T_8, T_3, T_{10}, T_5, T_0$$

Comment/question: Why do you suppose some variations (like T_4, as seen earlier) only run through a few different transpositions when repeated over and over, while others (such as T_7) run through all twelve?

4. Which musical transpositions have generate a cyclic subgroup consisting of exactly four variations?

Answers: Recall that any variation which involves an inversion (i.e. T_nI or T_nIR) is its own opposite. So, a variation that generates more than two variations must be either a transposition or a transposition followed by a retrograde.

Since 3 goes into 12 four times, we can see pretty quickly that four repetitions of T_3 will result in transposition by 3+3+3+3=12 semitones; that is, $T_3T_3T_3T_3 = T_0$. Similarly, four repetitions of T_3R has the same effect as four repetitions of T_3 and four retrogrades.

The other variations with this property are T_9 and T_9R. This isn't as readily apparent as the other two answers, but they both work: $T_9T_9 = T_{18} = T_6$; $T_9T_9T_9 = T_{27} = T_3$; $T_9T_9T_9T_9 = T_{36} = T_0$. Similarly, T_9R generates a subgroup of size four as well.

Comment: The mathematical reason why T_9 generates a subgroup of size 4 is that $9 + 9 + 9 + 9$ - that is, 9×4 – is the smallest multiple of 9 that is also a multiple of 12. That is, $9 \times 4 = 36$, which is a multiple of 12, and no smaller multiple of 9 is a multiple of 12. In other words, the "least common multiple" of 9 and 12 is $9 \times 4 = 36$. Contrast this result with #3(d) above, in which T_7 turns out to generate a subgroup of size 12; this occurs because the "least common multiple" of 7 and 12 is $7 \times 12 = 84$; no smaller multiple of 7 turns out to also be a multiple of 12.

5. For each of the following sets of variations (using the usual rules for combining variations) determine whether the set is a group. (Remember: to be a group, you need to have an identity, opposites, and closure.)

a) $\{T_0, R, T_4, T_4R, T_8, T_8R\}$
b) $\{T_0, I, T_4, T_4I\}$
c) $\{T_0, T_3, T_4, T_6, T_8, T_9\}$
d) $\{T_2, T_4, T_6, T_8, T_{10}\}$

Answers: (a) is a group; (b), (c) and (d) are not groups.

For (a), we'll use a table to show that all of the group criteria are satisfied:

	T_0	R	T_4	T_4R	T_8	T_8R
T_0	T_0	R	T_4	T_4R	T_8	T_8R
R	R	T_0	T_4R	T_4	T_8R	T_8
T_4	T_4	T_4R	T_8	T_8R	T_0	R
T_4R	T_4R	T_4	T_8R	T_8	R	T_0
T_8	T_8	T_8R	T_0	R	T_4	T_4R
T_8R	T_8R	T_8	R	T_0	T_4R	T_4

Note that we have the identity (T_0 is an element of the set), closure (since every entry in the table was also in the original set), and opposites (since the identity, T_0, appears in each row).

Comment: This set of variations also happens to be the cyclic subgroup generated byh T_4R (see exercise #3(b)). Since cyclic subgroups are always groups, this would be a valid alternative way of showing that this set is a group.

b) $\{T_0, I, T_4, T_4I, T_8\}$

This is not a group because it is not closed. For example, $T_4T_4I = T_8I$, which is not in the set.

Comment: Note that it's not necessary to make a complete operation table (as we did in part a) to show that a set under an operation is NOT a group; to invalidate one of the criteria for a group, all we need to do is find *one* single example to the contrary. (The point of making a complete table is that it's a way to prove that no such contrary examples exist.)

c) $\{T_0, T_3, T_4, T_6, T_8, T_9\}$

This is not a group because it is not closed. For example: T_3 and T_4 are both in the set; however, $T_3T_4 = T_7$ but T_7 is not in the set. (There are other examples we could use here, but one is sufficient.)

Comment: Notice that while we don't have "closure," this set *does* satisfy the other two criteria for a group – it has an identity, T_0, and every element of the group has an opposite: T_3 and T_9 are opposites, T_4 and T_8 are opposites, T_6 is its own opposite, and T_0 is its own opposite.

d) $\{T_2, T_4, T_6, T_8, T_{10}\}$

We know that for any group of variations, the identity element will be T_0. Since T_0 is not included in this set, we can immediately determine that it is not a group (since it does not contain an identity element).

(Note: Adding T_0 to this set *would* make it into a group – verify this for yourself.)

6. Determine whether each of the following is a group.
   a) The set {0, 2, 4} under addition modulo 5
   b) The set {0, 2, 4} under addition modulo 6
   c) The set {1, 2, 3, 4} under multiplication modulo 7
   d) The set {1, 2, 4} under multiplication modulo 7

   Answers: (b) and (d) are groups; (a) and (c) are not groups.
   (a) This set is not a group under addition mod 5 because it is not closed. For example, 2+4=1 (mod 5), but 1 isn't in the set. (Also, neither 2 nor 4 has an opposite in the set.)
   (c) This set is not a group under multiplication mod 7 because it is not closed. For example, 2*3=6 (mod 7), but 7 isn't in the set. (Also, 3 has no opposite in the set.)

   For each of (b) and (d), you can make a table to verify that each set is a group under the given operation. (Ask if you need help with this!)

7. For each group, find the indicated cyclic subgroups.
a) For the group {0, 1, 2, 3, 4, 5} under mod 6 addition, find the subgroups $\langle 2 \rangle$, $\langle 3 \rangle$, and $\langle 4 \rangle$.

- Adding 2 repeatedly gives us 2, 4, 0, 2, 4, 0, …, so $\langle 2 \rangle = \{0, 2, 4\}$.
- Adding 3 repeatedly gives us 3, 0, 3, 0, 3, 0, …, so $\langle 3 \rangle = \{0, 3\}$.
- Adding 4 repeatedly gives us 4, 2, 0, 4, 2, 0, …, so $\langle 4 \rangle = \{0, 2, 4\}$.

Comment: Note that 2 and 4 generate the same subgroup. This happens because 4 is in the subgroup genereated by 2 (and vice-versa).

b) For the group {0, 1, 2, …, 14} under mod 15 addition, find the subgroups $\langle 5 \rangle$ and $\langle 6 \rangle$.

- Adding 5 repeatedly gives us 5, 10, 0, 5, 10, 0, …, so $\langle 5 \rangle = \{0, 5, 10\}$.
- Adding 6 repeatedly gives us 6, 12, 3, 9, 0, …, so $\langle 6 \rangle = \{0, 3, 6, 9, 12\}$.

c) For the group {1, 2, 3, 4, 5, 6} under mod 7 multiplication, find the subgroups $\langle 2 \rangle$ and $\langle 3 \rangle$.

- Multiplying by 2 gives us 2, 4, 1, 2, 4, 1, …, so $\langle 2 \rangle = \{1, 2, 4\}$.
- Multiplying by 3 gives us 3, 2, 6, 4, 5, 1, …, so $\langle 3 \rangle = \{1, 2, 3, 4, 5, 6\}$.

Comment: note that $\langle 3 \rangle$ is the same as the entire group. This is fine, since any group is technically a subgroup of itself. In a case such as this, the element 3 is called a “generator” for the entire group {1, 2, 3, 4, 5, 6}. This group actually contains one other such “generator;” can you find it?

d) For the group {1, 2, 3, … , 12} under mod 13 multiplication, find the subgroups $\langle 3 \rangle$ and $\langle 4 \rangle$.

- Multiplying by 3 gives us 3, 9, 1, 3, 9, 1, …, so $\langle 3 \rangle = \{1, 3, 9\}$.
- Multiplying by 4 gives us 4, 3, 12, 9, 10, 1, …, so $\langle 4 \rangle = \{1, 3, 4, 9, 10, 12\}$.

Comment: Notice that every element of $\langle 3 \rangle$ is also contained in $\langle 4 \rangle$, but not vice-versa. Thus, $\langle 3 \rangle$ is a *subgroup* of $\langle 4 \rangle$.