Certain Lambert Series Identities and their Proof via Trigonometry: Part 1

Introduction

This is yet another post based on a paper of Ramanujan titled "*On certain arithmetical functions*" which appeared in *Transactions of the Cambridge Philosophical Society* in 1916. In this paper Ramanujan provided a lot of identities concerning Lambert series and thereby deduced many relations between various divisor functions. Apart from the amazing results proved in this paper, what I liked most is the very elementary approach followed by Ramanujan compared to the methods of modern authors who are seduced by the *modular form*.

Ramanujan's Functions P, Q, R

Ramanujan introduced the following Lambert series and used them extensively in deriving many identities in elliptic function theory:

$$\begin{aligned} P(q) &= 1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \cdots\right) \\ &= 1 - 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} \quad (1) \\ Q(q) &= 1 + 240\left(\frac{q^2}{1-q^2} + \frac{2^3q^4}{1-q^4} + \frac{3^3q^6}{1-q^6} + \cdots\right) \\ &= 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}} \quad (2) \\ R(q) &= 1 - 504\left(\frac{q^2}{1-q^2} + \frac{2^5q^4}{1-q^4} + \frac{3^5q^6}{1-q^6} + \cdots\right) \\ &= 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}} \quad (3) \end{aligned}$$

We have already met $P(q)$ in a previous post regarding series for $1/\pi$. Also the $R(q)$ is not to be confused with the Rogers-Ramanujan continued fractions introduced in the last post. Ramanujan also used the alternative notation L, M, N instead of P, Q, R. Again to simplify matters regarding manipulation of above series Ramanujan used the variable $x = q^2$ and hence we get the following notation which will be used subsequently in this post:

$$\begin{aligned}P(x) &= 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right)\\ &= 1 - 24\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} \qquad (4)\\ Q(x) &= 1 + 240\left(\frac{x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right)\\ &= 1 + 240\sum_{n=1}^{\infty}\frac{n^3x^n}{1-x^n} \qquad (5)\\ R(x) &= 1 - 504\left(\frac{x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \cdots\right)\\ &= 1 - 504\sum_{n=1}^{\infty}\frac{n^5x^n}{1-x^n} \qquad (6)\end{aligned}$$

The Lambert series above can be easily expressed as generating functions for divisor function. In general for any positive integer r, we can see that

$$\sum_{n=1}^{\infty}\frac{n^rx^n}{1-x^n} = \sum_{n=1}^{\infty}\sigma_r(n)x^n$$

where $\sigma_r(n)$ denotes sum of r^{th} powers of divisors of n.

Ramanujan considered these sums and its generalization below:

$$\begin{aligned}S_r(x) &= E_r + \frac{1^rx}{1-x} + \frac{2^rx^2}{1-x^2} + \frac{3^rx^3}{1-x^3} + \cdots\\ &= E_r + \sum_{n=1}^{\infty}\frac{n^rx^n}{1-x^n}\\ &= E_r + \sum_{n=1}^{\infty}\sigma_r(n)x^n \qquad (7)\\ \Phi_{r,s}(x) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^rn^sx^{mn} \qquad (8)\end{aligned}$$

where E_r is a suitably chosen constant related with Bernoulli's numbers as we shall see later. It is easy to see that

$$\begin{aligned}\Phi_{r,s}(x) = \Phi_{s,r}(x) &= \sum_{n=1}^{\infty}n^s\sigma_{r-s}(n)x^n \qquad (9)\\ S_r(x) &= E_r + \Phi_{0,r}(x) \qquad (10)\end{aligned}$$

Ramanujan was able to express $\Phi_{r,s}(x)$, with $r+s$ an odd positive integer, in terms of P, Q, R in a very elementary manner using trigonometrical series. This is one of the truly amazing proofs which Ramanujan provided. In a way the proof shows that a lot more can be achieved with elementary stuff than people think. To understand the proof we need to develop the series for $\cot(x)$ and there we will see the use of Bernoulli's numbers.

Expansion of $\cot x$

We know that the Bernoulli's numbers B_n are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{11}$$

Now we can see that

$$\frac{x}{e^x - 1} = \frac{1}{1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \cdots} = 1 - \frac{x}{2} + \cdots$$

and it is easy to verify that $\{x/(e^x - 1)\} + (x/2)$ is an even function therefore it follows that:

$$B_0 = 1, B_1 = \frac{-1}{2}, B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_3 = B_5 = \cdots = B_{2n+1} = 0 \tag{12}$$

Next we proceed to find expansion of $\cot x$ in powers of x. We have

$$\begin{aligned}\cot x &= \frac{\cos x}{\sin x} = i \cdot \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = i \cdot \frac{e^{2ix} + 1}{e^{2ix} - 1} = i \cdot \left(1 + \frac{2}{e^{2ix} - 1}\right)\\ &= i + \frac{1}{x} \cdot \frac{2ix}{e^{2ix} - 1}\\ &= i + \frac{1}{x}\left(1 - \frac{2ix}{2} + B_2 \cdot \frac{(2ix)^2}{2!} + B_4 \cdot \frac{(2ix)^4}{4!} + \cdots\right)\\ &= \frac{1}{x} - B_2 \frac{2^2 x}{2!} + B_4 \frac{2^4 x^3}{4!} - B_6 \frac{2^6 x^5}{6!} + \cdots \end{aligned} \tag{13}$$

Differentiating with respect to x we get

$$\begin{aligned}-\operatorname{cosec}^2 x &= -\frac{1}{x^2} - B_2 \frac{2^2}{2!} + 3B_4 \frac{2^4 x^2}{4!} - 5B_6 \frac{2^6 x^4}{6!} + \cdots\\ \Rightarrow 1 + \cot^2 x &= \frac{1}{x^2} + \frac{1}{3} - 3B_4 \frac{2^4 x^2}{4!} + \cdots\\ \Rightarrow \cot^2 x &= \frac{1}{x^2} - \frac{2}{3} - 3B_4 \frac{2^4 x^2}{4!} + 5B_6 \frac{2^6 x^4}{6!} - 7B_8 \frac{2^8 x^6}{8!} + \cdots \end{aligned} \tag{14}$$

A Trigonometrical Identity

Ramanujan next uses the formula for sum of cosines of angles in arithmetic progression in the following manner

$$\begin{aligned}2\{\cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta\} &= \frac{2\cos\dfrac{n\theta}{2}\sin\dfrac{(n-1)\theta}{2}}{\sin\dfrac{\theta}{2}}\\ &= \frac{\sin\left(n - \dfrac{1}{2}\right)\theta - \sin\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}\\ &= \frac{\sin n\theta\cos\dfrac{\theta}{2} - \cos n\theta\sin\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} - 1\\ &= \cot\frac{\theta}{2}\sin n\theta - \cos n\theta - 1\end{aligned}$$

to get

$$\cot\frac{\theta}{2}\sin n\theta = 1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos(n-1)\theta + \cos n\theta \tag{15}$$

Out of the blue Ramanujan now sets out to consider the expression

$$S = \left(\frac{1}{4}\cot\frac{\theta}{2} + \frac{x\sin\theta}{1-x} + \frac{x^2\sin 2\theta}{1-x^2} + \frac{x^3\sin 3\theta}{1-x^3} + \cdots\right)^2$$

Let $u_m = x^m/(1-x^m)$ and then the above expression can be written as

$$\begin{aligned} S &= \left(\frac{1}{4}\cot\frac{\theta}{2} + u_1\sin\theta + u_2\sin 2\theta + u_3\sin 3\theta + \cdots\right)^2 \\ &= \left(\frac{1}{4}\cot\frac{\theta}{2} + \sum_{m=1}^{\infty} u_m\sin m\theta\right)^2 \\ &= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \frac{1}{2}\cot\frac{\theta}{2}\sum_{m=1}^{\infty} u_m\sin m\theta + \left(\sum_{m=1}^{\infty} u_m\sin m\theta\right)^2 \\ &= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + T_1 + T_2 \end{aligned}$$

where using (14) we can express T_1 in terms of cosines as:

$$T_1 = \sum_{m=1}^{\infty} u_m\left\{\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(m-1)\theta + \frac{1}{2}\cos m\theta\right\}$$

and T_2 can also be expressed in terms of cosines as follows:

$$\begin{aligned} T_2 &= \sum_{m=1}^{\infty} u_m\sin m\theta\sum_{n=1}^{\infty} u_n\sin n\theta \\ &= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\cos(m-n)\theta - \cos(m+n)\theta\}u_m u_n \end{aligned}$$

and thus $T_1 + T_2$ can be arranged in a series of the form

$$T_1 + T_2 = \sum_{k=0}^{\infty} C_k\cos k\theta$$

Now it is clear that the contribution to C_0 from T_1 is $(1/2)\sum u_m$ and from T_2 the contribution is $(1/2)\sum u_m^2$ and therefore

$$\begin{aligned}
C_0 &= \frac{1}{2}\sum_{m=1}^{\infty} u_m + \frac{1}{2}\sum_{m=1}^{\infty} u_m^2\\
&= \frac{1}{2}\sum_{m=1}^{\infty} u_m(1+u_m)\\
&= \frac{1}{2}\sum_{m=1}^{\infty} \frac{x^m}{(1-x^m)^2}\\
&= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nx^{mn}\\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \frac{1}{2}\sum_{n=1}^{\infty} nu_n
\end{aligned}$$

For $k > 0$, the part of C_k coming from T_1 is

$$\frac{1}{2}u_k + \sum_{m=k+1}^{\infty} u_m = \frac{1}{2}u_k + \sum_{l=1}^{\infty} u_{k+l}$$

and the part of C_k coming from T_2 is

$$\frac{1}{2}\sum_{m-n=k} u_m u_n + \frac{1}{2}\sum_{n-m=k} u_m u_n - \frac{1}{2}\sum_{m+n=k} u_m u_n = \sum_{l=1}^{\infty} u_l u_{k+l} - \frac{1}{2}\sum_{l=1}^{k-1} u_l u_{k-l}$$

Thus it follows that for $k > 0$

$$\begin{aligned}
C_k &= \frac{1}{2}u_k + \sum_{l=1}^{\infty} u_{k+l} + \sum_{l=1}^{\infty} u_l u_{k+l} - \frac{1}{2}\sum_{l=1}^{k-1} u_l u_{k-l}\\
&= \frac{1}{2}u_k + \sum_{l=1}^{\infty} u_{k+l}(1+u_l) - \frac{1}{2}\sum_{l=1}^{k-1} u_l u_{k-l}\\
&= \frac{1}{2}u_k + \sum_{l=1}^{\infty} u_k(u_l - u_{k+l}) - \frac{1}{2}\sum_{l=1}^{k-1} u_k(1+u_l+u_{k-1})\\
&= u_k\left\{\frac{1}{2} + \sum_{l=1}^{\infty}(u_l - u_{k+l}) - \frac{1}{2}\sum_{l=1}^{k-1}(1+u_l+u_{k-l})\right\}\\
&= u_k\left\{\frac{1}{2} + u_1 + u_2 + \cdots + u_k - \frac{k-1}{2} - (u_1 + u_2 + \cdots + u_{k-1})\right\}\\
&= u_k\left\{1 + u_k - \frac{k}{2}\right\}
\end{aligned}$$

The crucial part in the above proof are the easily verifiable identities:

$$u_{k+l}(1+u_l) = u_k(u_l - u_{k+l}),\ \ u_l u_{k-l} = u_k(1+u_l+u_{k-l})$$

Finally putting all the pieces together we can see that

$$\begin{aligned}
S &= \left(\frac{1}{4}\cot\frac{\theta}{2} + \frac{x\sin\theta}{1-x} + \frac{x^2\sin 2\theta}{1-x^2} + \frac{x^3\sin 3\theta}{1-x^3} + \cdots\right)^2\\
&= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} nu_n + \sum_{k=1}^{\infty} u_k\left(1 + u_k - \frac{k}{2}\right)\cos k\theta\\
&= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{m=1}^{\infty} u_m(1+u_m)\cos m\theta + \frac{1}{2}\sum_{m=1}^{\infty} mu_m(1-\cos m\theta)\\
&= \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \sum_{m=1}^{\infty}\frac{x^m\cos m\theta}{(1-x^m)^2} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{mx^m}{1-x^m}(1-\cos m\theta)
\end{aligned}$$

Ramanujan proved the above without using the symbols $\sum$ and u_m and expressed his formula directly as:

$$\begin{aligned}
&\left(\frac{1}{4}\cot\frac{\theta}{2} + \frac{x\sin\theta}{1-x} + \frac{x^2\sin 2\theta}{1-x^2} + \frac{x^3\sin 3\theta}{1-x^3} + \cdots\right)^2\\
&\quad = \left(\frac{1}{4}\cot\frac{\theta}{2}\right)^2 + \frac{x\cos\theta}{(1-x)^2} + \frac{x^2\cos 2\theta}{(1-x^2)^2} + \frac{x^3\cos 3\theta}{(1-x^3)^2} + \cdots\\
&\quad + \frac{1}{2}\left\{\frac{x(1-\cos\theta)}{1-x} + \frac{2x^2(1-\cos 2\theta)}{1-x^2} + \frac{3x^3(1-\cos 3\theta)}{1-x^3} + \cdots\right\} \qquad (16)
\end{aligned}$$

The presentation we have offered above is from G. H. Hardy's *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*. The proof above may look complicated because of symbolism, but in reality it involves basic algebraic manipulations.

In a similar manner Ramanujan uses the trigonometric identity:

$$\begin{aligned}
&\cot^2\frac{\theta}{2}(1-\cos n\theta)\\
&\quad = (2n-1) + 4(n-1)\cos\theta + 4(n-2)\cos 2\theta + \cdots\\
&\qquad + 4\cos(n-1)\theta + \cos n\theta \qquad (17)
\end{aligned}$$

and establishes the following result:

$$\begin{aligned}
&\left\{\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12} + \frac{x(1-\cos\theta)}{1-x}\right.\\
&\left. + \frac{2x^2(1-\cos 2\theta)}{1-x^2} + \frac{3x^3(1-\cos 3\theta)}{1-x^3} + \cdots\right\}^2\\
&\quad = \left(\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12}\right)^2\\
&\qquad + \frac{1}{12}\left\{\frac{1^3x(5+\cos\theta)}{1-x} + \frac{2^3x^2(5+\cos 2\theta)}{1-x^2}\right.\\
&\qquad \left. + \frac{3^3x^3(5+\cos 3\theta)}{1-x^3} + \cdots\right\} \qquad (18)
\end{aligned}$$

The algebraic manipulations in this case are of similar nature but bit more complicated and hence will not be presented here. From (14) it is clear that

$$\frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12} = \frac{1}{2}\left(\frac{1}{\theta^2} - \frac{3B_4}{4!}\theta^2 + \frac{5B_6}{6!}\theta^4 - \frac{7B_8}{8!}\theta^6 + \cdots\right) \qquad (19)$$

and

$$\frac{mx^m}{1-x^m}(1-\cos m\theta) = \frac{mx^m}{1-x^m}\left(\frac{m^2\theta^2}{2!} - \frac{m^4\theta^4}{4!} + \cdots\right)$$

Hence the LHS of (18) can be written as

$$\left\{\frac{1}{2\theta^2} + \frac{\theta^2}{2!}\left(-\frac{B_4}{2\cdot 4} + \sum_{m=1}^{\infty}\frac{m^3x^m}{1-x^m}\right) - \frac{\theta^4}{4!}\left(-\frac{B_6}{2\cdot 6} + \sum_{m=1}^{\infty}\frac{m^5x^m}{1-x^m}\right) + \cdots\right\}^2$$

It is now time to relate E_r of (7) with Bernoulli's numbers as follows:

$$E_r = -\frac{B_{r+1}}{2(r+1)}$$

and then using (7) we can write the LHS of (18) as:

$$\left\{\frac{1}{2\theta^2} + \frac{\theta^2}{2!}S_3(x) - \frac{\theta^4}{4!}S_5(x) + \cdots\right\}^2$$

To compute the RHS of (18) we need to first square the equation (19). Instead of squaring the series expansion on right of (19) I prefer to use derivatives of the expansion given in (14). Thus on differentiating (14) we get

$$\begin{aligned}2\cot x(-1-\cot^2 x) &= -\frac{2}{x^3} - 2\cdot 3B_4\frac{2^4x}{4!} + 4\cdot 5B_6\frac{2^6x^3}{6!} - \cdots\\ \Rightarrow \cot x + \cot^3 x &= \frac{1}{x^3} + 2\cdot 3B_4\frac{2^3x}{4!} - 4\cdot 5B_6\frac{2^5x^3}{6!} + \cdots\end{aligned}$$

and using (13) we get

$$\cot^3 x = \frac{1}{x^3} - \frac{1}{x} + \frac{2^2}{4!}(2\cdot 2\cdot 3B_4 + 3\cdot 4B_2)x - \frac{2^4}{6!}(2\cdot 4\cdot 5B_6 + 5\cdot 6B_4)x^3 + \cdots$$

Differentiating the above equation we get

$$\begin{aligned}3\cot^2 x(-1-\cot^2 x) &= -\frac{3}{x^4} + \frac{1}{x^2} + \frac{2^2}{4!}(2\cdot 2\cdot 3B_4 + 3\cdot 4B_2)\\ &\quad - \frac{2^4}{6!}\cdot 3(2\cdot 4\cdot 5B_6 + 5\cdot 6B_4)x^2 + \cdots\\ \Rightarrow 3\cot^2 x + 3\cot^4 x &= \frac{3}{x^4} - \frac{1}{x^2} - \frac{2^2}{4!}(2\cdot 2\cdot 3B_4 + 3\cdot 4B_2)\\ &\quad + \frac{2^4}{6!}\cdot 3(2\cdot 4\cdot 5B_6 + 5\cdot 6B_4)x^2 - \cdots\end{aligned}$$

Adding this equation and (14) (and values of B_2, B_4) we get

$$\begin{aligned}3\cot^4 x + 4\cot^2 x = \frac{3}{x^4} &- \frac{14}{15} + \frac{2^5B_6}{6}\cdot\frac{x^2}{2!}\\ &- \frac{2^7B_8}{8}\cdot\frac{x^4}{4!}\\ &+ \frac{2^9B_{10}}{10}\cdot\frac{x^6}{6!} - \cdots\end{aligned}\tag{20}$$

Again going back to the equation (19) we see that square of its LHS is given by

$$\begin{aligned}\left(\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12}\right)^2 &= \frac{1}{3\cdot 2^6}\left(3\cot^4\frac{\theta}{2}+4\cot^2\frac{\theta}{2}\right)+\frac{1}{144}\\ &= \frac{1}{4\theta^4}+\frac{1}{480}+\frac{1}{24}\left(\frac{B_6}{6}\cdot\frac{\theta^2}{2!}-\frac{B_8}{8}\cdot\frac{\theta^4}{4!}+\cdots\right)\\ &= \frac{1}{4\theta^4}-\frac{B_4}{16}+\frac{1}{24}\left(\frac{B_6}{6}\cdot\frac{\theta^2}{2!}-\frac{B_8}{8}\cdot\frac{\theta^4}{4!}+\cdots\right)\\ &= \frac{1}{4\theta^4}+\frac{E_3}{2}-\frac{1}{12}\left(E_5\cdot\frac{\theta^2}{2!}-E_7\cdot\frac{\theta^4}{4!}+\cdots\right)\end{aligned}$$

We can now clearly see that the RHS of (18) is given by

$$\frac{1}{4\theta^4}+\frac{S_3(x)}{2}-\frac{1}{12}\left(\frac{\theta^2}{2!}S_5(x)-\frac{\theta^4}{4!}S_7(x)+\cdots\right)$$

and finally the equation (18) is transformed into

$$\begin{aligned}&\left\{\frac{1}{2\theta^2}+\frac{\theta^2}{2!}S_3(x)-\frac{\theta^4}{4!}S_5(x)+\cdots\right\}^2\\ &\quad=\frac{1}{4\theta^4}+\frac{S_3(x)}{2}-\frac{1}{12}\left(\frac{\theta^2}{2!}S_5(x)-\frac{\theta^4}{4!}S_7(x)+\cdots\right)\end{aligned}\tag{21}$$

For even integer $n>2$ we equate the coefficients of θ^n on both sides to obtain the following euqation

$$\begin{aligned}\frac{(n-2)(n+5)}{12(n+1)(n+2)}S_{n+3}(x) &= \binom{n}{2}S_3(x)S_{n-1}(x)\\ &\quad+\binom{n}{4}S_5(x)S_{n-3}(x)+\cdots\\ &\quad+\binom{n}{n-2}S_{n-1}(x)S_3(x)\end{aligned}\tag{22}$$

where

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

is the usual binomial coefficient.

Now it is easy to see that we have

$$P(x)=-24S_1(x), Q(x)=240S_3(x), R(x)=-504S_5(x)$$

and hence using the relation (22) we can evaluate $S_{2n-1}(x)$ in terms of P, Q, R for all integers $n>0$. For small values of n it is easy to apply the formula and derive the following:

$$\begin{aligned}1+480\Phi_{0,7}(x)&=Q^2\\ 1-264\Phi_{0,9}(x)&=QR\\ 691+65520\Phi_{0,11}(x)&=441Q^3+250R^2\\ 1-24\Phi_{0,13}(x)&=Q^2R\end{aligned}$$

Ramanujan does not stop here and actually uses the relation (22) to evaluate $S_7(x), S_9(x), \ldots, S_{31}(x)$ and gives his results in terms of the function $\Phi_{r,s}(x)$. Ramanujan did all this for his love of numbers and we show one example here which sheds light on the nature of numbers he dealt with:

$$\begin{aligned}7709321041217 + 32640\Phi_{0,31}(x) &= 764412173217Q^8(x)\\ &+ 5323905468000Q^5(x)R^2(x)\\ &+ 1621003400000Q^2(x)R^4(x)\end{aligned}$$

In the next post we will analyze the equation (16) and the results derived from it.

Postscript: L. C. Shen provided another proof of (16) in 1993 using derivatives of theta functions which we reproduce below:

We have from these posts

$$\begin{aligned}\theta_1(z, q) &= 2q^{1/4}\sin z\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2z+q^{4n})\\ &= 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin(2n+1)z \end{aligned}\tag{23}$$

Differentiating twice the infinite series representation of $\theta_1(z, q)$ with respect to z we get

$$\frac{\partial^2\theta_1}{\partial z^2} = -2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(2n+1)^2\sin(2n+1)z$$

Again differentiating $\theta_1(z, q)$ with respect to q we get

$$\begin{aligned}\frac{\partial\theta_1}{\partial q} &= 2\sum_{n=0}^{\infty}(-1)^n(n+1/2)^2 q^{(n+1/2)^2-1}\sin(2n+1)z\\ &= \frac{1}{2}q^{-3/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(2n+1)^2\sin(2n+1)z\end{aligned}$$

and thus we arrive at the partial differential equation satisfied by θ_1

$$\frac{\partial^2\theta_1}{\partial z^2} = -4q\frac{\partial\theta_1}{\partial q}\tag{24}$$

Next by logarithmic differentiation of the product expansion of $\theta_1(z, q)$ with respect to z we get

$$\begin{aligned}\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial z} &= \cot z + 4\sum_{n=1}^{\infty}\frac{q^{2n}\sin 2z}{1-2q^{2n}\cos 2z+q^{4n}}\\
&= \cot z + \frac{4}{2i}\sum_{n=1}^{\infty}\frac{q^{2n}(e^{2iz}-e^{-2iz})}{(1-q^{2n}e^{2iz})(1-q^{2n}e^{-2iz})}\\
&= \cot z + \frac{4}{2i}\sum_{n=1}^{\infty}q^{2n}\left(\frac{e^{2iz}}{1-q^{2n}e^{2iz}}-\frac{e^{-2iz}}{1-q^{2n}e^{-2iz}}\right)\\
&= \cot z + \frac{4}{2i}\left(\sum_{n=1}^{\infty}\frac{q^{2n}e^{2iz}}{1-q^{2n}e^{2iz}}-\sum_{n=1}^{\infty}\frac{q^{2n}e^{-2iz}}{1-q^{2n}e^{-2iz}}\right)\\
&= \cot z + \frac{4}{2i}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{q^{2n}e^{2iz}\}^{m}-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{q^{2n}e^{-2iz}\}^{m}\right)\\
&= \cot z + \frac{4}{2i}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q^{2m}e^{2imz}q^{2m(n-1)}\right.\\
&\qquad\left.-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q^{2m}e^{-2imz}q^{2m(n-1)}\right)\\
&= \cot z + \frac{4}{2i}\left(\sum_{m=1}^{\infty}q^{2m}e^{2imz}\sum_{n=1}^{\infty}q^{2m(n-1)}\right.\\
&\qquad\left.-\sum_{m=1}^{\infty}q^{2m}e^{-2imz}\sum_{n=1}^{\infty}q^{2m(n-1)}\right)\\
&= \cot z + \frac{4}{2i}\left(\sum_{m=1}^{\infty}\frac{q^{2m}e^{2imz}}{1-q^{2m}}-\sum_{m=1}^{\infty}\frac{q^{2m}e^{-2imz}}{1-q^{2m}}\right)\\
&= \cot z + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nz \qquad (25)\end{aligned}$$

Noting that $1-2q^{2n}\cos 2z+q^{4n}=(1-q^{2n}e^{2iz})(1-q^{2n}e^{-2iz})$ and performing logarithmic differentiation with respect to q we get

$$\begin{aligned}\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial q} &= \frac{1}{4q}-\frac{2}{q}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-\frac{2}{q}\left(\sum_{n=1}^{\infty}\frac{nq^{2n}e^{2iz}}{1-q^{2n}e^{2iz}}+\sum_{n=1}^{\infty}\frac{nq^{2n}e^{-2iz}}{1-q^{2n}e^{-2iz}}\right)\\
&= \frac{1}{4q}-\frac{2}{q}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-\frac{2}{q}\left(\sum_{n=1}^{\infty}nq^{2n}e^{2iz}\sum_{m=1}^{\infty}\{q^{2n}e^{2iz}\}^{m-1}\right.\\
&\qquad\left.+\sum_{n=1}^{\infty}nq^{2n}e^{-2iz}\sum_{m=1}^{\infty}\{q^{2n}e^{-2iz}\}^{m-1}\right)\\
&= \frac{1}{4q}-\frac{2}{q}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-\frac{2}{q}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{2mn}e^{2imz}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{2mn}e^{-2imz}\right)\\
&= \frac{1}{4q}-\frac{2}{q}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-\frac{2}{q}\left(\sum_{m=1}^{\infty}e^{2imz}\sum_{n=1}^{\infty}nq^{2mn}+\sum_{m=1}^{\infty}e^{-2imz}\sum_{n=1}^{\infty}nq^{2mn}\right)\\
&= \frac{1}{4q}-\frac{2}{q}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-\frac{2}{q}\left(\sum_{m=1}^{\infty}e^{2imz}\frac{q^{2m}}{(1-q^{2m})^2}+\sum_{m=1}^{\infty}e^{-2imz}\frac{q^{2m}}{(1-q^{2m})^2}\right)\\
&= \frac{1}{4q}-\frac{2}{q}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}-\frac{4}{q}\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\cos 2nz\end{aligned}$$

Using differential equation (24) we get

$$\frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial z^2} = -1 + 16\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\cos 2nz + 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} \tag{26}$$

Now differentiating equation (25) with respect to z we get

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial z}\right) &= -1 - \cot^2 z + 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz\\ \Rightarrow \frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial z^2} - \left(\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial z}\right)^2 &= -1 - \cot^2 z + 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz\\ \Rightarrow \left(\frac{1}{\theta_1}\frac{\partial\theta_1}{\partial z}\right)^2 &= 1 + \cot^2 z - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2nz + \frac{1}{\theta_1}\frac{\partial^2\theta_1}{\partial z^2}\end{aligned}$$

Now using (25) and (26) we get

$$\begin{aligned}&\left(\cot z + 4\sum_{n=1}^{\infty}\frac{q^{2n}\sin 2nz}{1-q^{2n}}\right)^2\\ &= \cot^2 z + 16\sum_{n=1}^{\infty}\frac{q^{2n}\cos 2nz}{(1-q^{2n})^2} + 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}(1-\cos 2nz)\end{aligned}$$

If we replace z by $\theta/2$ and q^2 by x and divide resulting equation by 16 we get the identity (16) obtained by Ramanujan using algebraic manipulation.

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