Practice Exercises: Variations and Groups (these will *not* be collected, but will be discussed in class)

Note: Each question about variations is based on the 48 variations discussed in class, which are defined based on the usual 12-tone scale.

1. Find the opposite of each of the following.
a) T_8 b) T_4R c) T_4I d) T_9IR e) T_6R

2. Which variations (out of the set of 48 musical variations) are their own opposites?

3. Find the cyclic subgroup generated by each of the following variations:
a) T_4 b) T_4R c) T_3IR d) T_7

4. Which musical transpositions have generate a cyclic subgroup consisting of exactly four variations?

5. Determine whether each of the following is a group.
a) The set {0, 2, 4} under addition modulo 5
b) The set {0, 2, 4} under addition modulo 6
c) The set {1, 2, 3, 4} under multiplication modulo 7
d) The set {1, 2, 4} under multiplication modulo 7

6. For each of the following sets of variations (using the usual rules for combining variations) determine whether the set is a group. (Remember: to be a group, you need to have an identity, opposites, and closure.)
a) $\{T_0, R, T_4, T_4R, T_8, T_8R\}$
b) $\{T_0, I, T_4, T_4I\}$
c) $\{T_0, T_3, T_4, T_6, T_8, T_9\}$
d) $\{T_2, T_4, T_6, T_8, T_{10}\}$

SOLUTIONS

1. Find the opposite of each of the following.
a) The opposite of T_8 is T_4. This is because $T_8T_4 = T_0$, which is the identity.

Comment: in general, the opposite of T_n is T_{12-n}.

b) The opposite of T_4R is T_8R. This is because $T_4 \underbrace{R\, T_8}_{T_8R} R = \underbrace{T_4\ T_8}_{T_0} \underbrace{R\ R}_{T_0} = T_0$, which is the identity.

Comment: in general, the opposite of T_nR is $T_{12-n}R$.

c) The opposite of T_4I is T_4I. (Strangely enough, it is its *own* opposite!)

$$T_4 \underbrace{I\ T_4}_{T_8I} I = \underbrace{T_4\, T_8}_{T_0} \underbrace{I\, I}_{T_0} = T_0$$

Comment: In fact, it turns out that T_nI is always its own opposite. This is an interesting "side-effect" of the rule for switching the order of inversions and transpositions. Actually, the underlying reason behind this property is that every variation of the form T_nI is actually another inversion – that is, an inversion centered somewhere other than C. (For example, T_2I is the inversion centered at F.)

Here's how it works in general:

$$T_n \underbrace{I\, T_n}_{T_{12-n}I} I = \underbrace{T_nT_{12-n}}_{T_0} \underbrace{II}_{T_0} = T_0$$

d) The opposite of T_9IR is T_9IR.

Comment: As is the case with variations of the form T_nI, each variation of the form T_nIR is also its own opposite:

$$T_nI \underbrace{R\ T_n}_{T_nR} \underbrace{IR}_{RI} = T_n \underbrace{I\ \ T_n}_{T_{12-n}I} \underbrace{R\ R}_{T_0} I = \underbrace{T_nT_{12-n}}_{T_0} \underbrace{II}_{T_0} = T_0$$

e) The opposite of T_6R is T_6R.

Comment: See part (b) – this just applies the same rule for finding the opposite of T_nR, since $6 = 12 - 6$. It's interesting to note, though, that T_6R is yet another variation which is its own opposite. This observation leads us to…

2. Which variations (our of all of the 48 possible variations) are their own opposites?

Solution: This is easier to answer if you work through #1 first (see above). We find that all variations of the form T_nI or T_nIR are their own opposites. There are 12 of each of these (one for each value of n between 0 and 11, inclusive), which gives us 24 variations which are their own opposites. As seen in part (e) of #1, there are others as well – T_6R is its own opposite. Similarly, T_6 is its own opposite as well. The only other variations which are their own opposites are R and T_0. This gives us a total of 28 (out of 48) variations which are their own opposites.

3. Find the cyclic subgroup generated by each of the following variations:

a) T_4

Answer: the cyclic subgroup generated by T_4 is $\{T_4, T_8, T_0\}$. This is because combining T_4 with itself repeatedly gives us T_4, then T_8, then T_0.

b) T_4R

Answer: The cyclic subgroup generated by T_4R is $\{T_4R, T_8, R, T_4, T_8R, T_0\}$. See below for details:

One repetition: T_4R

Two repetitions:

$$T_4R\ T_4R = T_8$$

Three repetitions: Note that we know two repetitions give us T_8, so we don't need to do that again – just "add" another T_4R to the previous result, which was T_8:

$$\underbrace{T_8\ T_4}_{T_0}\ R = R$$

Four repetitions: As before, just "add" another T_4R to the preceding result:

$$\underbrace{R\ T_4}_{T_4R}\ R = T_4R\ R = T_4$$

Five repetitions: Proceed as before:

$$\underbrace{T_4\ \ T_4}_{T_8}\ R = T_8R$$

Six repetitions:

$$T_8\ \underbrace{R\ T_4}_{T_4R}\ R = \underbrace{T_8T_4}_{T_0}\ \underbrace{RR}_{T_0} = T_0$$

We see that six repetitions of T_4R result in the identity, and this is the smallest number of repetitions which give us this result.

c) T_3IR

Answer: As we noted above (in the solution for #1(d), and again in the solution for #2), any variation of the form T_nIR is its own opposite. Therefore, $T_3IR\ T_3IR = T_0$, so the cyclic subgroup generated by T_3IR only has two variations: $\{T_3IR, T_0\}$.

d) T_7

You would need to repeat T_7 12 times to end up with the identity, T_0. You should verify this for yourself. I won't show all the calculations here, but you should end up with – in order (relative to the number of times you've repeated T_7) – the following results:

$$T_7, T_2, T_9, T_4, T_{11}, T_6, T_1, T_8, T_3, T_{10}, T_5, T_0$$

Comment/question: Why do you suppose some variations (like T_4, as seen earlier) only run through a few different transpositions when repeated over and over, while others (such as T_7) run through all twelve?

4. Which musical transpositions have generate a cyclic subgroup consisting of exactly four variations?

Answers: Recall that any variation which involves an inversion (i.e. T_nI or T_nIR) is its own opposite. So, a variation that generates more than two variations must be either a transposition or a transposition followed by a retrograde.

Since 3 goes into 12 four times, we can see pretty quickly that four repetitions of T_3 will result in transposition by 3+3+3+3=12 semitones; that is, $T_3T_3T_3T_3 = T_0$. Similarly, four repetitions of T_3R has the same effect as four repetitions of T_3 and four retrogrades.

The other variations with this property are T_9 and T_9R. This isn't as readily apparent as the other two answers, but they both work: $T_9T_9 = T_{18} = T_6$; $T_9T_9T_9 = T_{27} = T_3$; $T_9T_9T_9T_9 = T_{36} = T_0$. Similarly, T_9R generates a subgroup of size four as well.

Comment: The mathematical reason why T_9 generates a subgroup of size 4 is that $9 + 9 + 9 + 9$ - that is, 9×4 – is the smallest multiple of 9 that is also a multiple of 12. That is, $9 \times 4 = 36$, which is a multiple of 12, and no smaller multiple of 9 is a multiple of 12. In other words, the "least common multiple" of 9 and 12 is $9 \times 4 = 36$. Contrast this result with #3(d) above, in which T_7 turns out to generate a subgroup of size 12 since the "least common multiple" of 7 and 12 is $7 \times 12 = 84$; no smaller multiple of 7 turns out to also be a multiple of 12.

5. Determine whether each of the following is a group.
a) The set {0, 2, 4} under addition modulo 5
b) The set {0, 2, 4} under addition modulo 6
c) The set {1, 2, 3, 4} under multiplication modulo 7
d) The set {1, 2, 4} under multiplication modulo 7

Answers: (b) and (d) are groups; (a) and (c) are not groups.
(a) This set is not a group under addition mod 5 because it is not closed. For example, 2+4=1 (mod 5), but 1 isn't in the set. (Also, neither 2 nor 4 has an opposite in the set.)
(c) This set is not a group under multiplication mod 7 because it is not closed. For example, 2*3=6 (mod 7), but 7 isn't in the set. (Also, 3 has no opposite in the set.)

For each of (b) and (d), you can make a table to verify that each set is a group under the given operation.

6. For each of the following sets of variations (using the usual rules for combining variations) determine whether the set is a group. (Remember: to be a group, you need to have an identity, opposites, and closure.)
a) $\{T_0, R, T_4, T_4R, T_8, T_8R\}$
b) $\{T_0, I, T_4, T_4I\}$
c) $\{T_0, T_3, T_4, T_6, T_8, T_9\}$
d) $\{T_2, T_4, T_6, T_8, T_{10}\}$

Answers: (a) is a group; (b), (c) and (d) are not groups.

For (a), we'll use a table to show that all of the group criteria are satisfied:

a)

	T_0	R	T_4	T_4R	T_8	T_8R
T_0	T_0	R	T_4	T_4R	T_8	T_8R
R	R	T_0	T_4R	T_4	T_8R	T_8
T_4	T_4	T_4R	T_8	T_8R	T_0	R
T_4R	T_4R	T_4	T_8R	T_8	R	T_0
T_8	T_8	T_8R	T_0	R	T_4	T_4R
T_8R	T_8R	T_8	R	T_0	T_4R	T_4

Note that we have the identity (T_0 is an element of the set), closure (since every entry in the table was also in the original set), and opposites (since the identity, T_0, appears in each row).

b) $\{T_0, I, T_4, T_4I\}$

This is not a group because it is not closed. For example, $T_4T_4I = T_8I$, which is not in the set. (There are other examples we could use here, but one is sufficient.)

c) $\{T_0, T_3, T_4, T_6, T_8, T_9\}$

This is not a group because it is not closed. For example: T_3 and T_4 are both in the set; however, $T_3T_4 = T_7$ but T_7 is not in the set. (There are other examples we could use here, but one is sufficient.)

Comment: Note that it's not necessary to make a complete operation table (as we did in parts a and b) to show that a set under an operation is NOT a group; to invalidate one of the criteria for a group, all we need to do is find one single example to the contrary. (The point of making a complete table is that it's a way to prove that no such contrary examples exist.)

Comment: Notice that while we don't have "closure," this set does satisfy the other two criteria for a group – it has an identity, and every element of the group has an opposite in the set: T_3 and T_9 are opposites, T_4 and T_8 are opposites, T_6 is its own opposite, and T_0 is its own opposite.

d) $\{T_2, T_4, T_6, T_8, T_{10}\}$

We know that for any group of variations, the identity element will be T_0. Since T_0 is not included in this set, we can immediately determine that it is not a group (since it does not contain an identity element).