Proof of Chudnovsky Series for 1/π(PI)

In 1988 D. V. Chudnovsky and G. V. Chudnovsky (now famous as "Chudnovsky Brothers") established a general series for π by extending Ramanujan's ideas (presented in this series of posts). It can be however shown that their general series can be derived using Ramanujan's technique. Chudnovsky's approach has the advantage that using class field theory the algebraic nature of parameters in the general series can be determined and this greatly aids in the empirical evaluation of the parameters and thereby providing an actual series consisting of numbers.

In this post we shall not focus on the algebraic nature of parameters in the Chudnovsky's general series as it involves the deep and complicated class field theory. Rather we will show that the Chudnovsky series can be obtained via Ramanujan's technique and thereby furnish a proof of the Chudnovsky general series.

The Chudnovsky Series for $1/\pi$

We use the notation from the last two posts for Ramanujan's Lambert series:

$$\begin{aligned}P(q) &= 1 - 24\left(\frac{q}{1-q} + \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \cdots\right)\\ Q(q) &= 1 + 240\left(\frac{q}{1-q} + \frac{2^3q^2}{1-q^2} + \frac{3^3q^3}{1-q^3} + \cdots\right)\\ R(q) &= 1 - 504\left(\frac{q}{1-q} + \frac{2^5q^2}{1-q^2} + \frac{3^5q^3}{1-q^3} + \cdots\right)\end{aligned}$$

Let $n > 3$ be an integer and we write

$$\begin{aligned}P_n &= P(-e^{-\pi\sqrt{n}}),\ Q_n = Q(-e^{-\pi\sqrt{n}}),\ R_n = R(-e^{-\pi\sqrt{n}})\\ j_n &= 1728\frac{Q_n^3}{Q_n^3 - R_n^2}\\ b_n &= \sqrt{n(1728 - j_n)}\\ a_n &= \frac{b_n}{6}\left\{1 - \frac{Q_n}{R_n}\left(P_n - \frac{6}{\pi\sqrt{n}}\right)\right\}\end{aligned}$$

then Chudnovsky brothers establish the following general series for $1/\pi$:

$$\boxed{\frac{1}{\pi} = \frac{1}{\sqrt{-j_n}}\sum_{m=0}^{\infty}\frac{(6m)!}{(3m)!(m!)^3}\frac{a_n + mb_n}{j_n^m}} \tag{1}$$

We will show that this series is the same as series (21) in this post namely

$$\boxed{\frac{1}{\pi} = \sum_{m=0}^{\infty}\frac{(1/6)_m(5/6)_m(1/2)_m}{(m!)^3}(A + mB)J_n^m} \tag{2}$$

where

$$\begin{aligned}J_n &= \frac{-27G_n^{48}}{(G_n^{24}-4)^3}\\ A &= 2\sqrt{n}\cdot\sqrt{\frac{G_n^{24}-1}{(G_n^{24}-4)^3}}+\frac{G_n^{12}}{\sqrt{G_n^{24}-4}}\left\{\frac{\sqrt{n}}{3}(1-2k^2)-\frac{R_n(k,k')}{6}\right\}\\ B &= \sqrt{n}\sqrt{1-J_n}\end{aligned}$$

and this series is derived using Ramanujan's technique.

The verification that both the series (1) and (2) are same (but written in completely different notation) requires that we establish the following relations:

$$\frac{(6m)!}{(3m)!} = 12^{3m}(1/6)_m(5/6)_m(1/2)_m = 1728^m(1/6)_m(5/6)_m(1/2)_m \tag{3}$$

$$J_n = \frac{1728}{j_n} = \frac{Q_n^3-R_n^2}{Q_n^3} \tag{4}$$

$$A = \frac{a_n}{\sqrt{-j_n}} \tag{5}$$

$$B = \frac{b_n}{\sqrt{-j_n}} \tag{6}$$

The first identity regarding factorials is easy to establish once we observe that for each factor in denominator $(3m)!$ we have twice that number as a factor in numerator and thus by cancellation we get

$$\begin{aligned}\frac{(6m)!}{(3m)!} &= 2^{3m}1\cdot 3\cdot 5\cdot 7\cdots(6m-1)\\ &= 12^{3m}\cdot\frac{1}{6}\cdot\frac{3}{6}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots\cdot\frac{6m-1}{6}\\ &= 12^{3m}(1/6)_m(5/6)_m(1/2)_m = 1728^m(1/6)_m(5/6)_m(1/2)_m\end{aligned}$$

In what follows we will put $x = G_n^{24}$ whenever suitable to simplify the calculations (as well as typing). Using identities $(21),(22)$ from the last post we can see that

$$\begin{aligned}\frac{Q_n^3-R_n^2}{Q_n^3} &= \frac{(1-4G_n^{-24})^3-(1-2k^2)^2(1+8G_n^{-24})^2}{(1-4G_n^{-24})^3}\\ &= \frac{(1-4G_n^{-24})^3-(1-G_n^{-24})(1+8G_n^{-24})^2}{(1-4G_n^{-24})^3}\\ &= \frac{(G_n^{24}-4)^3-(G_n^{24}-1)(G_n^{24}+8)^2}{(G_n^{24}-4)^3}\\ &= \frac{(x-4)^3-(x-1)(x+8)^2}{(x-4)^3}\\ &= \frac{x^3-12x^2+48x-64-(x-1)(x^2+16x+64)}{(x-4)^3}\\ &= \frac{x^3-12x^2+48x-64-(x^3+15x^2+48x-64)}{(x-4)^3}\\ &= \frac{-27x^2}{(x-4)^3} = \frac{-27G_n^{48}}{(G_n^{24}-4)^3} = J_n\end{aligned}$$

so that relation (4) is established.

We next establish (6) as it is easier to handle compared to (5). We have then

$$\begin{aligned}\frac{b_n}{\sqrt{-j_n}} &= \sqrt{\frac{n(1728 - j_n)}{-j_n}}\\ &= \sqrt{n\left(1 - \frac{1728}{j_n}\right)} = \sqrt{n(1 - J_n)} = B\end{aligned}$$

To establish (5) we need to proceed in multiple steps. The idea is to express a_n, b_n in terms of G_n. We start with the simpler b_n and we have:

$$\begin{aligned}b_n &= \sqrt{n(1728 - j_n)}\\ &= \sqrt{1728n\left(1 - \frac{j_n}{1728}\right)}\\ &= \sqrt{1728n\left(1 - \frac{1}{J_n}\right)}\\ &= \sqrt{1728n\left(1 + \frac{(G_n^{24} - 4)^3}{27G_n^{48}}\right)}\\ &= \sqrt{1728n\left(1 + \frac{(x - 4)^3}{27x^2}\right)}\\ &= \sqrt{1728n\left(\frac{(x - 4)^3 + 27x^2}{27x^2}\right)}\\ &= \sqrt{1728n\left(\frac{(x - 1)(x + 8)^2}{27x^2}\right)}\\ &= \frac{8(x + 8)}{x}\sqrt{n(x - 1)}\\ &= \frac{8(G_n^{24} + 8)}{G_n^{24}}\sqrt{n(G_n^{24} - 1)}\end{aligned}$$

Next we need to handle P_n. Let $q = e^{-\pi\sqrt{n}}$ and then $P_n = P(-q)$. By equation (9) of [this post](#) (note that the definition of $P(q)$ given there is different from what is being used in this post and accordingly $P(q)$ of that post matches with $P(q^2)$ of the current post) we have:

$$\begin{aligned}P(q^2) &= \left(\frac{2K}{\pi}\right)^2 \frac{R_n(k, k')}{2\sqrt{n}} + \frac{3}{\pi\sqrt{n}}\\ &= \left(\frac{2K}{\pi}\right)^2 \left(\frac{R_n(k, k')}{2\sqrt{n}} + \frac{3\pi}{4K^2\sqrt{n}}\right)\end{aligned}$$

and from equation (14) of the [previous post](#) we have:

$$P(q^2) = \left(\frac{2K}{\pi}\right)^2 \left(\frac{3E}{K} + k^2 - 2\right)$$

so that

$$\begin{aligned}\frac{3E}{K}+k^{2}-2&=\frac{R_{n}(k,k')}{2\sqrt{n}}+\frac{3\pi}{4K^{2}\sqrt{n}}\\ \Rightarrow\frac{6E}{K}+2k^{2}-4&=\frac{R_{n}(k,k')}{\sqrt{n}}+\frac{3\pi}{2K^{2}\sqrt{n}}\\ \Rightarrow\frac{6E}{K}+4k^{2}-5&=\frac{R_{n}(k,k')}{\sqrt{n}}+\frac{3\pi}{2K^{2}\sqrt{n}}+2k^{2}-1\end{aligned}$$

and using equation (23) of last post we get:

$$\begin{aligned}P_{n}=P(-q)&=\left(\frac{2K}{\pi}\right)^{2}\left(\frac{R_{n}(k,k')}{\sqrt{n}}+\frac{3\pi}{2K^{2}\sqrt{n}}+2k^{2}-1\right)\\ &=\left(\frac{2K}{\pi}\right)^{2}\left(\frac{R_{n}(k,k')}{\sqrt{n}}+2k^{2}-1\right)+\frac{6}{\pi\sqrt{n}}\\ \Rightarrow P_{n}-\frac{6}{\pi\sqrt{n}}&=\left(\frac{2K}{\pi}\right)^{2}\left(\frac{R_{n}(k,k')}{\sqrt{n}}+2k^{2}-1\right)\end{aligned}$$

From equations $(21),(22)$ of last post we now have:

$$\begin{aligned}\frac{Q_{n}}{R_{n}}\left(P_{n}-\frac{6}{\pi\sqrt{n}}\right)&=\frac{(1-4G_{n}^{-24})\left(\frac{R_{n}(k,k')}{\sqrt{n}}+2k^{2}-1\right)}{(1-2k^{2})(1+8G_{n}^{-24})}\\ &=\frac{x-4}{(1-2k^{2})(x+8)}\frac{R_{n}(k,k')}{\sqrt{n}}-\frac{x-4}{x+8}\end{aligned}$$

and therefore

$$\begin{aligned}a_{n}&=\frac{b_{n}}{6}\left\{1-\frac{Q_{n}}{R_{n}}\left(P_{n}-\frac{6}{\pi\sqrt{n}}\right)\right\}\\ &=\frac{4(x+8)}{3x}\sqrt{n(x-1)}\left(\frac{2(x+2)}{x+8}-\frac{x-4}{(1-2k^{2})(x+8)}\frac{R_{n}(k,k')}{\sqrt{n}}\right)\\ &=\frac{8}{3x}(x+2)\sqrt{n(x-1)}-\frac{4}{3x}\frac{x-4}{1-2k^{2}}R_{n}(k,k')\sqrt{x-1}\\ &=\frac{8}{3x}(x+2)\sqrt{n(x-1)}-\frac{4}{3x}\frac{(x-4)\sqrt{x}}{\sqrt{x-1}}R_{n}(k,k')\sqrt{x-1}\\ &=\frac{8}{3x}(x+2)\sqrt{n(x-1)}-\frac{4(x-4)}{3\sqrt{x}}R_{n}(k,k')\end{aligned}$$

Note that in the above derivation we have used the fact that

$$1-2k^{2}=\sqrt{1-G_{n}^{-24}}=\sqrt{\frac{x-1}{x}}$$

and the same relation would be used in derivation which follows.

Finally we can see that

$$\begin{aligned}\frac{a_n}{\sqrt{-j_n}} &= \frac{a_n}{\sqrt{-1728/J_n}}\\ &= \frac{a_n x}{8\sqrt{(x-4)^3}}\\ &= \frac{x}{8\sqrt{(x-4)^3}}\left(\frac{8}{3x}(x+2)\sqrt{n(x-1)} - \frac{4(x-4)}{3\sqrt{x}}R_n(k,k')\right)\\ &= \frac{x+2}{3}\sqrt{\frac{n(x-1)}{(x-4)^3}} - \sqrt{\frac{x}{x-4}}\cdot\frac{R_n(k,k')}{6}\\ &= \frac{x-4+6}{3}\sqrt{\frac{n(x-1)}{(x-4)^3}} - \sqrt{\frac{x}{x-4}}\cdot\frac{R_n(k,k')}{6}\\ &= \frac{1}{3}\sqrt{\frac{n(x-1)}{x-4}} + 2\sqrt{n}\sqrt{\frac{x-1}{(x-4)^3}} - \sqrt{\frac{x}{x-4}}\cdot\frac{R_n(k,k')}{6}\\ &= \frac{\sqrt{n}}{3}\sqrt{\frac{x}{x-4}}\cdot(1-2k^2) + 2\sqrt{n}\sqrt{\frac{x-1}{(x-4)^3}} - \sqrt{\frac{x}{x-4}}\cdot\frac{R_n(k,k')}{6}\\ &= 2\sqrt{n}\sqrt{\frac{x-1}{(x-4)^3}} + \sqrt{\frac{x}{x-4}}\left(\frac{\sqrt{n}}{3}(1-2k^2) - \frac{R_n(k,k')}{6}\right)\\ &= 2\sqrt{n}\sqrt{\frac{G_n^{24}-1}{(G_n^{24}-4)^3}} + \frac{G_n^{12}}{\sqrt{G_n^{24}-4}}\left(\frac{\sqrt{n}}{3}(1-2k^2) - \frac{R_n(k,k')}{6}\right) = A\end{aligned}$$

and thus the relation (5) is established. This completes the proof of Chudnovsky's series (1).

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