Math 105 – Music & Mathematics

CYCLE NOTATION

First, a definition: given an ordered list of n distinct objects, a “permutation” is an action by which these objects are rearranged in a systematic way.

For example, the act of rearranging the list 1 2 3 4 5 into the list 2 4 5 1 3 is a permuation of five objects.

> (Note: sometimes the rearrangements themselves are referred to as “permutations” as well, but technically the word should refer to the action, not the result. It’s one of those words that is often used both as a noun and as a verb – for example, if you walk (verb) somewhere, one might say you are taking a walk (noun).)

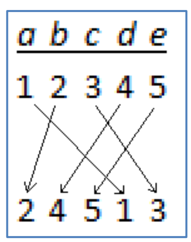
Any permutation can be described formally using “cycle notation.” This is a notation we use to keep track of how each entry of the list is being used, based on its initial and final position in the list. For consistency, we’ll use lower-case letters to designate relative positions: a for 1st position, b for 2nd, c for 3rd, etc.

We’ll introduce the notation by way of the preceding example, in which 1 2 3 4 5 is rearranged into 2 4 5 1 3. Let’s start by lining up these two lists, so it’s clear which entry is in which relative position before and after:

a	*b*	*c*	*d*	*e*
1	2	3	4	5
2	4	5	1	3

The movements are as follows:

- 1: moves from a → d
- 2: moves from b→ a
- 3: moves from c → e
- 4: moves from d → b
- 5: moves from e → c



The key to analyzing a permutation is to see which positions in the list are grouped together. For example, we see that we have the movements a→d, d→ b, and b→ a. This forms a *cycle* among these three positions in the list – that is, the entries that start out in positions a, b, and d will be rearranged among themselves, but will end up occupying (in a different order) these same three positions. We express this observation in cycle notation as the “3-cycle” (adb). This cycle is called a 3-cycle because it includes 3 positions, and also because it “resets” itself if it is repeated 3 times.

The other two positions, c and e, are swapped by this permutation – this is expressed by the “2-cycle” (ce).

So, our permutation can be described by the cycles $(adb)(ce)$. This is a complete description of how the permutation rearranges the objects in any list.

Let’s continue our example by again applying the permutation $(adb)(ce)$, this time to the list 2 4 5 1 3:

$\underline{a\ b\ c\ d\ e}$
2 4 5 1 3
_ _ _ _ _

To “fill in the blanks,” we track the movement of each number in our list by referring to the cycles that make up the permuation. We’ll start with the 3-cycle (adb). This cycle tells us to move the entries in the list, as before, so that the entry in position a is moved to position d, the entry in position d is moved to b, and the entry in position b is moved to a. Proceed one entry at a time

- $a \rightarrow d$: This time, 2 starts in position a, so 2 should end up in position d:

$\underline{a\ b\ c\ d\ e}$
2 4 5 1 3
_ _ _ 2 _

- $d \rightarrow b$: Next, note that 1 starts in position d, so 1 should end up in position b:

$\underline{a\ b\ c\ d\ e}$
2 4 5 1 3
_ 1 _ 2 _

- $b \rightarrow a$: Finally, note that 4 starts in position b, so 4 should end up in position a:

$\underline{a\ b\ c\ d\ e}$
2 4 5 1 3
4 1 _ 2 _

That takes care of the e 3-cycle (adb). If that were the entire permutation, then we’d just fill in the 5 and the 3 in positions c and e, respectively. However, remember that the permutation we’re applying right now is $(adb)(ce)$, so we’re not done yet! Note that the 2-cycle (ce) is just a “swap” – we trade the contents of these two positions. Since we start with 5 in position c and 3 in position e, the (ce) cycle tells us to just swap these two entries, leaving us with :

$\underline{a\ b\ c\ d\ e}$
2 4 5 1 3
4 1 3 2 5

Thereofre, applying $(adb)(ce)$ to the list 2 4 5 1 3 gives us the list 4 1 3 2 5.

A few more examples:

Example: Use cycle notation to describe the permutation that rearranges 1 2 3 4 5 6 into 6 4 2 5 3 1.

Answer: Let's look at the list before and after the permutation is applied, with each position labeled:

a b c d e f
1 2 3 4 5 6
6 4 2 5 3 1

Our cycles are as follows: $a \to f \to a$, or (af); and $b \to c \to e \to d \to b$, or $(bced)$.

So, the permutation shown here could be described in cycle notation as $(af)(bced)$.

Example: Apply the permutation $(ace)(bgdf)$ to the scale C D E F G A B.

Answer: (note – don't confuse capital letters used for notes with lower case letters used for positions!)

Start by noting the starting position of each note:

a b c d e f g
C D E F G A B
_ _ _ _ _ _ _

The cycle (ace) moves note C from $a \to c$, note E from $c \to e$, and note G from $e \to a$. This gives us

a b c d e f g
C D E F G A B
G _ C _ E _ _

Next, the cycle $(bgdf)$ moves note D from $b \to g$, note B from $g \to d$, note F from $d \to f$, and note A from $f \to b$. This gives us the result:

a b c d e f g
C D E F G A B
G A C B E F D

Therefore, the new melody created when we appliy $(ace)(bgdf)$ to the scale C D E F G A B is G A C B E F D.