Finding Limits with Mathematica

Defining Functions

To define a function in *Mathematica* you first type the name of the function followed by a [then the variable (like x) followed by the underscore (_) then] := and finally the function expression. For example,

```
f[x_] := 10 x - 1.86 x^2
```

Enter the above line into a *Mathematica* worksheet, then press Shift+Enter. (There is no output since you are defining a function rather than carrying out a computation.) Now you can evaluate this function just like any of *Mathematica's* built-in functions.

Try the following examples:

```
f[1]

f[2]

f[1 / 2]

f[Pi]
```

In fact you can even evaluate the function at a variable or an expression. For example,

```
f[2 x]

f[1 + h]

f[u - v]
```

You can also use the commands Plot and Solve in conjunction with your new function:

```
Plot[f[x], {x, 0, 5}]

Solve[f[x] == 0, x]
```

Here are a few other examples illustrating how to define a function:

```
g[x_] := Sqrt[x] Sin[x]

Plot[g[x], {x, 0, 12}]

h[x_] := Sin[x] / x

Plot[h[x], {x, -10, 10}]
```

Limits Using Tables

This is the numerical (rather than algebraic) approach to limits. Say we want to find the limit of the function h[x] (as defined above) as x approaches 0. (What does this limit appear to be according to the plot?) We might use x-values such as 0.1, 0.01, 0.001, 0.0001, then -0.1, -0.01, -0.001, -0.0001, to see what happens as x approaches x from above, then below.

Try it! Evaluate h[0.1], h[0.01], h[0.001], and h[0.0001], and see what happens...

```
h[0.1]
```

```
h[0.01]
```

```
h[0.001]
```

```
h[0.0001]
```

Notice the last two outputs. These are only approximations, not exact answers. These results are an indication that we should probably use more decimal places.

To set the number of output decimal places to something larger than 6, go to the *Mathematica* menu at the top of the screen and select Edit > Preferences... Click on the "Appearance" tab, then the "Numbers" tab, then the "Formatting" tab. Under "Displayed precision" change the number of digits to something larger, say 25. Then you can close this window by clicking the X in the upper right corner of the window.   Now, notice what happened to the numbers in your previous outputs.

Limits Using Graphs

You already know all of the commands you need here, just Plot. Say we wanted to find the limit of the following function as x approaches 0.

```
Plot[Sin[x] / Abs[x], {x, -2, 2}]
```

Clearly, the limit as x approaches 0 from the left is -1 and the limit as x approaches 0 from the right is 1 hence the two sided limit does not exist.

Algebraic Method

The syntax to have *Mathematica* calculate a limit algebraically is "Limit" followed by the function you are taking the limit of and the value which you want x to "approach." The way you tell *Mathematica* what x is approaching is with the arrow notation. Type - > (minus sign then greater than symbol) to enter the arrow; *Mathematica* will convert it automatically. For example:

```
Limit[Sin[x] / x, x → 0]
```

```
Limit[(Sqrt[x] - 2) / (x - 4), x → 4]
```

We can also find one sided limits using *Mathematica.* The notation for doing this is a little strange. To find a limit from the left we use the option Direction ⟶ 1 and to find the limit from the right we use the option Direction ⟶ -1. So to find the limit from the left we use

```
Limit[1 / x, x → 0, Direction → 1]
```

and to find the limit from the right we use

```
Limit[1 / x, x → 0, Direction → -1]
```

CAUTION! The standard "Limit" command actually gives you the right hand limit. So, if you're not sure (say from looking at a graph) that the left-hand and right-hand limits are equal, then you should check using the above "Direction" option.

For example, enter:

```
Limit[1 / x, x → 0]
```

This limit is technically undefined, since it's positive infinity from the right but negative infinity from the left. *Mathematica* returns positive infinity. By our definition of "limit," this is not correct. This isn't an error in *Mathematica*; just a cautionary comment to make sure you know what it's telling you with its "Limit" function!

■ Advanced Topic : Making Tables in *Mathematica*

When finding limits numerically, it is convenient to create a table of values, rather than entering one computation at a time. You will need to use two table commands together: "Table" and "TableForm." command. (The TableForm command makes the columns of the table.)

In the Table command you start with a list of two items: the first is the x coordinate, and the second is the y coordinate. In our example below, 4 + 0.1^p is the x coordinate; since our function is k,
k[4 + 0.1^p] is the corresponding y coordinate. The second list {p, 1, 10} tells *Mathematica* to let the value of p range between 1 and 10. Since p is in the expression for our x coordinate, the x values are going to be 4+0.1^1, 4+0.1^2, 4+0.1^3, ..., 4+0.1^10. That is, 4.1, 4.01, 4.001, ... , 4.0000000001. Our y values will then consist of the function, k, evaluated at each of these x values.

```
k[x_] := (Sqrt[x] - 2) / (x - 4)
```

```
TableForm[Table[{4 + 0.1^p, k[4 + 0.1^p]}, {p, 1, 10}]]
```

In this Table command our x values are 4 - 0.1^p so they are 3.9, 3.99, ..., 3.9999999999.

```
TableForm[Table[{4 - 0.1^p, k[4 - 0.1^p]}, {p, 1, 10}]]
```

In general if you want to create a sequence of numbers which approach a given number, say *a*, from the right, you can use *a* + 0.1^*p* (where p = 1, 2, 3, ...). To approach *a* from the left, we can use a similar strategy: *a* - 0.1^*p* (where p = 1, 2, 3, ...). This is not the only way to create such sequences of numbers, but it's probably the most convenient.

LAB ASSIGNMENT:
Due Friday, September 17 (printout)

1. Define the function f(x)=(5^x - 1)/x. Use *Mathematica* to find limits, as directed below, then answer each question by typing a sentence or two (or more) in response.
a) Use a graph to estimate the limit of f(x) as x approaches 0. What does the limit appear to be? Explain.
b) Use a table to estimate the limit of f(x) as x approaches 0; follow the example shown in the examples, where you make a table and let "p" (the exponent) run from 0 to 10. What does the lmiit appear to be? Explain.
c) Find the limit of f(x) as x approaches 0 algebraically.
d) What is the exact limit of f(x) as x approaches zero? Do all three approaches seem to be in agreement? (If not, why do you think that is?)

2. Define the function g(x) = (tan(x) - x)/x^3. Use *Mathematica* to find limits, as directed below, then answer each question by typing a sentence or two (or more) in response.
a) Use a graph to estimate the limit of g(x) as x approaches 0. What does the limit appear to be? Explain.
b) Use a table to estimate the limit of g(x) as x approaches 0; follow the example shown in the examples, where you make a table and let "p" (the exponent) run from 0 to 10. What does the lmiit appear to be? Explain.
c) Find the limit of g(x) as x approaches 0 algebraically.
d) What is the exact limit of g(x) as x approaches zero? Do all three approaches seem to agree? (If not, why do you think that is?)