

## Math 105, Spring 2017 – Test #1 (Tuning Systems) Review Sheet

This document summarizes the concepts you should understand, and some of the types of problems you should be able to solve, for the test that will be given on Friday, February 24.

**Pitch** (or “tone”) – a sound with a specific, sustained frequency of vibration, which is measured in “hertz” (Hz), which stands for vibrations per second. For example, if a tone has a frequency of 500 Hz, this means we are hearing a regular oscillation (back-and-forth change) in the ambient air pressure that cycles at a rate of 500 vibrations per second. The higher the frequency, the higher the pitch that is heard.

**Interval** – when two tones are sounded in unison, they create an “interval.” The sound of this interval depends mainly on the ratio (or quotient) between the frequencies of the two tones. This ratio is referred to as the interval’s **frequency ratio**. Generally speaking, if an interval’s frequency ratio is equal (or, at least, extremely close) to a fraction with a small denominator – such as  $3/2$ ,  $4/3$ ,  $5/3$ ,  $5/4$ , etc. – then the interval will tend to be pleasing to the ear. Roughly speaking, the greater the denominator of the frequency ratio (when written as a fraction), the worse the interval sounds.

### Frequency Ratios

In the table below, each interval (up to the octave) is defined in terms of its “ideal” frequency ratio (under just intonation), and also in terms of the number of semitones separating the top and bottom notes in the interval (on a standard keyboard with the 12-tone octave). Names and numbers you’ll be expected to know by memory are in **bold** type below – these are the **octave**, **perfect fifth**, **perfect fourth**, **major third**, and **major sixth**. The other interval names and ratios in the table below do *not* need to be memorized.

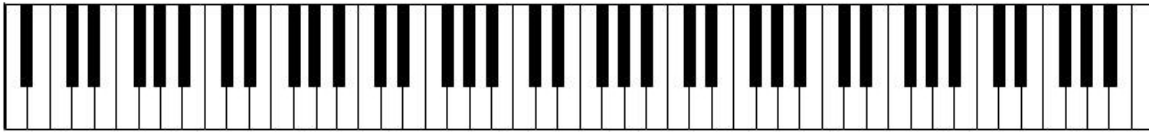
Interval name	Number of semitones	Just intonation frequency ratio
minor second	1	$16/15$
major second	2	$9/8$
minor third	3	$6/5$
<b>major third</b>	<b>4</b>	<b><math>5/4</math></b>
<b>perfect fourth</b>	<b>5</b>	<b><math>4/3</math></b>
diminished fifth	6	$45/32$
<b>perfect fifth</b>	<b>7</b>	<b><math>3/2</math></b>
minor sixth	8	$8/5$
<b>major sixth</b>	<b>9</b>	<b><math>5/3</math></b>
minor seventh	10	$9/5$
major seventh	11	$15/8$
<b>octave</b>	<b>12</b>	<b>2</b>

Reminder: To *raise* a pitch by one of these intervals (from a given base frequency), we *multiply* its frequency by the corresponding frequency ratio. To *lower* a pitch by one of these intervals, we *divide* its frequency by the corresponding frequency ratio.

## Tuning Systems - Different tunings of the keyboard:

The standard piano keyboard:

\* 88 keys total, starting from a low A, and including 7 full octaves from low C to high C.



- You are expected to know the layout of the keyboard, and how to identify which notes are A, B, C, etc. based on the pattern of white keys and black keys. Further, you should know the meaning of “sharp” – that is, it raises a note by one semitone (for example, A# is one semitone higher than A). For the time being, we simply refer to the black keys on the keyboard as “sharp” keys – starting from a C, they are (in order) C#, D#, F#, G#, A#.
- Be aware of the mathematical near-equivalence between 7 octaves and 12 fifths, and how those go together to motivate the design of the standard keyboard. As a reminder: starting from the lowest C, we can raise by 7 octaves – *or* by 12 fifths – to reach the highest C at the right end of the keyboard.
- Pythagorean tuning – based entirely on the octave and on preserving perfect fifths, starting from the “base note.” This tuning method starts at a base note, raises by six “fifths,” and (starting over again at the base note) lowers by five “fifths,” (also raising/lowering by octaves as necessary) to construct a twelve-tone scale. The standard pentatonic scale includes the base note, R1, R2, R3, and R4; the standard twelve-tone scale includes the base note, R1 through R6, and L1 through L5.
- Just intonation – rather than focusing on fifths exclusively, under just intonation we tune all of the “fundamental” intervals (major third, perfect fourth, perfect fifth, major sixth) relative to one specific “base” note, then fill in the other frequency ratios in the octave so that as many intervals as possible are given the correct ratios. (Note: there will *always* be some “broken” intervals, no matter what tuning system we use; fixing one will invariably break another.)
- Equal temperament – preserves the 2/1 ratio of the octave; but, rather than tuning any of the other intervals to their “ideal” frequency ratios, we instead specify that every semitone must have the same frequency ratio. For a 12-tone octave, this ratio is forced to be  $2^{1/12}$ , so that 12 semitones will give us a perfect octave. Under this intonation, several intervals turn out to have frequency ratios that are very close (but not equal) to their “ideal” frequency ratios – for example, the frequency ratio for a seven-semitone interval under equal temperament is  $2^{7/12} \approx 1.498$ , which is very close (but not equal) to the perfect fifth’s frequency ratio of 3/2.

For each of these tuning methods, you should know: how the tuning is carried out; what each method is designed to preserve (its strengths), and what it does not necessarily preserve (its weaknesses); and how to find the frequency of any note under any tuning method, given the frequency of a base note.

Important note: certain questions on the test will be written to assess your understanding of a specific tuning system. This will be clearly indicated in the instructions for these questions. If you answer a question about a tuning system by using the wrong system – for example, if you use just intonation frequency ratios to answer a question about equal temperament – then that question will be marked wrong, with little to no partial credit given.

## “Cents” measurement of intervals

Under equal temperament, we can measure intervals in terms of “cents.” A “cent” is 1/100 of a 12-TET semitone – that is, each semitone is equal to 100 cents, and so an octave is equal to 1200 cents. In order to measure non-equally tempered intervals (such as just intonation intervals) in terms of “cents,” logarithms must be used.

### Converting from cents to a frequency ratio:

If  $c$  = the width of an interval in cents, and  $r$  = that interval’s frequency ratio, you can solve for  $r$  by first converting the number of cents to semitones (divide by 100), then by using the 12-TET formula

$$r = 2^{n/12},$$

where  $n$  is the number of semitones. (We can streamline this a bit: if you replace  $n$  with  $c/100$ , which is the number of semitones, then we end up with the formula

$$r = 2^{c/1200}.$$

### Converting from a frequency ratio to cents:

If  $r$  = the frequency ratio of the interval to be measured, and  $c$  = the width of that interval in “cents,” then we need to solve the above equation,  $r = 2^{c/1200}$ , for  $c$ . To solve this, we use logarithms:

\* First, solve for the exponent:  $\frac{c}{1200} = \frac{\log(r)}{\log(2)}$  (using the “log” function key on your calculator);

\* Then, multiply by 1200 to solve for  $c$ . Round  $c$  to the nearest whole number. The resulting formula for converting from a frequency ratio for cents is

$$c = 1200 \cdot \frac{\log(r)}{\log(2)}$$

Note: You will be expected to know (or be able to figure out) the formulas necessary to solve problems involving conversion to and from cents measurement.

## **Rational Approximation** (also referred to as “continued fractions”)

Used to find fractions (rational numbers) with relatively small denominators which are “very close” to given irrational numbers (e.g. square roots, logarithms, fractional powers of 2, etc). The procedure is discussed in detail in the “Rational Approximations” handout on the class notes page.

Final thoughts:

- Prepare for this test by working through all of the sets practice problems, or at least as many as time will allow. The way to study for a math test is by working on math problems – that’s as true for this math class as for any other.
- This document is meant to *summarize* what we’ve covered to this point. While it is hopefully pretty thorough, I do not guarantee that it is exhaustive. In general, anything we covered in class, or anything that was covered in the assigned reading, is fair game for the test.
- On the day of the test, you may (and, in fact, *must*) use a calculator. No notes, books, or other references will be allowed. Further, your calculator may not be a device capable of receiving information from external sources – in particular, **you may not use your phone as your calculator.**

Reminder: You **MUST** make sure that YOU know how to carry out on YOUR calculator the types of calculations we’ve done in class and which you had to perform on the practice exercises. It will help if you do the practice exercises using the same calculator that you plan to bring along to the test!