MATH 105: Counting Problems

Note:
P(n,k) denotes the number of ways to select a permutation of k elements from a set of n elements.
C(n,k) denotes the number of ways to select a combination of k elements from a set of n elements.

<u>Practice Exercises</u>

1. Evaluate each of the following: P(8,4), P(9,5), P(12,4)
Answers:

$$P(8,4) = 8 \times 7 \times 6 \times 5 = 1680$$
$$P(9,5) = 9 \times 8 \times 7 \times 6 \times 5 = 15120$$
$$P(12,4) = 12 \times 11 \times 10 \times 9 = 11880$$

2. How many different 5-note melodies are possible, if each note can be any of the 12 notes from the standard 12-tone scale, and no note may be repeated?
Answer: A melody with no repeated notes is a permutation, so this is just

$$P(12,5) = 12 \times 11 \times 10 \times 9 \times 8 = 95{,}040.$$

Note: We don't necessarily need to think or write "P(12,5)" to solve this problem. Instead, just use the multiplication principle: since no repetition is allowed, we can just select one note at a time... since there are 12 ways to select the first note, then 11 ways to select the second note (no matter what the first note was), then 10 ways to select the third note (no matter what the first two notes were), and so on, we end up with the result $12 \times 11 \times 10 \times 9 \times 8$, as shown above.

3. How does the answer to #2 change if notes may be repeated?
Answer: If unlimited repetition of notes is allowed, then there are 12 ways to select each note in the melody. Therefore, the number of ways to select the entire melody would be $12 \times 12 \times 12 \times 12 \times 12$ (or 12^5), which is equal to 248,832.

Comments: Compare the last two answers. With unlimited repetition of notes, there are 248,832 ways to select a five-note melody from the twelve-tone scale. If no repetition is allowed, there are "only" 95,040 melodies possible. This is a pretty big difference – 248,832 is about 2.6 times as big as (or "1.6 times bigger than") 95,040.

Another fact we can infer from these results is there are 153,872 (that's 248,832 minus 95,040) melodies with at least one repeated note.

4. How many ways are there to rearrange the letters of the word "MUSICAL"?
Answer: Since "musical" contains seven distinct letters, we can just use the factorial rule: $7! = 5040$.
(Another way to describe this: we're counting ordered selections, without any repetition, of seven letters from a set of seven letters; these are permutations, so we're looking for $P(7,7)$. This is the same thing as 7!; in general, $P(n,n) = n!$.)

5. Evaluate each of the following: C(8,4), C(9,5), C(12,4)
Answers:

$$C(8,4) = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = \frac{\overset{2}{\cancel{8}} \times 7 \times \overset{1}{\cancel{6}} \times 5}{\underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} = 2 \times 7 \times 5 = 70$$

$$C(9,5) = \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2 \times 1} = \frac{9 \times \overset{2}{\cancel{8}} \times 7 \times \overset{1}{\cancel{6}} \times \overset{1}{\cancel{5}}}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} = 9 \times 2 \times 7 = 126$$

$$C(12,4) = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = \frac{\overset{1}{\cancel{12}} \times 11 \times \overset{5}{\cancel{10}} \times 9}{\underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} = 11 \times 5 \times 9 = 495$$

6. How many 5-note chords are possible, if we're selecting from the 12-tone scale?
Answer: A "chord" is a combination – that is, an unordered selection without repetition – so we count these using the combinations formula:

$$C(12,5) = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = \frac{\overset{1}{\cancel{12}} \times 11 \times \overset{1}{\cancel{10}} \times 9 \times 8}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} = 11 \times 9 \times 8 = 792$$

7. a) The word "committee" implies that the order of selection is not relevant – i.e., all members of the committee are just members, with no notion of "first," "second," etc. Thus, we're selecting a combination of 4 students from 14 students; there are

$$C(14,4) = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = \mathbf{1001} \text{ different ways to make this selection.}$$

b) The difference between (a) and (b) is that the designation of "officers" implies that the order of selection matters – that is, it matters which student is "president," which is "vice president," etc. Thus, this is an ordered selection. We're given that no repetition is allowed; thus, this is a permutation (rather than a combination, as in part (a)). The number of permutations of 4 students selected from 14 students is

$$P(14,4) = 14 \times 13 \times 12 \times 11 = \mathbf{24,024} \text{ ways to select four officers.}$$

c) As in part (a), "committee" implies an unordered selection without repetition – that is, a combination. In this case, though, we're really choosing two combinations: 2 male students from 5, and 2 female students from 9. The number of ways to make each of these selections, respectively, is $C(5,2) = 10$, and $C(9,2) = 36$. If our selection process is to first choose the males (10 options) and to then choose the females (36 options), then we see that there are

$$C(5,2) \times C(9,2) = 10 \times 36 = \mathbf{360} \text{ ways to select the committee.}$$

8. Since there's some limited repetition allowed, this is not a basic combination or permutation counting problem. So, we need to formulate some sort of "selection process" to methodically count the number of ways to create such a melody. We'll show one valid selection process below (there are others, which, if done correctly, will give us the same result).

Selection Process:

First, decide which four (out of nine) notes will be D's. This is an unordered selection without repetition, so there are $C(9,4) = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$ options.

Next, decide which two (out of the remaining five) notes will be F#'s. This is an unordered selection without repetition, so there are $C(5,2) = \frac{5 \times 4}{2 \times 1} = 10$ options.

Now, there are three notes left – these can be selected in order from among the other 10 notes ({C, C#, D#, E, F, G, G#, A, A#, B}) of the 12-tone scale. This will be an ordered selection without repetition, so there are $P(10,3) = 10 \times 9 \times 8 = 720$ options.

Total: There are $126 \times 10 \times 720 = 907{,}200$ different melodies of this type.

9. How many 7-note melodies are there with three C's, one D, one E, one F and one G?

(e.g., CCCDEFG, FGDECCC, CGDFCEC, FCCDECG, etc… there are a lot more!)

Answer: Since there's some limited repetition allowed, this is not a basic combination or permutation counting problem. So, we need to formulate some sort of "selection process" to methodically count the number of ways to create such a melody. We'll demonstrate two valid processes below.

Solution #1:
Step 1: Decide which three notes (out of the seven notes of the melody) will be C's. This is an unordered selection without replacement, so there are $C(7,3) = \frac{7\times6\times5}{3\times2\times1} = 7 \times 5 =$ 35 options.
Step 2: Decide which of the remaining notes will be a D (4 options)
Step 3: Decide which of the remaining notes will be an E (3 options)
Step 4: Decide which of the remaining notes will be an F (2 options)
… at which point, the one remaining note will be a G.

Total: $35 \times 4 \times 3 \times 2 = 840$ such melodies.

(Comment: Steps 2, 3, and 4 can be combined, if we make the observation that, after placing the three C's, what remains is just placing notes D, E, F, G in four available spots, with no repetition. This is just a permutation of these four notes, and we know there are 4! = 24 ways to do that. This is another way to end up with $35 \times 24 = 840$ melodies.)

Solution #2:
Step 1: First decide which note (out of the seven notes of the melody) will be a D. There are seven notes to choose from, so there are 7 options.
(Note that we're not placing the C's first this time; in fact, this time we'll save those until last!)
Step 2: Decide which of the remaining six notes will be an E (6 options)
Step 3: Decide which of the remaining five notes will be an F (5 options)
Step 4: Decide which of the remaining four notes will be a G (4 options)
…at this point, only three notes are left, all of which will have to be C's. So, we're done!

Total: $7 \times 6 \times 5 \times 4 = 840$ such melodies.

10. (counting rearrangements)
MELODY: consists of 6 distinct letters, so the answer is 6!, or 720 rearrangements of “melody.”

RHYTHM: consists of 6 letters, one of which is repeated twice. So, we first place the two H’s; there are $C(6,2) = \frac{6\times5}{2\times1} = 15$ ways to do this. Thereafter, since the other four letters (R, Y, T, M) are distinct, there are 4! = 24 ways to place those four letters. Thus, there are $15 \times 24 = 360$ different rearrangements of the word “rhythm.”

(Alternate solution – different from the method discussed in class, but also valid: First write “RHYTHM” as “RHYThM” – that is, temporarily treat the two H’s as distinct symbols – to get 6! = 720 rearrangements. Then observe that each actually rearrangement of “RHYTHM” is being counted twice – for example, the single rearrangement “HMHTRY” is being counted as both “HMhTRY” and “ hMHTRY” – so we need to divide by two to offset this double-counting. In this way, we also get the answer of $720 \div 2 = 360$ rearrangements. You don’t necessarily need to know or use this alternate method, but it’s always interesting to observe when there are multiple ways to solve a problem.)

AARDVARK: First note that there are 3A’s, two R’s, and three other distinct letters (D, V, K). So, we may proceed as follows: First, decide where to put the three A’s – since there are eight locations to choose from, there are $C(8,3) = \frac{8\times7\times6}{3\times2\times1} = 56$ ways to choose. After that there are five locations remaining; we’ll next decide which two of those will be R’s. There are $C(5,2) = 10$ ways to make this choice. After that, there are three distinct letters remaining, which can be placed in 3! = 6 ways. Therefore, there are $56 \times 10 \times 6 = 3360$ rearrangements of “aardvark.”

TWITTER: This uses essentially the same selection process as #9 above – replace the notes C, D, E, F, G with the letters T, W, I, E, R, respectively. In each case, one note/letter is repeated three times, and each of the other four notes/letters is included once each. (Compare this problem to #9 to make sure you can see the connection.) Since the setup will be essentially the same, the answer will come out the same: there are 840 ways to rearrange “twitter.”

MISSISSIPPI: Note there are four S’s, four I’s, two P’s and one M. Therefore, by similar reasoning to the previous examples, there will be

$$\begin{aligned} &C(11,4) \times C(7,4) \times \ C(3,2) = \frac{11\times10\times9\times8}{4\times3\times2\times1} \times \frac{7\times6\times5\times4}{4\times3\times2\times1} \times \frac{3\times2}{2\times1} \\ &= (11\times10\times3)\times(7\times5)\times3 \\ &= 330 \times 35 \times 3 \\ &= 34{,}650 \text{ rearrangements of "Mississippi."} \end{aligned}$$

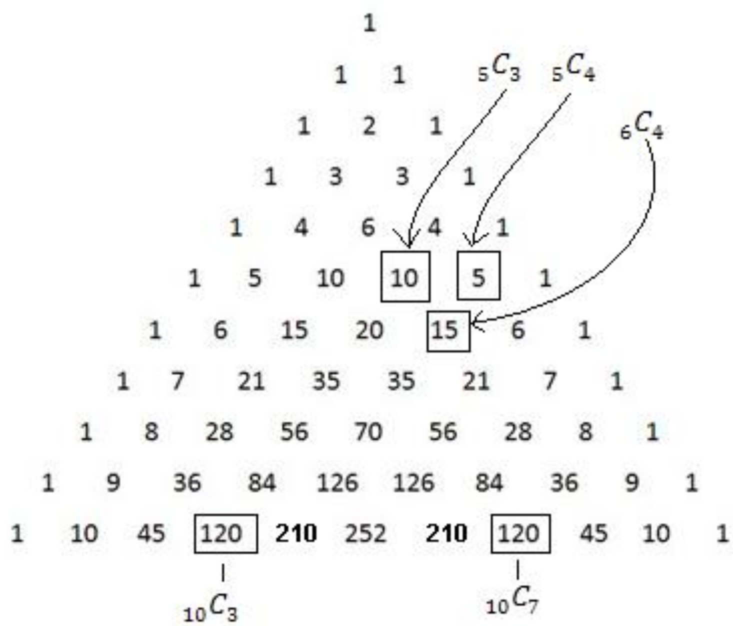
11. The first ten rows of Pascal's triangle are shown in the diagram below. (Find the 11th and 12th on your own!)

In the diagram, $C(n,k)$ is found by identifying the k^{th} entry of the n^{th} row of Pascal's triangle. Remember to start counting from 0, not from 1 – for example, to find $C(10,3)$, go to the 10th row (which starts with 1, then 10, then…), and count across to the third entry – counting the leading 1 as the "zeroth" entry, then 10 as the 1st entry, 45 as the 2nd entry, and finally 120 as the 3rd entry in that row. Thus, $C(10,3) = 120$. To check this, we'd use the combinations formula:

$$C(10,3) = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 5 \times 3 \times 8 = 120.$$

Make sure you can confirm the other answers using both Pascal's triangle and the combinations formula.

(Note: in the diagram below, $C(n,k)$ is being displayed as $_nC_k$. For example, "$_5C_3$" means $C(5,3)$. Same meaning, just different notation.)



Card problems:

The following few problems refer to the standard 52-card deck, as described in class. Recall that a "hand" is an unordered selection of cards, without replacement (i.e. a combination).

Hands discussed in class include: four-of-a-kind, full house, flush, ...

12. How many different ways are there to select a 5-card poker hand?
Answer:

$$C(52,5) = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960$$

13. How many ways are there to select a "suits full house," where three cards are all of one suit, and the other two cards share some other suit. (e.g., three clubs and two hearts would be a "suits full house.")

Solution: We need to devise a selection process for this.

Step 1: Select a suit from which we'll select three cards: <u>4 options</u>

Step 2: Select three cards from this suit. This selection is a combination of three cards from the thirteen cards in the suit; the number of ways to make such a selection is $C(13,3) = \frac{13\times12\times11}{3\times2\times1} = 13 \times 2 \times 11 =$ <u>286 options</u>

Step 3: Select a suit from which we'll select the other two cards: <u>3 options</u> (since we can't reuse the suit from step 1)

Step 4: Select two cards from this suit. This is a combination of two cards from the thirteen cards in the suit; the number of ways to make this selection is $C(13,2) = \frac{13\times12}{2\times1} = 13 \times 6 =$ <u>78 options</u>.

Therefore, by the multiplication principle for counting, there are $4 \times 286 \times 3 \times 78 =$ <u>267,696 ways</u> to select a "suits full house" from a standard deck of cards.

14. How many ways are there to select a three-card hand that consists entirely of face cards (jack, queen, king)?
Answer: We're looking for the number of ways to select three cards from the deck, but in such a way that all the cards are face cards. That means we're not selecting from the entire 52-card deck, but only from the 12 face cards in the deck. In other words, we need to find the number of ways to select a combination of 3 cards from a set of 12 cards. The number of ways to do this is given by $C(12,3) = \frac{12\times11\times10}{3\times2\times1} = 2 \times 11 \times 10 =$ <u>220 ways</u> to select a three-card hand that consists entirely of face cards.

15. How many ways are there to select a four-card hand which consists of only numbered cards (2, 3, 4, ..., 10)?
Answer: Note that there are nine numbered cards per deck, meaning there are $4 \times 9 = 36$ numbered cards in the deck. So, we're looking for the number of ways to select a combination of 4 cards from a set of 36 cards. The number of ways to do this is $C(36,4) = \frac{36\times35\times34\times33}{4\times3\times2\times1} = 3 \times 35 \times 17 \times 33 =$ <u>58,905 ways</u> to select a four-card hand that consists of only numbered cards.

16.How many ways are there to select "three of a kind" – that is, three cards all of one rank, and two other cards, each of different ranks? (e.g., Three kings, a jack, and a ten would count; three jacks and two tens would not count, since that's a "full house," as described in class)

Answer: 54,912.

(Note: This is a fairly difficult counting problem! Math majors often have trouble with this one.)

One valid selection process (which counts each 3-of-a-kind hand, exactly once each) is as follows:

Step 1: Select the three-of-a-kind rank: 13 options.
Step 2: Select 3 of the 4 cards of this rank for your hand: $C(4,3) = \frac{4\times3}{2\times1} =$ 4 options.
Step 3: Select the ranks for the other 2 cards from the 12 remaining ranks in the deck. (It's 12 remaining ranks since you can't re-use the one from Step 1.) This selection is unordered, since the two cards remaining to be selected are unordered – there's no "first card" or "second card." So, this is a combination of 2 ranks from the remaining 12. This gives us $C(12,2) = \frac{12\times11}{2\times1} =$ 66 options.

Step 4: Select one card from each of these ranks. You're selecting one out of four cards in each case, so there are $4 \times 4 =$ 16 options for this step.

Total: By the Multiplication Principle, there are $13 \times 4 \times 66 \times 16 = 54{,}912$ ways to select a three-of-a-kind hand.