The basic importance of children's reasoning about mathematics

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Reasoning

- My aim is to talk about the role of mathematical reasoning in learning mathematics
- I will start with the distinction between (a) mathematical tools & procedures and (b) mathematical reasoning
- ***Mathematical tools and procedures*** include counting and measuring systems: they are invented and often differ between cultures.
- ***Mathematical reasoning*** is about quantitative relations which are universal and underlie all mathematical problems: such as invariance, cardinality, transitivity, and inversion.

Gerard Vergnaud’s two questions

- In 1982 Vergnaud, one-time colleague of Piaget’s, asked 6-year-old children two arithmetical questions which required the same solution (7+3):
  - *Pierre had 7 marbles. He played a game and won 3 marbles. How many marbles did he have after the game?*
  - *Bertrand played a game of marbles and lost 7 marbles. After the game he had 3 marbles left. How many marbles did he have before the game?*
- Nearly all the children answered the 1st question correctly: only 26% got the 2nd one right.
- The reason for this huge difference is that the 1st story is about addition which is the correct operation whereas the 2nd is not.

Gerard Vergnaud’s two questions (ctd)

- To sort out the 2nd question, the child has to work out the quantitative relations involved.
- This must depend on her understanding additive composition and also the inverse relation between addition and subtraction.
- Thus, you can design problems that measure children’s ability to reason about quantitative relations (***reasoning items***) as well as problems that measure how accurately they calculate (***procedural items***).
- In reasoning problems the child usually has to work out what is the appropriate sum to do: in procedural problems the sum is clear, and the issue is how well the child carries it out.

Theoretical background

- The central claim in Piaget's theory of the central importance of reasoning about quantitative relations.
- His evidence for this claim mainly took the form of demonstrations that young children often do not understand basic logical principles, like transitivity,
- Subsequently many people produced evidence that young children can reason successfully about, for example, correspondence, invariance & transitivity,
- But this later evidence does not destroy the claim that reasoning plays a central part in children's mathematics
- And it leaves open the question of ***individual differences*** in mathematical reasoning

Explaining individual differences in mathematical achievement

- There are large individual differences in children's mathematical achievement and these appear quite early on in school learning.
- Three types of explanation have been identified in previous attempts to explain these differences

➊Early number skills

➋Mathematical reasoning

➌Cognitive functioning (specifically working memory-WM)

1. Number sense and computation

- Several psychologists (Dehaene, Gelman) have claimed an innate understanding of number.
- This number sense, they argue, leads to successful computation and is the basis for all mathematical learning

Measures of number sense and computation

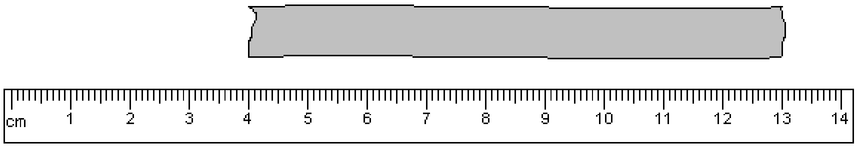
- An appropriate way to measure number sense in school children is to ask them to perform arithmetical computations; e.g.
  - *A shop had 25 milk cartons and sold 14. How many were left?*
  - *At £8 each, how much will 3 T-shirts cost?*
- In these problems, which are from the WISC Arithmetic subtest, the computations that have to be done are **transparent & obvious**. The problems test how **well** the child can do these **computations**

2. Mathematical Reasoning

- Another hypothesis is that the crucial factor in how well children learn mathematics is their ability to reason about quantitative relations.
- These relations include the additive composition of number and the inverse relation between addition & subtraction/multiplication & division.
- The solution to many arithmetical problems depends on reasoning about such relations.
- For example the solution to the problem *I started with 4 marbles and Tom gave me some more. So now I had 10. How many did Tom give me?* depends on reasoning about additive composition and on understanding inversion.

Measures of mathematical reasoning

- An appropriate way to measure children's mathematical reasoning is to give them arithmetical problems for which the correct computation is very easy, but the kind of computation that is needed is not at all transparent or obvious
- A reasoning test consisting entirely of items of this type was devised by Nunes & Bryant as part of a large scale longitudinal project (ALSPAC)

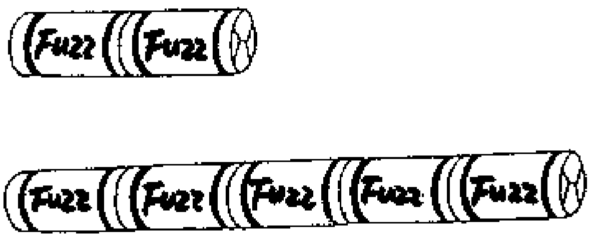


Answer ☐ cm

46% correct – 8y 9m

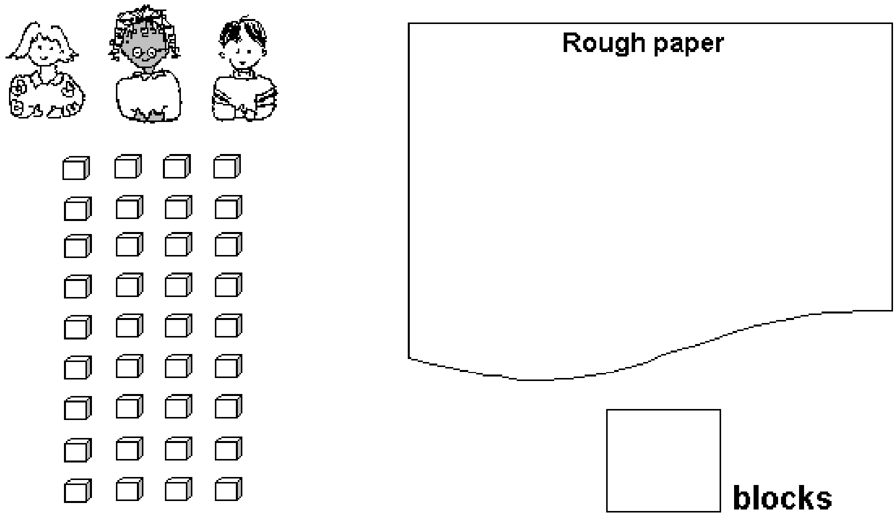
76% correct – 10y 7m

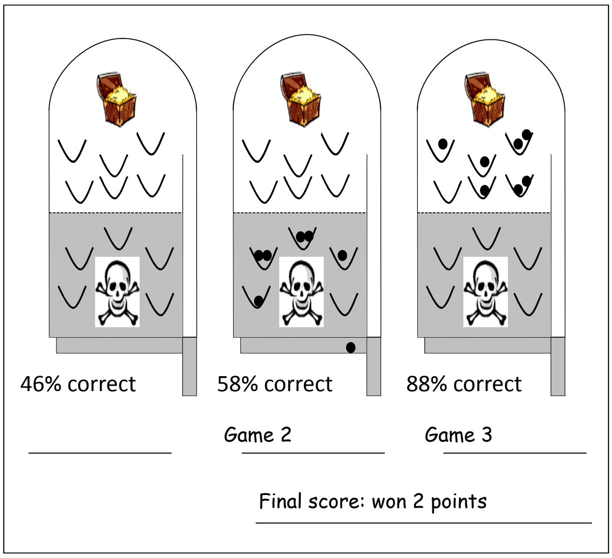
8



☐ sweets

8y 9m - 67% correct





Andrew played 3 games and can't remember his score for game 1.

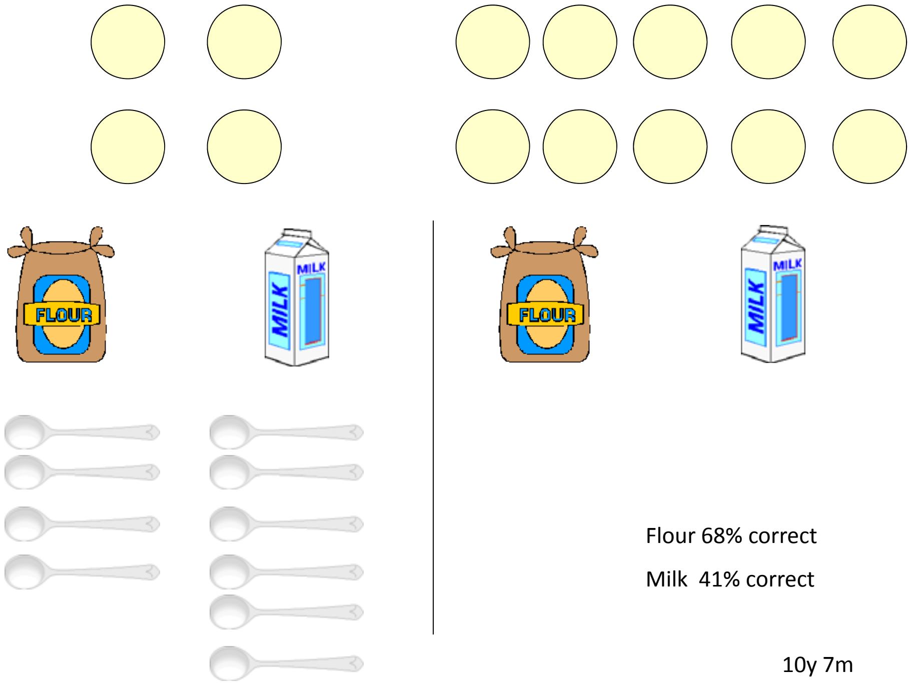
What was his score for Game 2?

What was his score for Game 3?

His final score counting all three games was a winning score of 2 points.

What happened in the first game?

10y 7m



Flour 68% correct

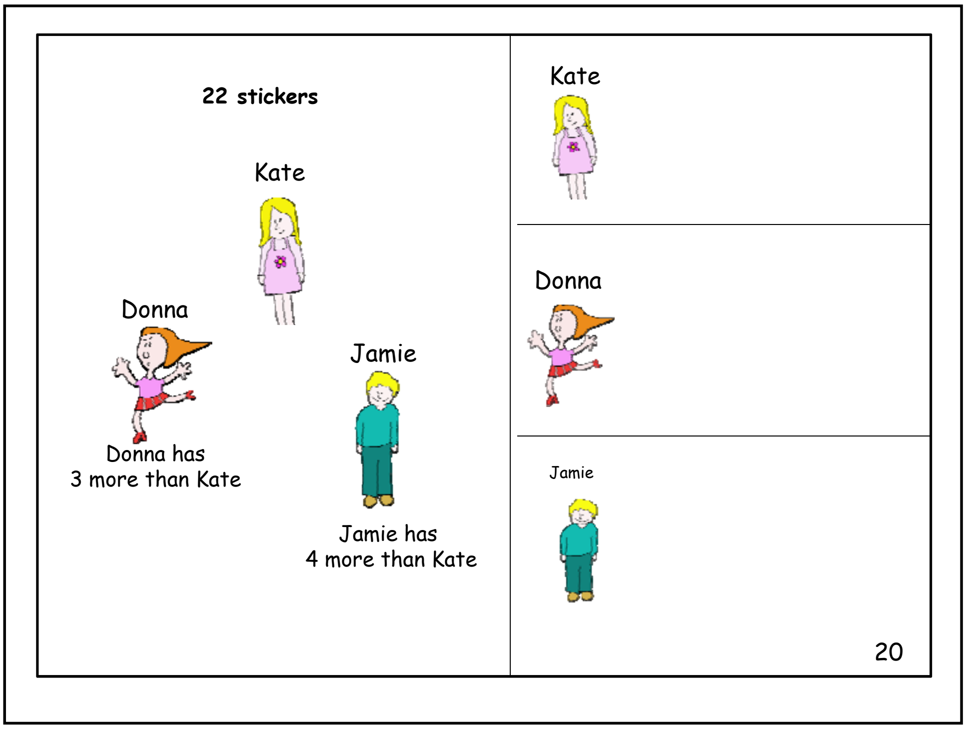
Milk 41% correct

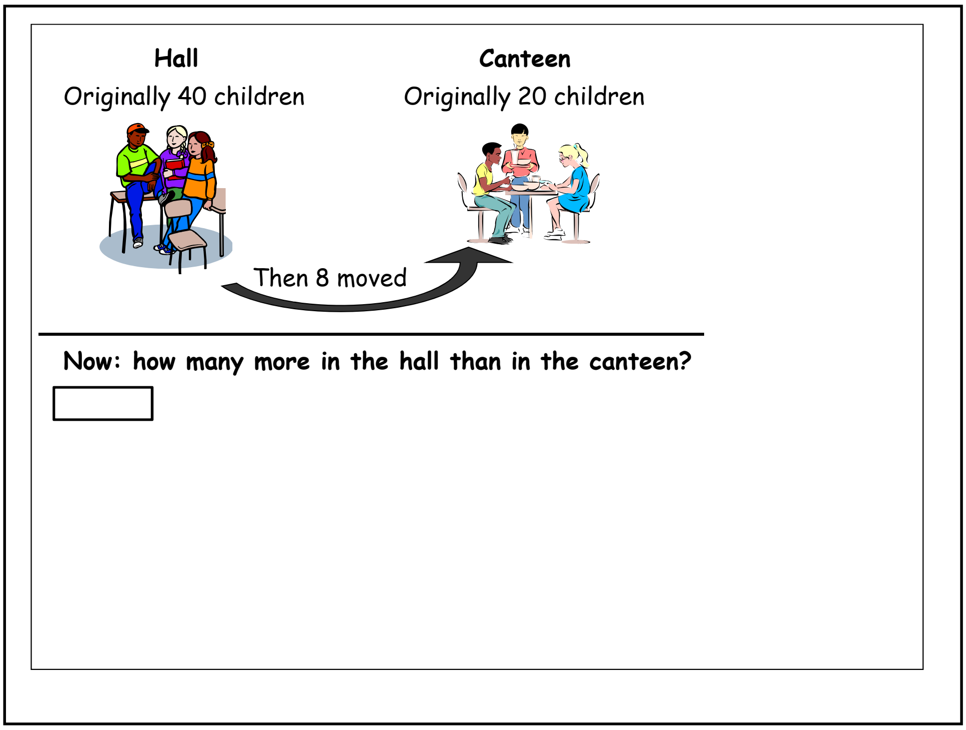
10y 7m

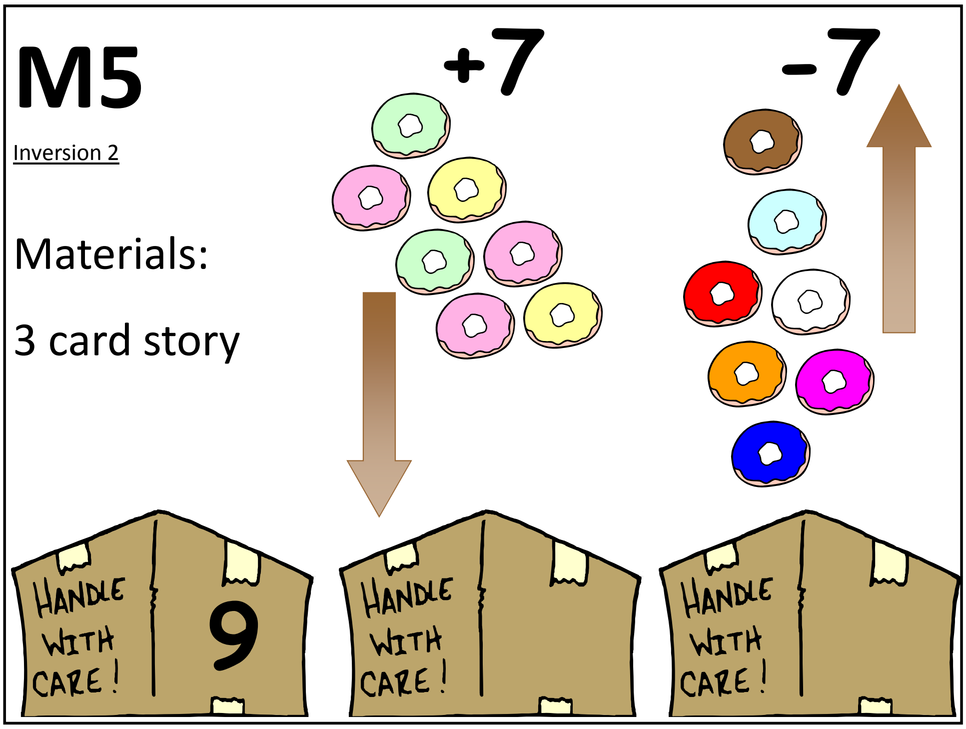
Will the colour of the mixed paint look the same on Monday and Tuesday? Circle yes or no.

70% correct

10y 7m


22 stickers
Kate
Donna
Donna has
3 more than Kate
Jamie
Jamie has
4 more than Kate
Kate
Donna
Jamie
20


Hall
Originally 40 children
Canteen
Originally 20 children
Then 8 moved
Now: how many more in the hall than in the canteen?



3. Cognitive functioning: specifically working memory (WM)

- A third view is that a crucial factor in children's learning about mathematics is their ability to process information (Hitch, Siegel).
- The most coherent form of this hypothesis focuses on children's working memory, which means the ability to store and re-organise information.
- One of the main measures of WM is the backward digit span (WISC)

The need to test all three hypotheses in the same study

- These three hypotheses are very different but they do not necessarily conflict. All three could be right.
- There is some positive evidence for all three hypotheses. However, no single longitudinal study has included measures of all three.
- So, we have no comparison of the power of the three factors or of any interaction between them.
- Recently we were able to make this kind of comparison in a large scale epidemiological study called the Avon Longitudinal Study of Parents and Children (ALSPAC)

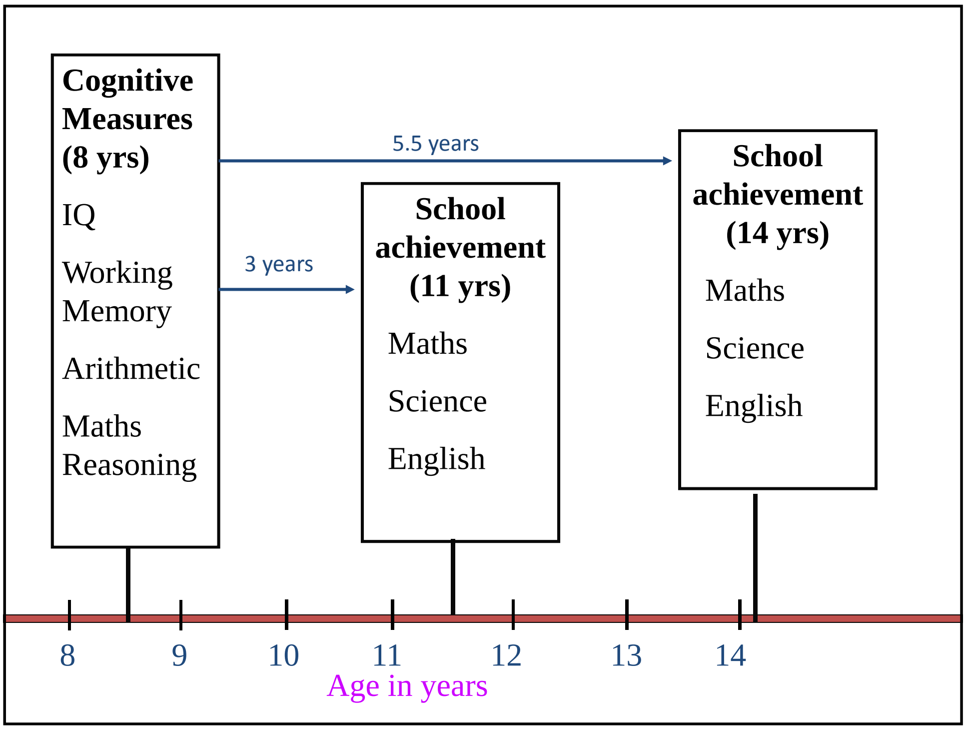
The Avon Longitudinal Study of Parents and Children (*ALSPAC*) : The Predictors

- This is a study of 14,000 children born in the Avon area in West England between April 1991 & Dec 1992
- The study involves medical, sociological, educational and psychological data.
- The data include:

1. a test of working memory at 8-years (WISC backward digit span)
2. a test of children’s knowledge of arithmetical procedures at 8 years in which the necessary calculation was obvious (WISC arithmetic)
3. a test of mathematical reasoning at 8yrs designed by Nunes and myself (additive & multiplicative reasoning) in which the calculations were easy and the main demand was to work out what kind of calculation was needed

The Avon Longitudinal Study of Parents and Children (***ALSPAC):*** The Outcome Measures

- The study included data from the standardised national tests of children's educational achievement given in schools across the country
- This gave us measures of the children's school achievement in Mathematics, Science and English at 11- and 14-years



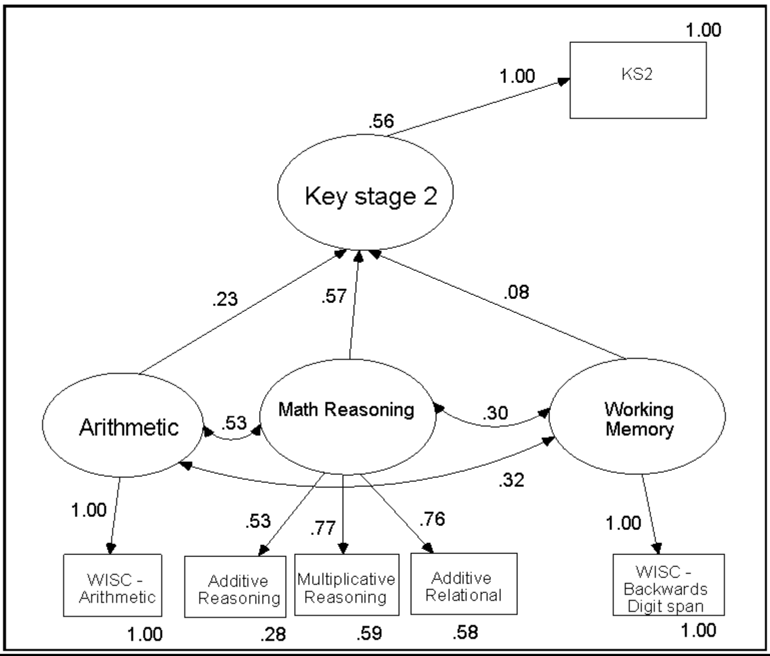
The method of analysis

- We ran Multiple Regression analyses with fixed steps to find out whether each factor contributed independently to mathematics achievement at 11- and 14-years
- We used structural equation models to test the relative importance of these contributions

How well **school maths achievement at 11yrs** was predicted by measures taken 3 years before (N=2413)

Step	β	Extra variance
1st: Age	.12	3%
2nd IQ	.32	37%
3rd Working Memory	.09	3%
4th Maths Reasoning	.35	12%
5th Arithmetic	.21	3%
		Total – 58%

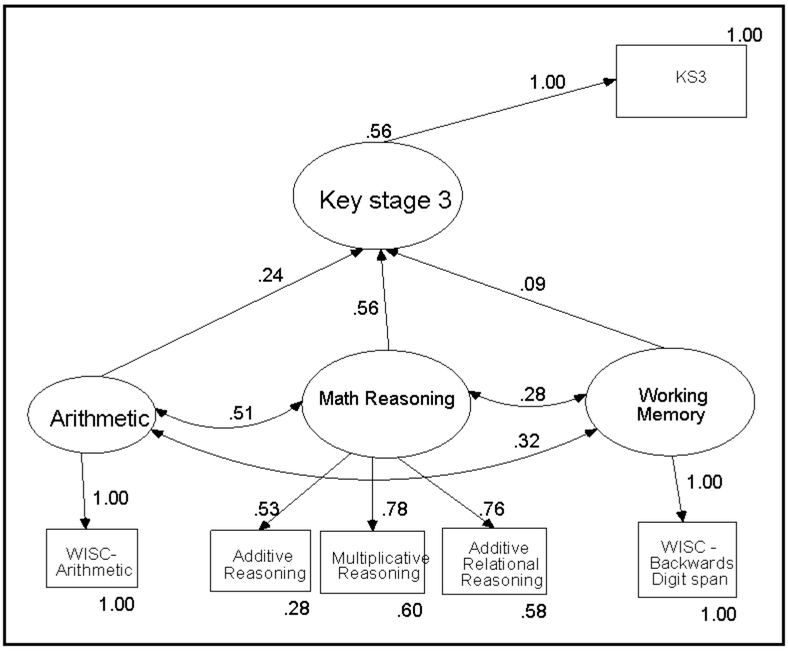
Predicting mathematical achievement at 11-years (KS2)



How well **school maths achievement at 14yrs** was predicted by measures taken 5.5 years before (N=1595)

Step	β	Extra variance
1st: Age	.09	1%
2nd IQ	.40	46%
3rd Working Memory	.06	2%
4th Maths Reasoning	.34	11%
5th Arithmetic	.18	2%
		Total - 62%

Predicting mathematical achievement at 14-years

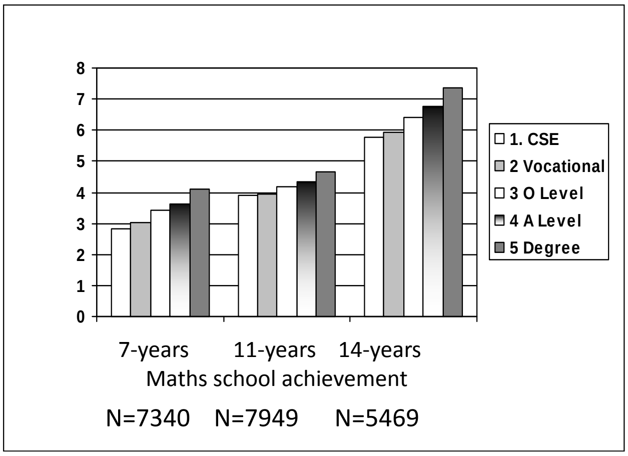


The **mathematical reasoning task** taken at 8 yrs predicted the children's achievement in Mathematics better than in Science and English at 11yrs and 14 yrs: the β co-efficients

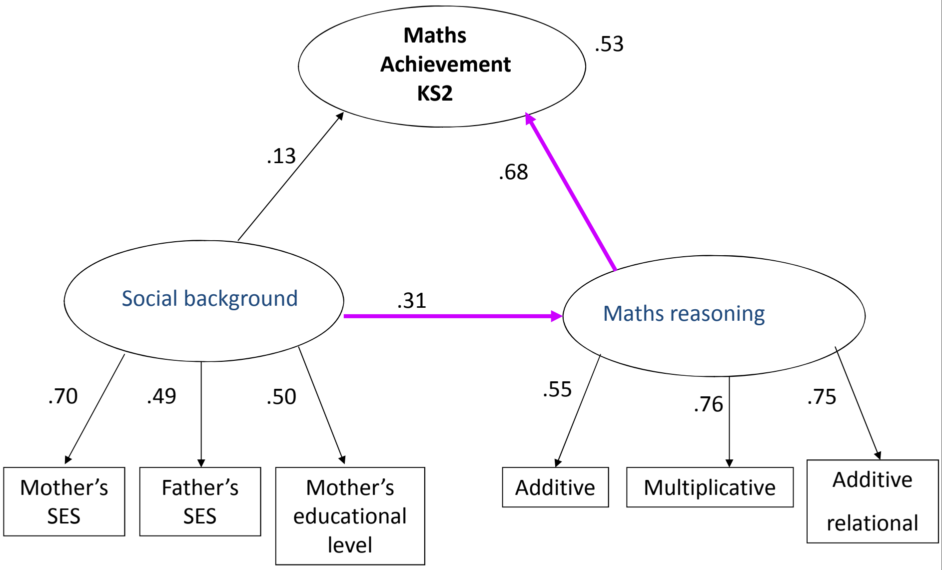
Predicting achievement in:

	Maths	>	Science	>	English
11 year achievement	.34		.19		.14
14 year achievement	.34		.21		.01

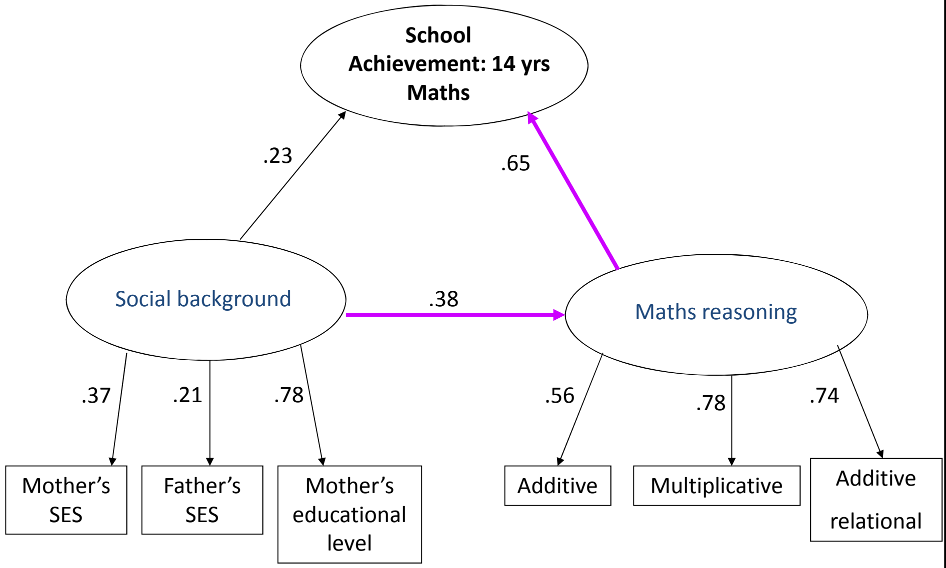
The children's achievement in mathematics in relation to their mothers' educational level



How social background and maths reasoning predict maths at school 3 years later on (N=3183)



How social background and maths reasoning predict maths at school 5.5 years later on (N=2083)



Conclusions about mathematical reasoning & SES

- We found a powerful and long-lasting relation between mathematical reasoning and mathematical progress
- The effects of SES on mathematical achievement may be mediated by mathematical reasoning: there is a strong indirect pathway **from** SES **via** mathematical reasoning **to** mathematical achievement

The next steps

- So far we have defined reasoning items in a negative way: they do not tell a child directly what move she must make.
- But, we should be more systematic than that. The reasoning in these items must be based on central mathematical principles. So, we should make sure that assessments cover these principles.
- This should provide a useful way of categorising reasoning tasks

Some widely used maths assessments are purely procedural

- In a review of standardised tests currently being used in a part of the UK, we found a marked bias towards procedural conventional items
- All 30 items in the WRAT Arithmetic sub-test are procedural & conventional
- So are all the items in the WISC Arithmetic sub-test
- So are all the items in the Number Diagnostic test (widely used in the UK)
- The NFER Progress in Mathematics test does include a some reasoning items, but several of the items categorised as reasoning problems in the manual do not test reasoning

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A preliminary list

- **One-to-one correspondence** as a basis for cardinal number and reasoning about the order of natural numbers
- **Transitivity** as a basis for understanding cardinal and ordinal relations between numbers and quantities
- The **inverse relation** between addition and subtraction and between multiplication and division
- The **additive composition of number**, and its relation to the understanding of the base-10 numeration system and to the decomposition of number in computation procedures
- Reasoning about **part-whole relations** (additive reasoning)
- Relations between the terms in division and the **equivalence** and order of **fractions**
- **Proportional reasoning** (multiplicative reasoning)

Which principles to include?

- The criterion should be the predictive power of scores on different kinds of reasoning items.
- It may be, for example, that scores in reasoning items based on transitivity do not predict children's future mathematical progress: in which case chuck them out.
- So item selection should be based partly on the curriculum and partly on longitudinal, predictive studies combined with intervention studies