Math 210, Fall 2015
Collected Homework #7

Prove each of the following using mathematical induction.

1. For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(4i-3) = 2n^2 - n$$

Proof: Let p_n denote the above proposition for each positive integer value of n.
First, we verify the initial step of the proof – $p_1: 1 = 2(1)^2 - 1$ is clearly true.
Next, we must carry out the inductive step of the proof.

Inductive hypothesis: Assume p_n is true for all values of n from 1 up to k, for some $k \geq 1$.
In particular, this means we will assume

$$p_k: \sum_{i=1}^{k}(4i-3) = 2k^2 - k$$

is true.
Adding $4(k+1)-3$ to both sides of this equation gives us the following (logically equivalent) equations:

$$\left(\sum_{i=1}^{k}(4i-3)\right) + 4(k+1) - 3 = (2k^2 - k) + 4(k+1) - 3$$

$$\begin{aligned}\sum_{i=1}^{k+1}(4i-3) &= 2k^2 - k + 4k + 4 - 3 \\ &= 2k^2 + 3k + 1 \\ &= 2k^2 + (4k - k) + (2 - 1) \\ &= 2k^2 + 4k + 2 - (k+1) \\ &= 2(k^2 + 2k + 1) - (k+1) \\ &= 2(k+1)^2 - (k+1)\end{aligned}$$

Thus, we conclude that

$$\sum_{i=1}^{k+1}(4i-3) = 2(k+1)^2 - (k+1).$$

This follows from our assumption that

$$\sum_{i=1}^{k}(4i-3) = 2k^2 - k;$$

that is, p_{k+1} follows from p_k.

We have completed the initial step and the inductive step of the proof by induction. Therefore,

$$p_n: \sum_{i=1}^{n}(4i-3) = 2n^2 - n$$

is true for all $n \in \mathbb{N}$.

2. For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Proof: Let p_n denote the above proposition for each positive integer value of n.
First, we verify the initial step of the proof – $p_1: \frac{1}{1(2)} = \frac{1}{1+1}$ is clearly true.
Next, we must carry out the inductive step of the proof.

Inductive hypothesis: Assume p_n is true for all values of n from 1 up to k, for some $k \geq 1$.
In particular, this means we will assume

$$p_k: \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

is true.
Adding $\frac{1}{(k+1)(k+2)}$ to both sides of this equation gives us the following (logically equivalent) equations:

$$\left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} \cdot \frac{k+2}{k+2} + \frac{1}{(k+1)(k+2)} \\ &= \frac{(k^2+2k)+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2}\end{aligned}$$

Thus, we conclude that

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1}.$$

This follows from our assumption that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1};$$

that is, p_{k+1} follows from p_k.

We have completed the initial step and the inductive step of the proof by induction. Therefore,

$$p_n: \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

is true for all $n \in \mathbb{N}$.