“Pascal’s Identity” and “Pascal’s Triangle”

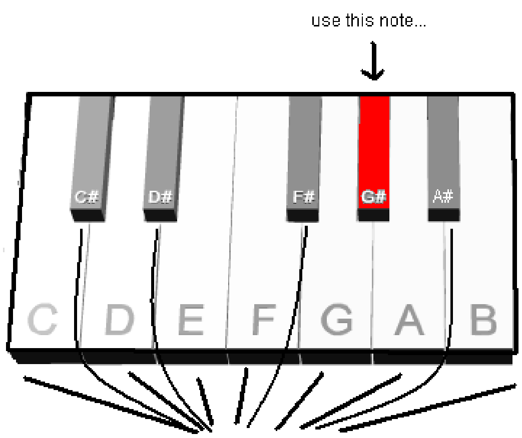
Question: How many different four-tone chords can be selected from the twelve-tone scale?

“Simple” answer: This is a combination of four tones selected from twelve, so the answer is found using the combinations formula:

$$C(12,4) = \frac{12\times 11\times 10\times 9}{4\times 3\times 2\times 1} = 495.$$

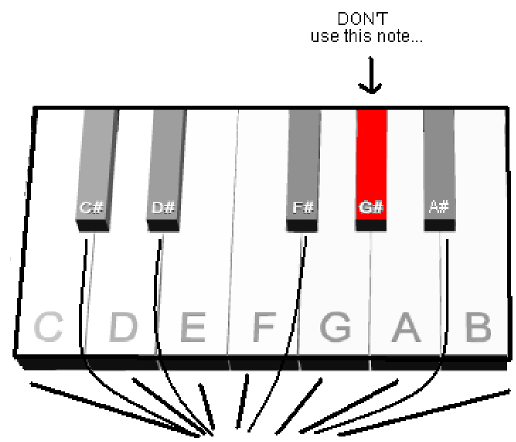
Another answer: Let’s count separately: (a) the chords which include G#, and (b) the chords which do not include G#. (Note that G# is arbitrary; any note selection would do here)

a) Including G#:



'There are $C(11,3) = \frac{11\times 10\times 9}{3\times 2\times 1} = 165$ four-note chords which <u>include</u> the G#.

b) Excluding G#:



There are $C(11,4) = \frac{11\times 10\times 9\times 8}{4\times 3\times 2\times 1} = 330$ four-note chords which <u>exclude</u> the G#.

Thus, there are 165+330=495 four-note chords possible.

Clearly the first counting method is easier, but observe:

$$C(12,4) = C(11,3) + C(11,4)$$

We could do something similar with any chord selection – or, indeed, with the selection of almost any combination of k objects from n objects – the same reasoning works as long as k is positive and n is at least 2. So, the above example generalizes to the following rule:

$$C(n,k) = C(n-1,k-1) + C(n-1,k)$$

This rule is known as “Pascal’s Identity.” It allows us to find the number of combinations from a set of size n based on the number of combinations from a set of size n-1. This recursive rule leads to a very handy array of numbers known as “Pascal’s Triangle,” in which each entry that isn’t a 1 is the sum of the two entries above it on either side. (The construction of Pascal’s Triangle will be discussed in more detail in class.)

PASCAL’S TRIANGLE

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 **210** 252 **210** 120 45 10 1

Theorem: The value of $C(n,k)$ is the k^{th}entry in row n.
Note: This works only if you start counting from $k = 0$ (so the “1” at the left end of each row is the “zeroth” entry in that row).

Example: To find the value of $C(8,3)$ using Pascal’s Triangle, first count down to the eighth row. (Note: for $n > 1$, the first entry other than 1 in the nth row will be n. So to quickly identify the eighth row, find the row that starts with 1, 8.) Then count across to the third entry in this list, keeping in mind that the 1 at the left end is the “zeroth” entry. So, counting across the eighth row, we find…