Proof that e squared is Not a Quadratic Irrationality

This post is based on the paper "Addition a la note sur l'irrationnalité du nombre e" by Joseph Liouville which contains proof of the fact that e^2 is not a quadratic irrationality.

In previous posts I covered that 1) e^2, e^4 are irrational and 2) e is not a quadratic irrationality. I now present the final chapter in this series namely the:

Proof that e^2 is Not a Quadratic Irrationality

Let's us assume on the contrary that there exist integers a, b, c (not all zero) such that

$$ae^4 + be^2 + c = 0$$

Since e^2 is irrational (as proved in previous post) we must have $a \neq 0 \neq c$. We can recast the above equation in the form:

$$ae^2 + ce^{-2} = -b \tag{1}$$

As we have seen in previous post if n is a power of 2 say $n = 2^m$ then $n!$ is divisible by 2^{n-1} and we can write

$$\begin{aligned} n!e^2 &= S_1 + R_1 \\ n!e^{-2} &= S_2 + R_2 \end{aligned}$$

where S_1, S_2 are integers divisible by 2^{n-1} and

$$\begin{aligned} R_1 &= \frac{2^{n+1}}{n+1} + \frac{2^{n+2}}{(n+1)(n+2)} + \cdots \\ R_2 &= (-1)^{n+1}\left(\frac{2^{n+1}}{n+1} - \frac{2^{n+2}}{(n+1)(n+2)} + \cdots\right) \end{aligned}$$

Also we have seen earlier that

$$|R_1| < \frac{2^{n+1}}{n-1},\ \ |R_2| < \frac{2^{n+1}}{n+1}$$

Hence it makes sense to multiply the equation (1) by $n!/2^{n-1}$ to arive at

$$\left(a \cdot \frac{S_1}{2^{n-1}} + c \cdot \frac{S_2}{2^{n-1}}\right) + \left(a \cdot \frac{R_1}{2^{n-1}} + c \cdot \frac{R_2}{2^{n-1}}\right) = -b \cdot \frac{n!}{2^{n-1}}$$

In the above equation the RHS is an integer and sum in first bracket on LHS is also an integer and therefore

$$R = a \cdot \frac{R_1}{2^{n-1}} + c \cdot \frac{R_2}{2^{n-1}}$$

is also an integer and

$$|R| < \frac{4|a|}{n-1} + \frac{4|c|}{n+1} < \frac{4(|a| + |c|)}{n-1} < 1$$

if n is a sufficiently large power of 2. This means that $R = 0$ for all sufficiently large values of $n = 2^m$. Thus we have

$$aR_1 + cR_2 = 0 \tag{2}$$

for all large values of $n = 2^m$.

Now we need to consider what happens when $n = 2^m + 1$. In this case we can clearly see that $n!$ is divisible by 2^{n-2} and hence the same argument can be repeated by multiplying the equation (1) by $n!/2^{n-2}$. The only difference would be that the bound for

$$R' = a \cdot \frac{R_1}{2^{n-2}} + c \cdot \frac{R_2}{2^{n-2}}$$

will be twice the bound for R and it will still tend to zero as $n \to \infty$. Hence we arrive at the conclusion that

$$aR_1 + cR_2 = 0 \tag{3}$$

for all large values of $n = 2^m + 1$.

Thus there will be two consecutive values of n, say 2^m and $2^m + 1$ where m is sufficiently large such that

$$aR_1 + cR_2 = 0$$

i.e.

$$\frac{R_1}{R_2} = -\frac{c}{a}$$

Now the RHS of the above equation is a constant and hence of constant sign. On the LHS R_1 is positive for all n whereas $R_2 < 0$ when $n = 2^m$ and $R_2 > 0$ if $n = 2^m + 1$. Thus we reach the desired contradiction arriving at an equation whose LHS alternates sign but RHS remains of constant sign. It now follows that we can't have integers a, b, c not all zero such that $ae^4 + be^2 + c = 0$.

Using continued fractions we can provide a simpler proof. If we assume that e^2 is a quadratic irrational then by algebraic manipulation it can be shown that $(e^2 - 1)/(e^2 + 1)$ is also a quadratic irrational. But from an earlier post we can see that

$$\frac{e^2 - 1}{e^2 + 1} = \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\cdots$$

and this is not periodic and hence not a quadratic irrational.

Note: The exposition of the proof from Liouville is presented in some papers online. However some of them have curious fallacies. In following Liouville's proof it is essential that we consider the behavior for $n = 2^m$ as well as $n = 2^m + 1$. Without this the sign of R_2 will not alternate and this is crucial to obtain a contradiction. Just considering the values $n = 2^m$ (as is done in this paper) will keep $R_2 < 0$ all the time.

In another instance the argument to prove $a = 0, c = 0$ was not very clear to me. Then I contacted the author Michel Waldschmidt by mail. Prof. Waldschmidt was considerate enough to provide me a very clear explanation of his approach which I now present below. We have

$$\begin{aligned} R &= \left(a \cdot \frac{R_1}{2^{n-1}} + c \cdot \frac{R_2}{2^{n-1}}\right) \\ &= 4\sum_{k=0}^{\infty}\{a + (-1)^{n+1+k}c\}\frac{n! \cdot 2^k}{(n+1+k)!} \\ &= 4(A_n + B_n + C_n) \end{aligned}$$

where

$$\begin{aligned} A_n &= \frac{a + (-1)^{n+1}c}{n+1} \\ B_n &= \frac{2\{a + (-1)^{n+2}c\}}{(n+1)(n+2)} \\ C_n &= \sum_{k=2}^{\infty}\{a + (-1)^{n+1+k}c\}\frac{n! \cdot 2^k}{(n+1+k)!} \end{aligned}$$

Since $R = 0$ for all large values of $n = 2^m$ it follows that

$$(n+1)(A_n + B_n + C_n) = 0$$

for all large $n = 2^m$. Thus we get

$$a + (-1)^{n+1}c + (n+1)(B_n + C_n) = 0$$

When we let $m \to \infty$ so that $n = 2^m \to \infty$ we see that $(n+1)(B_n + C_n) \to 0$ and thus $a - c = 0$ or that A_n is identically zero. Next from the equation

$$(n+1)(n+2)(B_n + C_n) = 0$$

for large $n = 2^m$ we can obtain $a + c = 0$ and thus we get $a = 0, c = 0$ contrary to our initial assumption. The advantage of this approach is that we don't need to consider values of $n = 2^m + 1$.

On the other hand Liouville simply analyzes the quantity $aR_1 + cR_2$ and says that depending upon the sign of a and c we need to choose $n = 2^m$ or $n = 2^m + 1$ such that aR_1 and cR_2 are of the same sign and thereby $aR_1 + cR_2$ remains non-zero and therefore non-integer.

I first tried to understand the expositions of Liouville's proof available online (because Liouville's paper is in French) and then I found the above mentioned papers. But the proof presented in these was not clear to me and hence I had to revert back to Liouville's paper via Google Translate and then I understood the proof. The same understanding has been presented in my post.

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Labels: Irrational Numbers , Mathematical Analysis