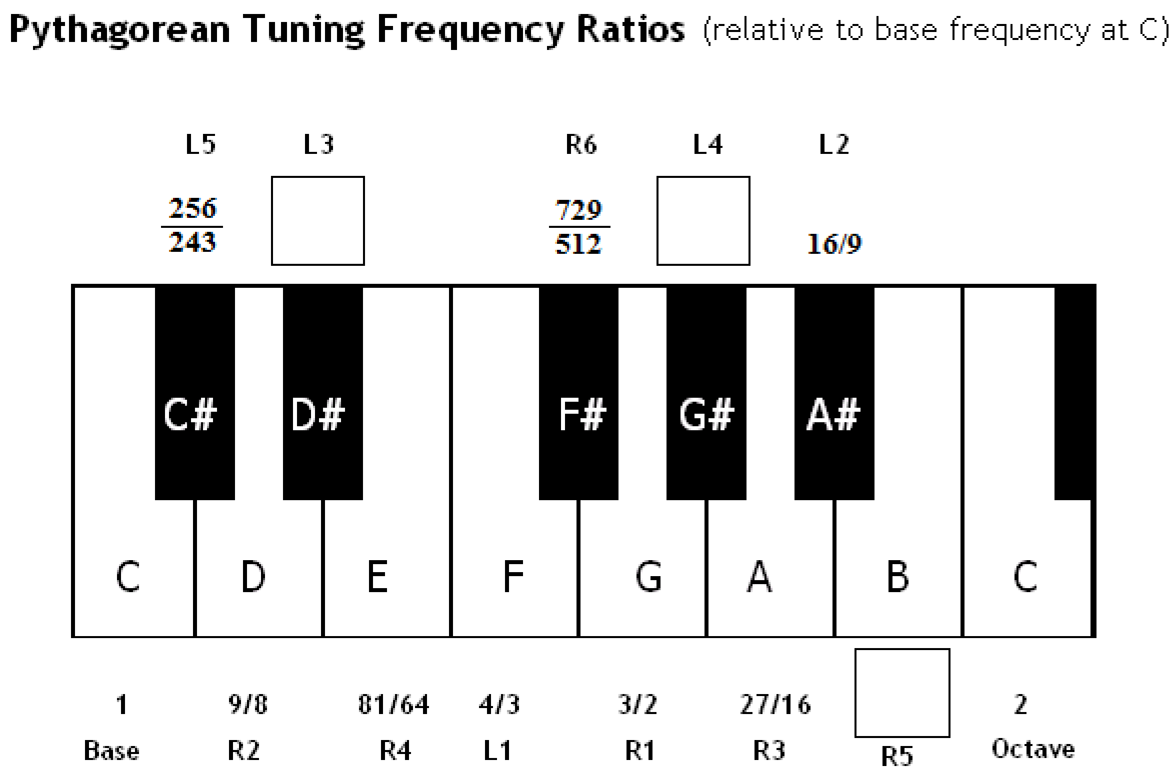
Pythagorean Tuning, continued…

Exercise: Complete the Pythagorean Tuning system (complete solution on following page)



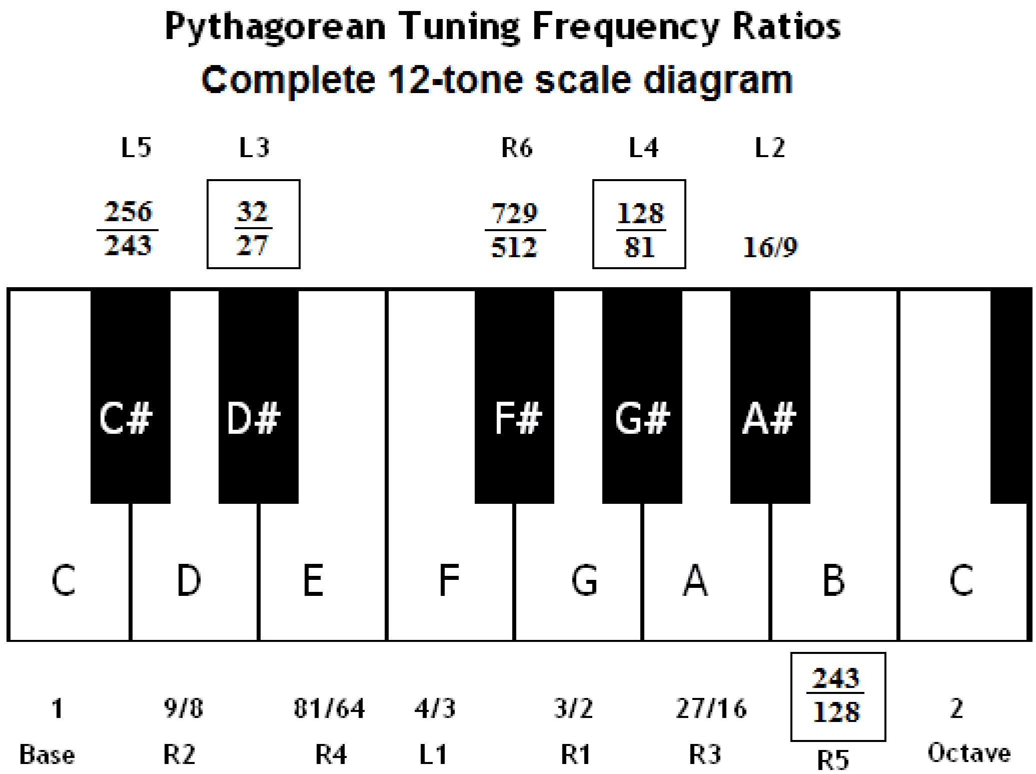
Note:

1. The fraction under each tone is the factor by which the base frequency must be multiplied to find the frequency of that tone in the given octave. For a tone below the base or above the octave, simply multiply or divide by 2 for each octave interval. For example, the multiplier for the A two octaves above the A shown in the diagram would be $\frac{27}{16} \times 2 \times 2 = \frac{27}{8} \times \frac{4}{1} = \frac{27}{4}$.

2. The notations R1, R2, etc. and L1, L2, etc are as defined in class. They are included to emphasize the fact that the above tuning system can be based on any tone in the twelve-tone scale, not necessarily on C. We tend to use C in diagrams, for convenience, since it's the basis for the standard keyboard layout, but we are not constrained to it. For example, if we were to elect to specify the base frequency as an "A", then R1 would correspond to "E", R2 would correspond to "B", R3 to F#, R4 to C#, etc.

3. A variation on the Pythagorean tuning system includes L6 rather than R6 in the twelve-tone scale. This would give us an alternate frequency ratio for the note F#.

Complete set of Pythagorean tuning multipliers:



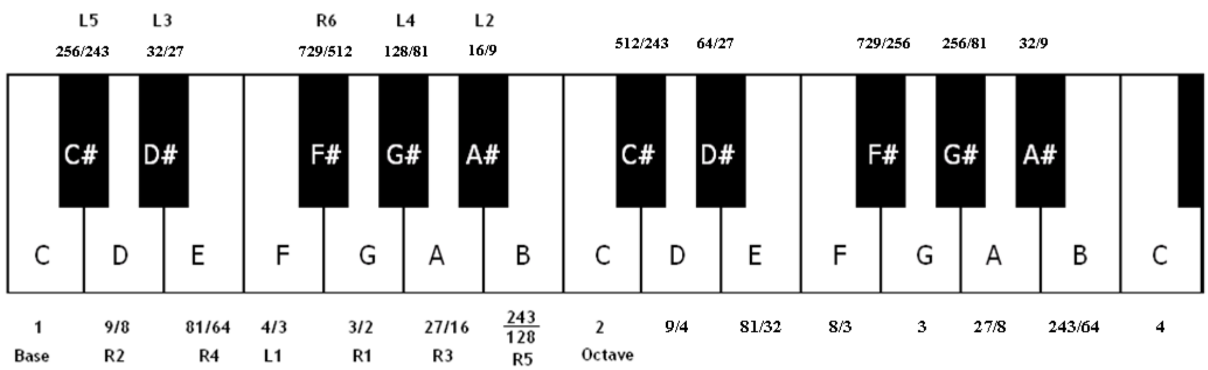
- Since note "R4" has frequency 81/64 of the base note's frequency, we multiply by 3/2 (raise by a fifth) to find the frequency of "R5:" $\frac{81}{64} \times \frac{3}{2} = \frac{243}{128}$, or approximately 1.898.

- Since "L2" has frequency 16/9 of the base frequency, we divide by 3/2 (lower by a fifth) to find the frequency of "L3:" $\frac{16}{9} \div \frac{3}{2} = \frac{16}{9} \times \frac{2}{3} = \frac{32}{27}$, or approximately 1.185.

- Starting from "L3," we divide by 3/2 (lower by a fifth) once again to find the frequency of "L4:" $\frac{32}{27} \div \frac{3}{2} = \frac{32}{27} \times \frac{2}{3} = \frac{64}{81}$. Now, since 64/81 is clearly less than 1, we must multiply by 2 (raise by an octave) to tune the note "L4" so that it is in the octave we're currently tuning. So, this gives us a result of $\frac{64}{81} \times \frac{2}{1} = \frac{128}{81}$, or approximately 1.580.

Here is a two-octave version of the diagram from the previous page:

Pythagorean Tuning Frequency Ratios (relative to base frequency at C)



We've added an extra octave to make it easier to compute frequency ratios for intervals whose higher and lower tones are above and below a "C", respectively (e.g. the perfect fifth A-E).

The above diagram reveals a flaw in Pythagorean tuning – if you know where to look! Recall that the objective of Pythagorean tuning is to preserves octaves and perfect fifths. The Pythagorean technique as summarized by the above diagram preserves every perfect fifth *except* the F#-C# perfect fifth (note that we are referring to the C# *above* F# in this case). More generally, the interval formed by the tones "L5" and "R6" is a "broken" fifth, since its frequency ratio is very close, but not equal, to 3/2.

For example, if we were to start from a base frequency of C: 500 Hz, we would obtain the approximate frequencies shown below:

F#: $\frac{729}{512} \times 440\ Hz \approx 711.914\ Hz$, and C#: $\frac{512}{243} \times 500\ Hz \approx 1053.498\ Hz$

The frequency ratio of this interval, then, is about $\frac{1053.498}{711.914} \approx 1.480$, which is noticeably less than the desired ratio of 1.5. This "broken" fifth, which always occurs for the L5-R6 interval under Pythagorean tuning, is often called the "**wolf fifth**" or "**wolf interval**."

Note that the *exact* "wolf interval" frequency ratio can be found without selecting a specific base frequency at all:

$$\frac{512}{243} \div \frac{729}{512} = \frac{512}{243} \times \frac{512}{729} = \frac{262144}{177147}, \text{or } \frac{2^{18}}{3^{11}}, \text{which is approximately } 1.480$$