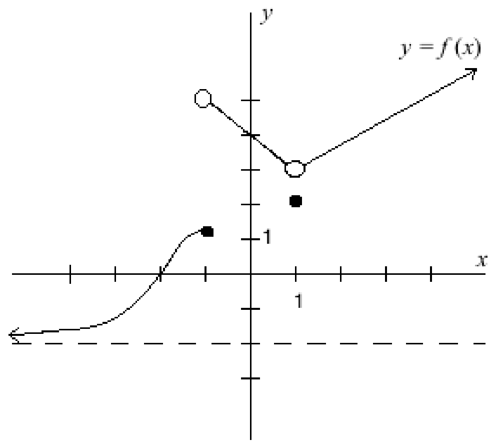
Math 160, Test #2B
Solutions

1. The following diagram shows the graph of $y = f(x)$. The domain of f is $(-\infty, \infty)$. The dashed line indicates a horizontal asymptote, which the graph approaches as it goes off to the left (but not to the right).



(a) Based on the diagram, write the apparent value of each of the following. If an answer does not exist, write "DNE" rather than a number. (Note: each of these should be clear from a quick look at the diagram; none should take much time to figure out. Don't spend more than a couple of minutes on this part!)

$f(-1) = 1$	$f(0) = 4$	$f(1) = 2$
$\lim_{x\to -1^-} f(x) = 1$	$\lim_{x\to 0} f(x) = 4$	$\lim_{x\to 1^-} f(x) = 3$
$\lim_{x\to -1^+} f(x) = 5$	$\lim_{x\to -\infty} f(x) = -2$	$\lim_{x\to 1^+} f(x) = 3$
$\lim_{x\to -1} f(x)\ DNE$	$\lim_{x\to \infty} f(x) = \infty$	$\lim_{x\to 1} f(x) = 3$

(b) Write the interval(s) on which $f'(x)$ appears to be positive, and the interval(s) on which $f'(x)$ appears to be negative. Briefly (one sentence) explain each of your answers.

Solution: $f'(x)$ is positive on $(-\infty, -1)$, negative on (-1, 1), and positive on $(1, \infty)$.

Explanation: Since $f(x)$ is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$, $f'(x)$ is positive on those intervals. Since $f(x)$ is decreasing on the interval (-1,1), $f'(x)$ is negative on this interval.

Note: each of these intervals is specifically an interval of x values. The y coordinates of points on the graph have no relevance in this discussion.

2. Evaluate each of the following according to the instructions. Show all of your work. You should not use your calculator for either part of this problem.

(a) Use algebra to find the exact value of $\lim_{x\to 4} \frac{2x^2 - 12x + 16}{3x - 12}$.
Solution: First, note that the top and bottom both approach 0 as $x \to 4$. This means both polynomials have $(x-4)$ as a factor. In particular, the top factors as $2(x-4)(x-2)$, and the bottom factors as $3(x-4)$. Therefore,

$$\frac{2x^2 - 12x + 16}{3x - 12} = \frac{2(x-4)(x-2)}{3(x-4)} = \frac{2(x-2)}{3}.$$

To find the limit of this fraction as $x \to 4$, we can now substitute $x = 4$:

$$\lim_{x\to 4} \frac{2x^2 - 12x + 16}{3x - 12} = \lim_{x\to 4} \frac{2(x-2)}{3} = \frac{2(4-2)}{3} = \frac{4}{3}.$$

(b) Use the *definition of the derivative* to find $f'(x)$ if $f(x) = 3x^2 + 2x$.
Solution: The definition of the derivative is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

For the function $f(x) = 3x^2 + 2x$, the numerator of this fraction is as follows:

$$\begin{aligned} f(x+h) - f(x) &= 3(x+h)^2 + 2(x+h) - (3x^2 + 2x) \\ &= 3(x^2 + 2xh + h^2) + 2x + 2h - 3x^2 - 2x \\ &= 3x^2 + 6xh + 3h^2 + 2x + 2h - 3x^2 - 2x \\ &= 6xh + 3h^2 + 2h \\ &= h(6x + 3h + 2) \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{h(6x + 3h + 2)}{h} \\ &= \lim_{h\to 0} (6x + 3h + 2) \\ &= 6x + 2. \end{aligned}$$

3. An object is moving along a straight line in such a way that its position at time t (in seconds) is given by $f(t) = 2t^3 + 6t$ (in feet). For each of the following, make sure to include the correct units of measurement with your answer.

   (a) Find the object's *average* velocity over the time interval $2 \le t \le 4$.
   Solution: Average velocity is defined as change in position divided by time elapsed:

$$\begin{aligned} v_{avg} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{152 - 28}{2} \\ &= \frac{124}{2} \\ &= 62. \end{aligned}$$

   Thus, the object's average velocity over the interval $2 \le t \le 4$ is 62 feet per second.

   (b) Find the object's *instantaneous* velocity at time $t = 3$.

   Solution: Instantaneous velocity is given by the derivative of the position function. If $f(t) = 2t^3 + 6t$, then $f'(t) = 2(3t^2) + 6(1) = 6t^2 + 6$. Therefore, the instantaneous velocity at time $t = 3$ is $f'(3) = 60$ feet per second.

4. Let $f(x) = \dfrac{2x+6}{5-x^2}$.

   (a) Use the quotient rule to show that $f'(x) = \dfrac{2x^2 + 12x + 10}{(5-x^2)^2}$. (Show your work!)

   Solution:

$$\begin{aligned} \frac{d}{dx}\left(\frac{2x+6}{5-x^2}\right) &= \frac{(5-x^2)(2) - (2x+6)(-2x)}{(5-x^2)^2} \\ &= \frac{10 - 2x^2 - (-4x^2) - (-12x)}{(5-x^2)^2} \\ &= \frac{2x^2 + 12x + 10}{(5-x^2)^2} \end{aligned}$$

   (b) Find an equation for the line tangent to the graph of f at the point (2,10).
   Solution: The slope of the tangent line is

$$f'(2) = \frac{2(2)^2 + 12(2) + 10}{5 - (2)^2)^2} = \frac{42}{(1)^2} = 42.$$

   So, using point-slope form, the line has equation

$$y - 10 = 42(x - 2),$$

   or, in slope-intercept form,

$$y = 42x - 74.$$

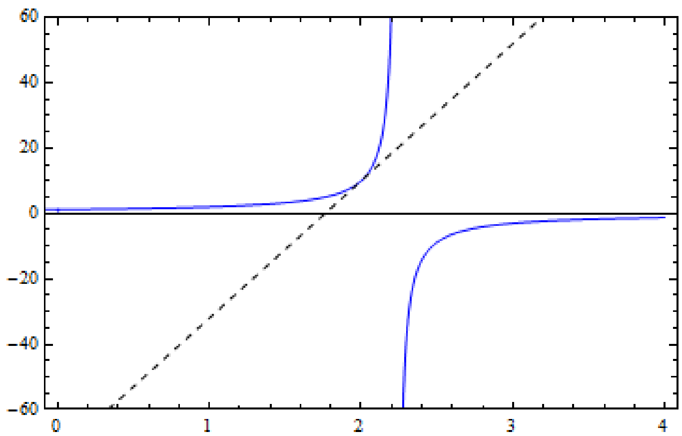
(c) Find the points on the graph of f where the tangent line is horizontal. For each of these points, find the x *and* y coordinates.

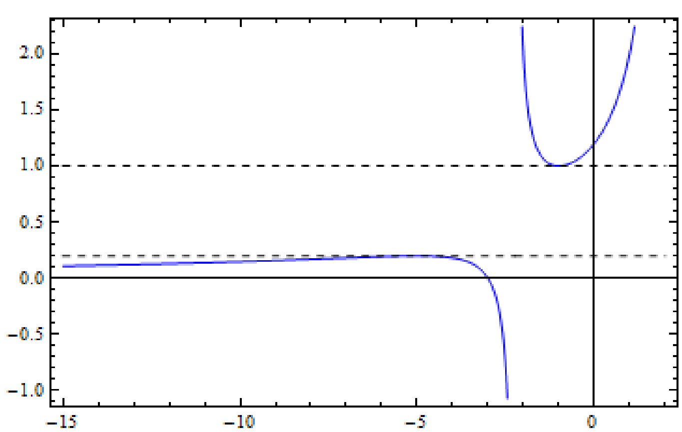
Solution: We must set $f'(x) = 0$ to find out where the tangent line is horizontal:

$$\begin{aligned} \frac{2x^2 + 12x + 10}{(5 - x^2)^2} &= 0 \\ 2x^2 + 12x + 10 &= 0 \\ 2(x^2 + 6x + 5) &= 0 \\ 2(x+1)(x+5) &= 0 \end{aligned}$$

This equation's solutions are $x = -1$ and $x = -5$. The corresponding points on the graph have y coordinates $f(-1) = 1$ and $f(-5) = 1/5$. So, the points on the graph where the tangent line is horizontal are $(-1, 1)$ and $\left(-5, \frac{1}{5}\right)$.

Comment: Shown below are graphs of f. The first shows the tangent line at $(2, 10)$; the second shows the horizontal tangents at $(-1, 1)$ and $(-5, 1/5)$.





5. Consider the function $f(x) = \dfrac{5^x - 25}{x - 2}$. (That's 5 to the x power, not 5 times x!)

   (a) Use your calculator to evaluate $f(2.01), f(2.0001)$, and $f(2.000001)$.
   Solutions: Your results should have been as follows:
   - $f(2.01) \approx 40.561$
   - $f(2.0001) \approx 40.239$
   - $f(2.000001) \approx 40.236$

   (b) Based on your results in part (a), estimate the value of $\lim_{x \to 2^+} f(x)$. Round your answer (if rounding is necessary) to the nearest thousandth. Briefly (one or two sentences) explain your reasoning.

   Solution: As $x \to 2$ from the right, the values of $f(x)$ seem to be getting closer and closer to 40.236 or so. To confirm this, we could look at x-values even closer to 2; for example, $f(2.000001)$ is also approximately 40.236 (to the nearest thousandth); this supports our estimated limit of 40.236 (to the nearest thousandth).

   Comment: Later in the semester, we will learn how to find this limit exactly. Its exact value turns out to be $25 \ln(5)$, or about 40.2359478.

6. Suppose $P(t) = 4000 - 60t + 8t\sqrt{t}$ is the population of the kingdom of Arandelle after t months, where $t = 0$ corresponds to January, 2010.

   Interpret the following values of $P(t)$ or $P'(t)$ in terms of what they mean about the population of Arandelle. For each, write at least one complete sentence of explanation; include correct dates and units of measurement, where appropriate, for full credit.

   (a) Find and interpret the value of $P(16)$.
   Solution: $P(16) = 4000 - 60(16) + 8(16)\sqrt{16} = 4000 - 640 + 512 = 3872$.
   This tells us that the population of Arandelle will be 3872 in May, 2011.
   (Note: If $t = 0$ in January, 2010, then $t = 12$ in January 2011, and so $t = 16$ in May, 2011.)

   (b) Find and interpret the value of $P'(16)$.
   Solution: First we must find $P'(t)$. Since $P(t) = 4000 - 60t + 8t^{3/2}$, we can use the Power Rule:

$$\begin{aligned} P(t) &= 4000 - 60t + 8t^{3/2} \\ P'(t) &= \quad 0 \ - 60(1) + 8 \cdot \frac{3}{2} t^{1/2} \\ &= -60 + 12\sqrt{t} \end{aligned}$$

   Therefore, $P'(16) = -60 + 12\sqrt{16} = -60 + 48 = -12$.
   This tells us that the population of Arandelle was *decreasing* at a rate of 12 people per month in May, 2011.