Math 105, Music & Mathematics

Equal Temperament

Both of the tuning systems we’ve considered thus far – Pythagorean and just intonation – use frequency ratios to determine which tones to include within each octave. Pythagorean is based entirely on preserving all octaves and (almost) all perfect fifths; just intonation also preserves all octaves, but allows certain fifths to become imperfect, or “broken,” in order to preserve some other desirable intervals (such as thirds and sixths). The next system we'll consider, called “equal temperament,” is based on an entirely different consideration.

One drawback of both systems considered thus far is the problem of inconsistent semitones – that is, each semitone (or “half step” in musical terminology) does not have the same frequency ratio as all of the others. For example, in a Pythagorean tuning system based on “C,” the first semitone, C-C#, has a frequency ratio of 256/243 (or about 1.0535), while the second semitone, C#-D, has frequency ratio of 2187/2048 (1.0679). This inconsistency – which persists throughout the scale – makes it impossible to “transpose” a melody or chord without fundamentally changing its sound. Just intonation has this drawback as well (as seen in the homework).

The desire for a tuning system that allows for transposition, while still preserving desirable just intonation intervals such as thirds, fourths, fifths, and sixths, is what led to the development of “equal temperament.” This new tuning system was designed with the following criteria:

* Preserve all octaves – that is, pairs of notes separated by an octave will always have a frequency ratio of exactly 2/1
* Consistent semitones –pairs of notes separated by 1 semitone will always have the same frequency ratio – no exceptions!

The trick to constructing 12-tone equal temperament (hereafter referred to as “12-TET”) is to figure out what semitone frequency ratio we need to use in order to preserve both consistent semitones and octaves. If we have twelve semitones to the octave, then raising a note by semitones twelve times must have the cumulative effect of raising it by an octave. Mathematically: if $r$ is the frequency ratio of a semitone, then $r$ multiplied by itself 12 times – that is, $r^{12}$ - must be equal to 2. The only frequency ratio with this desired property is the twelfth root of 2: $r=\sqrt[12]{2}$.