MATH 201, TEST #3 COMBINED A/B SOLUTIONS

(Solutions alternate by test form - first A, then B)

1. A pumpkin launched from ground level has altitude $f(t) = 128t - 16t^2$ feet after t seconds.

(a) Find the pumpkin's velocity and acceleration at time t. Include the correct units of measurement for each.

$v(t) = f'(t) = 128 - 32t$ ft./sec.

$a(t) = v'(t) = -32$ ft./sec./sec.

(or "ft./sec^2")

(b) After how many seconds will the pumpkin reach its maximum height? (Hint: Your answer should be a whole number.)

~~[illegible]~~ Max. height occurs when $v'(t) = 0$.

$128 - 32t = 0$

$128 = 32t$

$t = 4$ seconds

(c) What is the maximum height reached by the pumpkin?

Sub. $t = 4$ (from part (b)) into altitude function:

$f(4) = 128(4) - 16(4)^2$

$= 512 - 256$

$= 256$ feet

1. A pumpkin launched from ground level has altitude $f(t) = 96t - 16t^2$ feet after t seconds.

(a) Find the pumpkin's velocity and acceleration at time t. Include the correct units of measurement for each.

$v(t) = 96 - 32t$ ft/sec.

$a(t) = -32$ ft/sec^2

(b) After how many seconds will the pumpkin reach its maximum height? (Hint: Your answer should be a whole number.)

$t = 3$ seconds

(c) What is the maximum height reached by the pumpkin?

$f(3) = 288 - 144 = 144$ feet

2. A bacteria culture initially contains 20 cells and grows at a rate proportional to its size. After one hour, the population has increased to 70 cells.

Indicates exponential growth!

(a) Find an expression for the number of bacteria after t hours.

Let $y(t)$ = # bacteria at time t (hours)

Then, $y(t) = y(0)e^{kt}$, where $y(0) = 20$ and $y(1) = 70$ ← Given.

So, $70 = 20e^{k \cdot 1} \to e^k = \frac{7}{2} = 3.5 \to e^{kt} = (e^k)^t = 3.5^t$

Thus, $y(t) = 20(3.5)^t$

(b) After how many hours will the population reach 1000? (Give an exact answer, not a decimal approximation. Note that a calculator is *not* necessary for this.)

Solve: $y(t) = 1000$

$20(3.5)^t = 1000$

$(3.5)^t = \frac{1000}{20} = 50$

$t \ln(3.5) = \ln(50)$

$\to \boxed{t = \frac{\ln(50)}{\ln(3.5)}}$ ✓ EXACT ANSWER

(Note: Approx. 3.123 hrs.)

(c) Find the doubling time ~~for the~~ bacteria culture. (As in part (b), give an exact answer, not a decimal approximation.)

Solve: $40 = 20(3.5)^t$

$(3.5)^t = 2$

$t \ln(3.5) = \ln(2)$

$\boxed{t = \frac{\ln(2)}{\ln(3.5)}}$ ← EXACT ANSWER

(Note: Approx. 0.553 hrs)

2. A bacteria culture initially contains 40 cells and grows at a rate proportional to its size. After one hour, the population has increased to 100 cells.

(a) Find an expression for the number of bacteria after t hours.

$$y(t) = 40 \cdot \left(\frac{100}{40}\right)^{t}$$

$$y(t) = 40 \cdot (2.5)^{t}$$

(b) After how many hours will the population reach 800? (Give an exact answer, not a decimal approximation. Note that a calculator is *not* necessary for this.)

$$y(t) = 800$$

$$40(2.5)^{t} = 800$$

$$2.5^{t} = 20$$

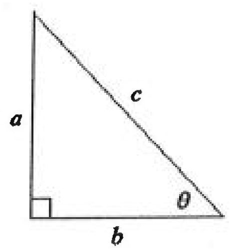
$$t = \frac{\ln(20)}{\ln(2.5)}$$

(c) Find the doubling time for the bacteria culture. (As in part (b), give an exact answer, not a decimal approximation.)

$$t = \frac{\ln(2)}{\ln(2.5)}$$

3. In the right triangle shown in the diagram, a, b, and c denote the lengths (in centimeters) of the three sides, and θ denotes the measure of the acute angle opposite the side of length a.

$a' = \frac{da}{dt} = 2$



$b' = \frac{db}{dt} = 1$

Suppose a is increasing at a rate of 2 cm per second and b is increasing at a rate of 1 cm per second. At the moment when $a = 8$, $b = 6$, and $c = 10$, find each of the following. Simplify each answer, and make sure to include the correct units of measurement.

NOTE: Your score on this problem will be based on your best two (out of three) parts. You will get *extra* credit if (and only if) you get all three parts completely correct.

(a) The rate at which c is increasing.

Start with $a^2+b^2=c^2$ ← Differentiate w.r.t. t

$2aa'+2bb'=2cc'$ ← Solve for c', given $a=8$, $b=6$, $c=10$, $a'=2$, $b'=1$

$2(8)(2)+2(6)(1)=2(10)c'$

$44 = 20c'$

→ $c' = 2.2$ cm/sec.

(b) The rate at which the area of the triangle is increasing.

Start: $A = \frac{1}{2}ab$ ← Differentiate w.r.t. t

$A' = \frac{1}{2}(ab'+ba')$

$A' = \frac{1}{2}(8(1)+6(2)) = \frac{1}{2}(20) = 10$ cm^2/sec.

(c) The rate at which θ is increasing.

Start: $\tan\theta = \frac{a}{b}$ ← Differentiate w.r.t. t

$\sec^2\theta\cdot\theta' = \frac{ba'-ab'}{b^2}$ Note: $\sec\theta = \frac{c}{b} = \frac{10}{6} = \frac{5}{3}$

$\left(\frac{5}{3}\right)^2\theta' = \frac{6(2)-8(1)}{6^2} = \frac{4}{36} = \frac{1}{9}$

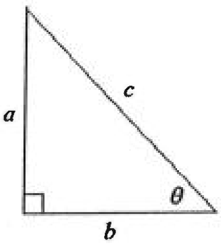
$\frac{25}{9}\theta' = \frac{1}{9}$

(or 0.04)

$\theta' = \frac{1}{9}\cdot\frac{9}{25}$ → $\theta' = \frac{1}{25}$ /second (or $\frac{1}{25}$ radian/second)

3. In the right triangle shown in the diagram, a, b, and c denote the lengths (in centimeters) of the three sides, and θ denotes the measure of the acute angle opposite the side of length a.

$a' = \frac{da}{dt} = 3$



$b' = \frac{db}{dt} = 1$

Suppose a is increasing at a rate of 3 cm per second and b is increasing at a rate of 1 cm per second. At the moment when $a = 6$, $b = 8$, and $c = 10$, find each of the following. Simplify each answer, and make sure to include the correct units of measurement.

NOTE: Your score on this problem will be based on your best two (out of three) parts. You will get *extra* credit if (and only if) you get all three parts completely correct.

(a) The rate at which c is increasing.

$a^2 + b^2 = c^2 \rightarrow 2aa' + 2bb' = 2cc'$

$aa' + bb' = cc'$

$6(3) + 8(1) = 10c'$

$26 = 10c'$

$c' = 2.6$ cm/sec.

(b) The rate at which the area of the triangle is increasing.

$A = \frac{1}{2}ab$

$A' = \frac{1}{2}(ab' + a'b)$

$= \frac{1}{2}(6(1) + 8(3)) = 15$ cm^2/sec

(c) The rate at which θ is increasing.

$\tan\theta = \frac{a}{b}$

$\sec^2\theta(\theta') = \frac{ba' - ab'}{b^2}$

$\left(\frac{10}{8}\right)^2 \theta' = \frac{8(3) - 6(1)}{8^2} = \frac{18}{64}$ (or $\frac{9}{50}$)

$\theta' = \frac{18}{64} \cdot \frac{64}{100} = \frac{18}{100} = .18$/second, or $.18$ radian/sec.

4. Use a linear approximation (or differentials) to estimate the value of $\sqrt{150}$. (Hint: $12^2 = 144$)

[redacted]

Let $f(x) = x^{1/2}$. Then, $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

At $x = 144$, the linearization of f

is $L(x) = f(144) + f'(144)(x-144)$

$$= 12 + \frac{1}{2 \cdot 12}(x-144)$$

$$= 12 + \frac{x-144}{24}.$$

So, $L(150) = 12 + \frac{150-144}{24} = 12 + \frac{6}{24} = \boxed{12\frac{1}{4}}$

or 12.25

So, $\sqrt{150} \approx 12\frac{1}{4}$. (Note: $(12\frac{1}{4})^2 = (\frac{49}{4})^2 = \frac{2401}{16} = 150\frac{1}{16}\ldots$ close!)

Differentials method: Let $y = x^{1/2}$, So $dy = \frac{1}{2\sqrt{x}}dx$

At $x = 144$, $y = 13$ let $dx = 150 - 144 = 6$.

Then $dy = \frac{1}{24}dx = \frac{6}{24} = \frac{1}{4}$.

Thus, $y + \Delta y \approx y + dy = 12 + \frac{1}{4} = 12\frac{1}{4}$.

4. Use a linear approximation (or differentials) to estimate the value of $\sqrt{234}$. (Hint: $15^2 = 225$)

$f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2\sqrt{x}}$

$L(x) = f(225) + f'(225)(x - 225)$

$L(x) = 15 + \frac{x-225}{30}$

$\rightarrow L(234) = 15 + \frac{234-225}{30} = 15 + \frac{9}{30} = \underline{\underline{15.3}}$

or: $y = \sqrt{x}$, $dy = \frac{1}{2\sqrt{x}}\, dx$

$dx = \cancel{}\, 234 - 225 = 9 \rightarrow dy = \frac{1}{2(15)}(9)$

$x = 225 \longrightarrow dy = \frac{9}{30} = 0.3$

So, $y + \Delta y \approx y + dy = 15 + 0.3 = \underline{\underline{15.3}}$

5. Find the absolute minimum and maximum values of $f(x) = 2\sin^2(x) + 2\cos(x)$ on $[0, \pi/2]$

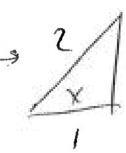
Need to evaluate $f(x)$ at $x=0$, $x=\frac{\pi}{2}$, and at all critical numbers between 0 and $\frac{\pi}{2}$...

Set $f'(x)=0$:

$$f'(x) = 4\sin(x)\cos(x) - 2\sin(x)$$

$$= 2\sin(x)(2\cos(x)-1)$$

This is zero if $\sin(x)=0$ or $\cos(x)=\frac{1}{2}$. $\longleftrightarrow$



$(x=60°)$

$\swarrow$ $x=0$ $\quad$ $\swarrow$ $x=\frac{\pi}{3}$

So, we need to evaluate: $f(0)$, $f(\frac{\pi}{2})$, and $f(\frac{\pi}{3})$.

$$f(0) = 2\sin^2(0) + 2\cos(0) = 0+2 = 2$$

$$f(\tfrac{\pi}{2}) = 2\sin^2(\tfrac{\pi}{2}) + 2\cos(\tfrac{\pi}{2}) = 2+0 = 2$$

$$f(\tfrac{\pi}{3}) = 2\sin^2(\tfrac{\pi}{3}) + 2\cos(\tfrac{\pi}{3})$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^2 + 2\left(\frac{1}{2}\right) = \cancel{2}\cdot\frac{3}{\cancel{4}_2} + 1 = \frac{5}{2} \text{ or } 2.5.$$

Solution:

The absolute minimum value of f is $\underline{\underline{2}}$ (at $x=0$ and at $x=\frac{\pi}{2}$)

The absolute maximum value of f is $\underline{\underline{2.5}}$ (at $x=\frac{\pi}{3}$).

5. Find the absolute minimum and maximum values of $f(x) = 4\sin^2(x) + 4\cos(x)$ on $[0, \pi/2]$

Note: This is nearly identical to #5 on Test #3A.

$f(x)$ on this exam is simply ~~[illegible]~~ two times the function in exam A, so all work is equivalent, and all results (for $f(x)$, not for x) are doubled.

Solutions: Abs. Min. is $f(x) = 4$ (at $x = 0$, $x = \pi/2$)

Abs. Max. is $f(x) = 5$ (at $x = \pi/3$)

6. Let $g(x) = 6x^{2/3} - x$.

(a) Explain why g satisfies the hypotheses of the Mean Value Theorem on the interval [0,8].

g is continuous on $[0,8]$,

and g is differentiable on $(0,8)$, since $g'(x) = 4x^{-1/3} - 1$

$$g'(x) = \frac{4}{x^{1/3}} - 1$$

(b) Find a number, c, that satisfies the conclusion of the Mean Value Theorem for g on the interval [0,8]. (Note: $8^{2/3}$ is a whole number; you should be able to find it without a calculator.)

For some number, c, between 0 and 8,

$$g'(c) = \frac{g(8)-g(0)}{8-0} = \frac{16-0}{8-0} = 2$$

$$\frac{4}{x^{1/3}} - 1 = 2 \rightarrow \frac{4}{c^{1/3}} = 3 \rightarrow c^{1/3} = \frac{4}{3}$$

$$c = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

(c) Explain why g does *not* satisfy the hypotheses of the Mean Value Theorem on the interval [-8,8]. Be specific!

$g(x) = 6x^{2/3} - x$, $\quad -8 \le x \le 8$

$$g'(x) = \frac{4}{x^{1/3}} - 1$$

Continuous on $[-8,8]$ ✓

Differentiable on $(-8,8)$ X

Not at $x=0$!

6. Let $h(x) = 12x^{2/3} - x$. $\longrightarrow h'(x) = 12 \cdot \frac{2}{3} x^{-1/3} - 1 = 8x^{-1/3} - 1$

(a) Explain why h satisfies the hypotheses of the Mean Value Theorem on the interval [0,8].

* See exam 3A Solution.

(b) Find a number, c, that satisfies the conclusion of the Mean Value Theorem for h on the interval [0,8]. (Note: $8^{2/3}$ is a whole number; you should be able to find it without a calculator.)

$$h'(c) = \frac{h(8) - h(0)}{8 - 0} = \frac{40 - 0}{8 - 0} = 5$$

$$\left(\begin{aligned} h(8) &= 12 \cdot 8^{2/3} - 8 \\ &= 12 \cdot 4 - 8 \\ &= 40 \end{aligned} \right)$$

$$8c^{-1/3} - 1 = 5 \rightarrow 8c^{-1/3} = 6$$

$$\frac{8}{c^{1/3}} = 6$$

$$c^{1/3} = \frac{8}{6} = \frac{4}{3}$$

$$c = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

(c) Explain why h does *not* satisfy the hypotheses of the Mean Value Theorem on the interval [-8,8]. Be specific!

* See 3A Solution.