Math 105. Music & Mathematics – Fall, 2013

Permutations & Change Ringing Practice Exercise Answers

Note: Only answers (no solutions) are provided below for #1-5. If you want to see full solutions for any of these, ask about them in class on Friday!

1. a. 2 1 4 3 6 5 b. 2 1 3 5 4 6 c. 4 5 3 1 2 6

   d. 4 5 1 3 2 6 e. 5 4 6 2 3 1 f. 3 5 6 2 4 1

2. a. (bed) b. (abd)(cegf)

   c. (afhd)(bge) d. (iadhfgec)

3. a. (ab) b. (abcdefg)

   c. (ac)(eg) d. (aceg)(bdf)

4. a. 7 5 1 2 6 4 3 b. 7 5 1 2 6 4 3

   c. 2 4 5 3 6 1 7 d. 3 4 5 1 6 2 7

5. a. ( ) b. (bdc)

   c. (afecdb) d. (adbfec)

6. The set contains the identity, ( ), which is also its own opposite. Also, (acd) and (adc) are opposites, as seen in #5a. Finally, if you check for closure (all possible combinations of permutations in the set), you'll find that all results are again in the set. (You can make a 3-by-3 table to verify this, if you like.) Therefore, this set is a group. (More specifically, it's a subgroup of the group of permutations of four objects.)

7. This set violates the "closure" property. As an example, it turns out that (ACB)(BCD)=(ADB) (verify this for yourself!); since (adb) is not in our set, this set cannot be a group.

Alternate answer (easier, IF you happen to notice it) – since "D" is the last letter used (no E, F, etc.), these are all permutations of four objects. We know that there are 4!=24 such permutations; that is, the permutation group contains 24 elements. The given set contains 5 permutations; however, since 5 is not a factor of 24, it can't be a subgroup! (Recall that this would imply 24/5=4.8 "cosets," which is impossible)

8. The subgroup generated by the 5-cycle (adbec) contains all permutations obtained by repeatedly combining (adbec) with itself. Since it's a 5-cycle, we know the subgroup it generates will contain five permutations.

To find the subgroup generated by (adbec), first find the five rearrangements of an ordered list – say, 1 2 3 4 5 – that are generated by repeatedly using the (adbec) permutation. (Note: in the diagram below, a right arrow is used to indicate that the (adbec) permutation has been applied to one list to rearrange it into another list…)

1 2 3 4 5 → 3 4 2 1 5 → 5 1 2 3 4 → 2 3 4 5 1 → 4 5 1 2 3 → 1 2 3 4 5

Now, we need to figure out what permutation would give us each of these rearrangements, starting from 1 2 3 4 5, all in one step…

1 2 3 4 5 → 3 4 2 1 5 -- this is the permutation (adbec)
1 2 3 4 5 → 5 1 2 3 4 -- this is the permutation (abcde)
1 2 3 4 5 → 2 3 4 5 1 -- this is the permutation (aedcb)
1 2 3 4 5 → 4 5 1 2 3 -- this is the permutation (acebd)
1 2 3 4 5 → 1 2 3 4 5 -- this is the identity permutation, ( )

So, the subgroup generated by (adbec) would be:
{ (adbec), (bcde), (aedcb), (acebd), ( )}

9. Find the subgroup generated by the permutation (adb)(ce).

This is similar to #8 – we want to find all permutations that can be obtained by repeatedly combining (adb)(ce) with itself. We'll proceed as in #8 above, by first identifying the rearrangements of an ordered list (we'll use 1 2 3 4 5 again, since it's the simplest choice), then finding the permutation which would have given us each permutation (starting from 1 2 3 4 5) in one step.

Note: in the diagram below, a right arrow is used to indicate that the (ADB)(CE) permutation has been applied…

1 2 3 4 5 → 2 4 5 1 3 → 4 1 3 2 5 → 1 2 5 4 3 → 2 4 3 1 5 → 4 1 5 2 3 → 1 2 3 4 5

Note that there are six distinct rearrangements, implying there are six permutations in the subgroup generated by (ADB)(CE)…
Now, find the permutations indicated by these rearrangements of 1 2 3 4 5:

1 2 3 4 5 → 2 4 5 1 3 -- this is the permutation (adb)(ce)
1 2 3 4 5 → 4 1 3 2 5 -- this is the permutation (abd)
1 2 3 4 5 → 1 2 5 4 3 -- this is the permutation (ce)
1 2 3 4 5 → 2 4 3 1 5 -- this is the permutation (adb)
1 2 3 4 5 → 4 1 5 2 3 -- this is the permutation (abd)(ce)

1 2 3 4 5 → 1 2 3 4 5 -- this is the identity permutation, ( )

So, the subgroup generated by the permutation (ADB)(CE) is

$$\{(ADB)(CE), (ABD), (CE), (ADB), (ABD)(CE), (\,)\}$$

Note that the second, fourth, and sixth permutations in the above list do not include the two-cycle (CE); this is because a two-cycle "resets" every other time it's repeated. Similarly, the third and sixth permutations do not contain a three-cycle, since the cycle (ADB) "resets" itself every three repetitions.

10. The answer to #10 is actually included in the work for #9. I'm not sure why I decided to include this as a separate practice exercise; it is redundant, so you can safely ignore it.

11. This problem is easier if we notice that it's VERY similar to #9. In particular, this is the subgroup generated by (ABC)(DE) (to answer part (e) first). Switch a few letters around, and this set is equivalent to the subgroup we came up with in #9 above.

On the other hand, if we don't make this observation, then we can show that the given set is a subgroup "the old-fashioned way" – by looking at all combinations of elements of the set, and checking for identity/opposites/closure. (Note: the following table takes some time to fill in; this is NOT something I'd ask for you to come up with on a test!)

	$(\,)$	(ABC)	(ACB)	(DE)	$(ABC)(DE)$	$(ACB)(DE)$
$(\,)$	$(\,)$	(ABC)	(ACB)	(DE)	$(ABC)(DE)$	$(ACB)(DE)$
(ABC)	(ABC)	(ACB)	$(\,)$	$(ABC)(DE)$	$(ACB)(DE)$	(DE)
(ACB)	(ACB)	$(\,)$	(ABC)	$(ACB)(DE)$	(DE)	$(ABC)(DE)$
(DE)	(DE)	$(ABC)(DE)$	$(ACB)(DE)$	$(\,)$	(ABC)	(ACB)
$(ABC)(DE)$	$(ABC)(DE)$	$(ACB)(DE)$	(DE)	(ABC)	(ACB)	$(\,)$
$(ACB)(DE)$	$(ACB)(DE)$	(DE)	$(ABC)(DE)$	(ACB)	$(\,)$	(ABC)

The above table demonstrates that each element of the set has an opposite in the set (since the identity appears in each row), the set has closure, and the identity is included. Therefore, this set is a group of permutations, and therefore it's a subgroup of the group of all permutations of five objects. This answers part (a) of #11.

b. To find the coset generated by (AB), combine each permutation in the set with (AB). For each, we should rewrite the permutation as a cycle, or as a combination of disjoint cycles, as needed. (Look at earlier exercises, e.g. #5, to review how this is done.)

$(\quad)(AB) = (AB)$
$(ABC)(AB) = (ACB)$
$(ACB)(AB) = (AC)$
$(DE)(AB) = (DE)(AB)$ (note that these cycles are already disjoint, so leave this as it is)
$(ABC)(DE)(AB) = (ACB)(DE)$
$(ACB)(DE)(AB) = (AC)(DE)$

So, the coset generated by (AB) is $\{(AB), (ACB), (AC), (DE)(AB), (ACB)(DE), (AC)(DE)\}$

c. Proceeding similarly to part b, the answer is as follows:

$$\{(BCD), (AC)(BD), (ADB), (BCDE), (AC)(BDE), (ADEB)\}$$

d. For this part of the problem, we need to remember two things:

- How do we determine the number of cosets a subgroup has?
  Since each coset is the same size, and there's no overlap, the number of cosets a subgroup has is the size of the group divided by the size of the subgroup.
  This subgroup has five elements, so our answer is something divided by five…

- How many elements are in our group?
  Remember, the larger group we're working in right now is the group of all possible permutations of five objects. We observed a few weeks ago that there are $n!$ (that's n factorial) ways to rearrange n objects. In this case, n would be 5, so there are $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ permutations of five objects.

Thus, the size of our group is 120, the size of the subgroup we're working with is 5, which means there are $120 \div 5 = 24$ cosets in all. That's the main idea; however, since the question asks "how many *other* cosets does this subgroup have," and we've already found three of them (remember that the subgroup itself is one of the cosets), there would technically be 24-3=21 *other* cosets.

e. Already answered (see above).

12. a. Applying permutation (AEBHD)(CFG) to 1 2 3 4 5 6 7 8 gives us 4 5 7 8 1 3 6 2.
b. Applying (AEBHD)(CFG) to 4 5 7 8 1 3 6 2 gives us 8 1 6 2 4 7 3 5.
c. We answered almost this exact question in class on 3/27… since the permutation consists of a 5-cycle and a 3-cycle, both cycles will "reset" simultaneously for the first time after 15 repetitions. Therefore, we'd get 15 different rearrangements of 1 2 3 4 5 6 7 8 by repeatedly applying the permutation (AEBHD)(CFG).

(Note: the answer to part c. would be the same for any permutation that consists of two disjoint cycles, where one is a 5-cycle and one is a 3-cycle.)

13. One way (not necessarily the best way) to use adjacent swaps to rearrange 1 2 3 4 5 6 7 8 into 4 5 7 8 1 3 6 2 would be as follows:

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1 2 3 4 5 6 7 8
(CD) (FG)
1 2 4 3 5 7 6 8
(BC)(DE)(GH)
1 4 2 5 3 7 8 6
(AB)(CD)(EF)
4 1 5 2 7 3 8 6
(BC)(DE)(FG)
4 5 1 7 2 8 3 6
   (CD)(EF)
4 5 7 1 8 2 3 6
      (DE)(FG)
4 5 7 8 1 3 2 6
              (GH)
4 5 7 8 1 3 6 2
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So, in order, we've used the sequence of adjacent swaps:
(CD)(FG)(BC)(DE)(GH)(AB)(CD)(EF)(BC)(DE)(FG)(CD)(EF)(DE)(FG)(GH)
to rearrange 1 2 3 4 5 6 7 8 into 4 5 7 8 1 3 6 2.

This rearrangement uses 16 swaps. This is just one sequence I found that works; however, I'm pretty sure that better (shorter) solutions exist. Can you find one? I'll give one extra credit point per swap to the first student who finds a shorter sequence – e..g, if you can accomplish the same rearrangement using 12 adjacent swaps, then I'll give you 16-12=4 extra points on the next test…

Note: this is another example of a practice exercise that is more time-consuming than any question I'd put on a test. I might ask something like this, but one that only requires, say, four or five swaps… (e.g., rearrange 1 2 3 4 into 4 2 1 3 using adjacent swaps; a solution to this problem would be (AB)(CD)(BC)(AB).)