Notes

Contents

1. *Rapport sur la formule des traces*

$p, q, \ell, \mathbf{F}_q, \mathbf{F}$: p is a prime number, $q = p^f$ is a power of p and $\mathbf{F}$ an algebraic closure of the field $\mathbf{F}_q$; ℓ is a prime number $\neq p$.

X_0, X: X_0 is a scheme on $\mathbf{F}_q$, $\mathrm{X} = \mathrm{X}_0 \times_{\mathbf{F}_q} \mathbf{F}$. If $\mathscr{F}_0$ is an (étale) sheaf on X_0, $\mathscr{F}$ denotes its inverse image on X.

1.1. Mise en garde. The exposition beginning with 1.3 is based on SGA 5, Exposé XIV by C. Houzel. This approach is explicit at the expense being somewhat anti-conceptual and overloaded with cumbersome notation, which makes it hard to remember. On the other hand, Deligne's approach to Frobenius is conceptually beautiful and easy to remember, but the first time I read it I couldn't understand it. Now that I do, I write these notes. Deligne uses φ to denote what he calls the 'Frobenius substitution'; it is the well-known topological generator of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$. He uses F to denote the 'Frobenius endomorphism', notated $\mathrm{fr}_{\mathrm{X}_0}$ below, which is the endomorphism of a scheme $\mathrm{X}_0/\mathbf{F}_q$ obtained by $x \mapsto x^q$ on the structure sheaf $\mathscr{O}_{\mathrm{X}_0}$. Finally, he uses F^* to denote the action of Frobenius on sheaves on a scheme over $\mathbf{F}_q$ (and base extensions of such), and their cohomology.

1.2. Frobenius following Deligne.

1.2.1. *Representing the Frobenius correspondence.* Let Q be any scheme and $\mathscr{G}$ an (étale) sheaf on Q. Let $\mathrm{Q}_{\mathrm{pet}}$ denote the category (Et/Q) of algebraic spaces étale over Q, equipped with the étale topology. We 'recall' the following

Proposition. — $\mathscr{G}$ is represented over $\mathrm{Q}_{\mathrm{pet}}$ *by l'espace étalé* $[\mathscr{G}]$ *of* $\mathscr{G}$*: an algebraic space, locally separated and étale over* Q. *If moreover $\mathscr{G}$ is*

(α) locally constant constructible
(β) locally constant,
(γ) constructible,

then $[\mathscr{G}]$ *may be taken to be*

(α) a scheme finite étale over Q *(SGAA Exp. IX 2.2).*

(β) *a scheme étale over* Q *(SGAA Exp. IX 2.2).*

(γ) *finitely presented as an algebraic space over* Q *(SGAA Exp. IX 2.7).*

In other words, $\mathscr{G}$ is the sheaf of local sections of the space $[\mathscr{G}]$: for each $\mathrm{V} \in \mathrm{Et}/\mathrm{Q}$,

$$\mathscr{G}(\mathrm{V}) = \mathrm{Hom}_{\mathrm{Q}}(\mathrm{V}, [\mathscr{G}]).$$

M. Artin constructs the espace étalé $[\mathscr{G}]$ associated to an arbitrary étale sheaf $\mathscr{G}$ on a scheme Q in *Théorèmes de représentabilité pour les espaces algébriques* VII §1. A sketch: put

$$(\mathscr{U} \to \mathscr{G}) := \coprod_{\mathrm{U},\xi\in\mathscr{G}(\mathrm{U})} (\mathrm{U} \xrightarrow{\xi} \mathscr{G}),$$

the sum executed over affine schemes U étale over Q and the $\xi \in \mathscr{G}(\mathrm{U})$. The canonical morphism $\mathscr{U} \times_{\mathscr{G}} \mathscr{U} \to \mathscr{U} \times_{\mathrm{Q}} \mathscr{U}$ induced by $\xi \in \mathscr{G}(\mathrm{U})$ and $\eta \in \mathscr{G}(\mathrm{V})$ (U, V affine étale schemes over Q) is an open immersion and defines an étale equivalence relation on $\mathscr{U}$. Let $[\mathscr{G}]$ be the quotient of $\mathscr{U}$ by this equivalence relation; it is an algebraic space over Q, in general not separated, but only locally separated; in other words, $\mathscr{U} \times_{\mathscr{G}} \mathscr{U} \to \mathscr{U} \times_{\mathrm{Q}} \mathscr{U}$ is not necessarily a closed immersion.

Now let X_0 be a scheme over $\mathbf{F}_q$ and $\mathscr{F}_0$ a sheaf on X_0. The formation of the espace étalé $[\mathscr{F}_0]$ of $\mathscr{F}_0$ yields the commutative square

$$\begin{array}{ccc} [\mathscr{F}_0] & \xrightarrow{\mathrm{F}} & [\mathscr{F}_0] \\ \downarrow & & \downarrow \\ \mathrm{X}_0 & \xrightarrow{\mathrm{F}} & \mathrm{X}_0. \end{array}$$

and hence a morphism $[\mathscr{F}_0] \to \mathrm{F}^*[\mathscr{F}_0]$. Replacing S with X_0, and X_0 with $[\mathscr{F}_0]$, 1.3.1 below finds that this morphism is relative Frobenius $\mathrm{Fr}_{[\mathscr{F}_0]/\mathrm{X}_0}$; as $[\mathscr{F}_0]$ is étale over X_0, 1.3.3 tells us that this morphism is an isomorphism $[\mathscr{F}_0] \xrightarrow{\sim} \mathrm{F}^*[\mathscr{F}_0]$ with inverse Deligne's Frobenius correspondence (1.2.1)

$$\mathrm{F}^* : \mathrm{F}^*\mathscr{F}_0 \xrightarrow{\sim} \mathscr{F}_0.$$

This morphism is the same as the one constructed with different notation in 1.3.4. The point is that by putting additional hypotheses on $\mathscr{F}_0$, one may assume that $[\mathscr{F}_0]$ is in fact a scheme étale over X_0.

1.2.2. *The Frobenius correspondence and the Frobenius endomorphism.* Let $\mathscr{F}_0$ be an abelian sheaf on X_0. We wish to elucidate Deligne's approach in 1.8 of *Rapport* to show that

$$F^{*-1} = \varphi \qquad (\text{on } H^i_c(X, \mathscr{F})).$$

Letting $Y_0 = \operatorname{Spec} \mathbf{F}_q$, as written $H^i_c(X, \mathscr{F}) = [R^i f_! \mathscr{F}_0](\mathbf{F})$, and, noting $F = \mathrm{id}$ on Y_0 we have the (stupid) diagram

$$\begin{tikzcd}
{[R^i f_! \mathscr{F}_0]} \arrow[dr, "F"] \arrow[drr, dashed, bend left, "F"] & & \\
& {[R^i f_! \mathscr{F}_0]} \arrow[r, "\mathrm{id}"'] \arrow[d] & {[R^i f_! \mathscr{F}_0]} \arrow[d] \\
& Y_0 \arrow[r, "\mathrm{id}"] & Y_0
\end{tikzcd}$$

which just serves to connect this discussion to that of the previous section and show that the morphism defining the inverse of the Frobenius correspondence F^* on $[R^i f_! \mathscr{F}_0]$ is indeed $F : [R^i f_! \mathscr{F}_0] \to [R^i f_! \mathscr{F}_0]$; as F acts on geometric points by φ, we see that

$$F^{*-1} : [R^i f_! \mathscr{F}_0](\mathbf{F}) \to [R^i f_! \mathscr{F}_0](\mathbf{F})$$

coincides with φ.

1.3. Frobenius following Houzel.

Definition. We denote by fr_{X_0} the morphism of schemes $X_0 \to X_0$ which is the identity on the underlying topological space $|X_0|$ and acts on the structure sheaf $\mathscr{O}_{X_0}$ by $g \mapsto g^q$.

This morphism is called *absolute Frobenius*.

1.3.1. If the structure morphism $X_0 \to \mathbf{F}_q$ factors through some scheme S, then we denote by $X_0^{(q/S)} := X_0 \times_{S, \mathrm{fr}_S} S$ the fiber product of $g : X_0 \to S$ by the morphism

$\mathrm{fr}_S : S \to S$ with projection $\pi_{X_0/S} : X_0^{(q/S)} \to X_0$. The absolute Frobenius fr_{X_0} then factors through the morphism $\pi_{X_0/S}$. We can form the diagram

$$\begin{tikzcd}
X_0 \arrow[dr, "\mathrm{Fr}_{X_0/S}"] \arrow[drr, dashed, bend left, "\mathrm{fr}_{X_0}"] \arrow[ddr, bend right, "g"'] & & \\
& X_0^{(q/S)} \arrow[r, "\pi_{X_0/S}"'] \arrow[d, "g^{(q)}"] & X_0 \arrow[d, "g"] \\
& S \arrow[r, "\mathrm{fr}_S"] & S
\end{tikzcd}$$

Definition. The morphism $\mathrm{Fr}_{X_0/S}$ is called *relative Frobenius*.

1.3.2. *Frobenius acts on geometric points.* Consider the set of geometric points $X(\mathbf{F}) = X_0(\mathbf{F})$. Frobenius acts on this set by $\varphi \in \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$, $\varphi(x) = x^q$. In particular, as $\mathbf{F}_q$ is perfect, $X^F = X_0(\mathbf{F}_q)$, where X^F denotes the geometric points fixed by Frobenius. In slightly more words, consider a geometric point $\bar{x} \to X_0$ centered on x. The fiber $X \times_{X_0} x$ is isomorphic to the spectrum of $A = \mathbf{F} \otimes_{\mathbf{F}_q} k(x)$. As $k(x)/\mathbf{F}_q$ is separable, $A \sim \prod_{[k(x):\mathbf{F}_q]} \mathbf{F}$, and $[k(x) : \mathbf{F}_q]$ is also equal to the number of $\mathbf{F}_q$-embeddings $k(x) \to \mathbf{F}$. Such an embedding is fixed by φ iff $k(x) = \mathbf{F}_q$, and by φ^f iff $k(x) \subset \mathbf{F}_{q^f}$. So for every point $x \in |X_0|$ with $[k(x) : \mathbf{F}_q] = f$, there are f geometric points centered on x; F acts transitively by φ on this set, and F^f fixes each of these geometric points.

1.3.3. *Behavior of relative Frobenius.* The relative Frobenius $\mathrm{Fr}_{X_0/S}$ is integral, surjective, and radicial, hence a universal homeomorphism. This is clear when $S = \mathbf{F}_q$; i.e. for $\mathrm{fr}_{X_0} = \pi_{X_0/S} \circ \mathrm{Fr}_{X_0/S}$; it follows that $\mathrm{Fr}_{X_0/S}$ is radicial [**EGA**, I 3.5.6 (ii)]. Moreover, $\pi_{X_0/S}$ is separated and radicial, therefore $\mathrm{Fr}_{X_0/S}$ is integral [**EGA**, II 6.1.5 (v)] and surjective.

Suppose moreover that $g : X_0 \to S$ is étale. The same is true of $g^{(q)} : X_0^{(q)} \to S$, and therefore $\mathrm{Fr}_{X_0/S} : X_0 \to X_0^{(q)}$ is étale. As $\mathrm{Fr}_{X_0/S}$ is also radicial and surjective, it is an isomorphism.

1.3.4. *Frobenius correspondence.* Let X_0 be a scheme over $\mathbf{F}_q$ and $\mathscr{F}_0$ a sheaf of sets on $(X_0)_{\text{ét}}$ (the small étale site of X_0 whose underlying category is the category of schemes étale over X_0). We have for all $U \to X$ étale $(\mathrm{fr}_{X_0})_* \mathscr{F}_0(U) = \mathscr{F}_0(U^{(q/X)})$.

The isomorphism $\mathscr{F}_0(\mathrm{Fr}_{\mathrm{U}/\mathrm{X}_0}) : (\mathrm{fr}_{\mathrm{X}_0})_*\mathscr{F}_0(\mathrm{U}) \to \mathscr{F}_0(\mathrm{U})$ is natural in U and induces an isomorphism of sheaves

$$\mathscr{F}_0(\mathrm{Fr}_{/\mathrm{X}_0}) : (\mathrm{fr}_{\mathrm{X}_0})_*\mathscr{F}_0 \xrightarrow{\sim} \mathscr{F}_0;$$

by adjunction applied to $\mathscr{F}_0(\mathrm{Fr}_{/\mathrm{X}_0})^{-1}$ we obtain a morphism

$$\mathrm{F}^* : \mathrm{fr}^*_{\mathrm{X}_0}\mathscr{F}_0 \to \mathscr{F}_0.$$

As $\mathrm{fr}_{\mathrm{X}_0}$ is integral, surjective, and radicial, $\mathrm{fr}^*_{\mathrm{X}_0} : (\mathrm{X}_0)_{\text{ét}} \to (\mathrm{X}_0)_{\text{ét}}$ is an equivalence of sites, and the functors

$$(\mathrm{fr}_{\mathrm{X}_0})_*, \mathrm{fr}^*_{\mathrm{X}_0} : \widetilde{(\mathrm{X}_0)_{\text{ét}}} \longrightarrow \widetilde{(\mathrm{X}_0)_{\text{ét}}}$$

are autoequivalences and quasi-inverses, where $\widetilde{(\mathrm{X}_0)_{\text{ét}}}$ denotes the étale topos on X_0 [**SGAA**, Exp. VIII, 1.1]. Therefore F^* is also an isomorphism.

Definition. The isomorphism $\mathrm{F}^* : \mathrm{fr}^*_{\mathrm{X}_0}\mathscr{F}_0 \longrightarrow \mathscr{F}_0$ is called the *Frobenius correspondence*.

1.3.5. *Frobenius acts on cohomology.* Consider $\mathscr{F}_0$ a sheaf of Λ-modules, for some commutative ring Λ. The canonical morphism $\alpha : \mathscr{F}_0 \to \mathrm{fr}_{\mathrm{X}_0 *}\mathrm{fr}^*_{\mathrm{X}_0}\mathscr{F}_0$ gives rise to

$$\Gamma(\mathrm{X}_0, \mathscr{F}_0) \xrightarrow{\alpha} \Gamma(\mathrm{X}_0, \mathrm{fr}^*_{\mathrm{X}_0}\mathscr{F}_0) \xrightarrow{\mathrm{F}^*} \Gamma(\mathrm{X}_0, \mathscr{F}_0);$$

we also denote the composition of these maps by F^*. When $\mathscr{F}_0 = \underline{\Lambda}$, this composition is easily seen to coincide with $\mathrm{id}_{\Gamma(\mathrm{X}_0,\underline{\Lambda})}$, and as every section $s \in \Gamma(\mathrm{X}_0, \mathscr{F}_0)$ corresponds to a morphism $s : \underline{\Lambda} \to \mathscr{F}_0$ and F^* is evidently functorial in $\mathscr{F}_0$, we find $\mathrm{F}^* \circ \Gamma(\mathrm{X}_0, s) = \Gamma(\mathrm{X}_0, s) \circ \mathrm{F}^* = \Gamma(\mathrm{X}_0, s)$, ergo $\mathrm{F}^* s = s$, so F^* induces the identity on $\Gamma(\mathrm{X}_0, \mathscr{F}_0)$. Recalling the definition of the Frobenius correspondence via adjunction, this action of Frobenius on $\Gamma(\mathrm{X}_0, \mathscr{F}_0)$ coincides with the composition

$$\Gamma(\mathrm{X}_0, \mathscr{F}_0) \xrightarrow{\mathscr{F}_0(\mathrm{Fr}_{/\mathrm{X}_0})^{-1}} \Gamma(\mathrm{X}_0, (\mathrm{fr}_{\mathrm{X}_0})_*\mathscr{F}_0) = \Gamma(\mathrm{X}_0, \mathscr{F}_0). \tag{$*$}$$

Considering $\mathscr{F}_0$ now as an object of $\mathrm{D}^+(\mathrm{X}_0, \Lambda)$, we have $\mathrm{fr}_{\mathrm{X}_0 *}$ preserves injective objects; hence the composition

$$\mathrm{F}^* : \mathrm{R}\Gamma(\mathrm{X}_0, \mathscr{F}_0) \xrightarrow{\mathscr{F}_0(\mathrm{Fr}_{/\mathrm{X}_0})^{-1}} \mathrm{R}\Gamma(\mathrm{X}_0, \mathrm{fr}_{\mathrm{X}_0 *}\mathscr{F}_0) \longrightarrow \mathrm{R}\Gamma(\mathrm{X}_0, \mathscr{F}_0)$$

can be computed by applying $(*)$ term-by-term to an injective resolution of $\mathscr{F}_0$, whence we see that F^* acts by identity on $\mathrm{R}\Gamma(\mathscr{F}_0)$, and hence

$$\mathrm{F}^* : \mathrm{H}^i(\mathrm{X}_0, \mathscr{F}_0) \longrightarrow \mathrm{H}^i(\mathrm{X}_0, \mathscr{F}_0)$$

is the identity for all i.

Suppose that X_0 is separated and of finite type over $\mathbf{F}_q$. Then we can replace Γ in the above discussion by Γ_c to find that Frobenius acts on compactly supported cohomology

$$\mathrm{F}^* : \mathrm{H}^i_c(\mathrm{X}_0, \mathscr{F}_0) \longrightarrow \mathrm{H}^i_c(\mathrm{X}_0, \mathscr{F}_0).$$

2.1. The trace formula for $\operatorname{Spec} \mathbf{F}_{q^n} \to \operatorname{Spec} \mathbf{F}_q$**.** Let $\mathrm{X}_0 = \operatorname{Spec} \mathbf{F}_{q^n}$, $q = p^f$, and $\mathscr{F}_0$ be a constructible $\mathbf{Q}_\ell$ sheaf on X_0, $\mathscr{F}$ its inverse image on X. In this case, the cohomological description of the L-function $\mathrm{L}(\mathrm{X}_0, \mathscr{F}_0)$ reads very simply

$$\det(1 - \mathrm{F}^*_x t^{d(x)}, \mathscr{F})^{-1} = \det(1 - \mathrm{F}^* t^f, \mathrm{H}^0_c(\mathrm{X}, \mathscr{F}))^{-1}, \tag{$*$}$$

where $d(x) = [k(x) : \mathbf{F}_p]$. We first need to make precise how Frobenius is acting on the left and right sides.

On the left side, we fix a geometric point $\overline{x} \to x = \mathrm{X}$ and construct the action of Frobenius on the fiber $\mathscr{F}_{\overline{x}}$ by picking the smallest power of $\mathrm{F}_{(q)}$ which actually fixes the geometric point $\overline{x}$, namely $\mathrm{F}_{(q^n)} = \mathrm{F}^n_{(q)}$. The notation F^*_x denotes the endomorphism of $\mathscr{F}_{\overline{x}}$ induced by $\mathrm{F}^*_{(q^n)}$. Up to isomorphism, $(\mathrm{F}_x, \mathscr{F}_{\overline{x}})$ do not depend on the choice of geometric point $\overline{x} \to x$, and the trace, determinant of this action are denoted by $\mathrm{Tr}(\mathrm{F}^*_x, \mathscr{F})$, etc.

Now, on the right side, we have the Frobenius correspondance on cohomology. We will make use of the identity

$$\mathrm{F}^{*-1} = \varphi,$$

where φ is the Frobenius considered as the topological generator of $\mathrm{Gal}(\mathbf{F}, \mathbf{F}_q)$ (c.f. remark below). The data of a $\mathbf{Q}_\ell$-sheaf on X is equivalent to the data of a finite-dimensional $\mathbf{Q}_\ell$-vector space $\mathrm{V} = \mathscr{F}_{\overline{x}}$ on which $\mathrm{Gal}(\overline{k}(x)/k(x)) = \pi_x$ acts continuously.

There is a canonical isomorphism

$$\pi_x = \mathrm{Gal}(\overline{k}(x), k(x)) \simeq \hat{\mathbf{Z}}$$

furnished by the Frobenius element

$$\varphi_x \in \mathrm{Gal}(\overline{k}(x), k(x)) = \mathrm{Gal}(\mathbf{F}/\mathbf{F}_{q^n}), \qquad \varphi_x(\lambda) = \lambda^{q^n},$$

so that the action of π_x on $\mathrm{V} = \mathscr{F}_{\overline{x}}$ is known once one knows the automorphism $(\varphi_x)_{\mathrm{V}}$ (under the one condition that $(\varphi_x)^\nu \longrightarrow \mathrm{id}_{\mathrm{V}}$ as $\nu \longrightarrow 0$ multiplicatively). If

$$\mathrm{pr}_x : \mathrm{X} = \mathrm{Spec}(\mathbf{F}_{q^n}) \longrightarrow \mathrm{Spec}\,\mathbf{F}_q = e$$

is the canonical morphism, π_x is identified via pr_{x*} with a subgroup of the analogous Galois group π_e for $e = \mathrm{Spec}(\mathbf{F}_q)$, itself topologically generated by φ, and via this identification we have the identity

$$\varphi_x = \varphi^n.$$

The sheaf $\mathrm{pr}_{x*}(\mathscr{F})$ is defined by the induced module

$$\mathrm{pr}_{x*}(\mathscr{F})_{\overline{e}} \simeq \mathscr{F}_{\overline{x}} \otimes_{\pi_x} \pi_e,$$

from which one deduces that, letting $f = \varphi_x^{-1}$, φ^{-1} acts on $\mathrm{pr}_{x*}(\mathscr{F})_{\overline{e}}$ by

$$f^{(n)} : (x_1, \ldots, x_n) \mapsto (f(x_n), x_1, \ldots, x_{n-1}),$$

where here we have written $\mathrm{pr}_{x*}(\mathscr{F})_{\overline{e}}$ with respect to a basis as a free π_x-module of rank n. Now the formula $(*)$ is a matter of verifying the formula

$$\det(1 - ft^n) = \det(1 - f^{(n)}t)$$

for f acting on a free module of rank n. This is Deligne's corollary 3.4.

Remark. Perhaps one way to think about the identification $F^* = \varphi^{-1}$ is by setting up the usual diagram

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{Fr}_{X/X_0}"] \arrow[drr, dashed, bend left, "\mathrm{fr}_X"] \arrow[ddr, bend right, "g"'] & & \\
& X^{(q/X_0)} \arrow[r, "\pi_{X/X_0}"'] \arrow[d, "g^{(q)}"] & X \arrow[d, "g"] \\
& X_0 \arrow[r, "\mathrm{fr}_{X_0}"] & X_0
\end{tikzcd}$$

with g the base extension of the map $\operatorname{Spec}\mathbf{F} \to \operatorname{Spec}\mathbf{F}_q$ that arises from fixing an algebraic closure of $\mathbf{F}_q$, and then observing that $\mathrm{Fr}_{X/X_0} = \varphi \times_{\mathbf{F}_q} \mathrm{id}_{X_0}$. Recalling that F^* on $\mathscr{F}_0/X_0$ is induced by Fr_{U/X_0}^{-1} for $U \to X_0$ étale, and by functoriality F^* on $\mathscr{F}/X$ is induced by pulling back the same, hence by Fr_{U/X_0}^{-1} for $U \to X$ étale, in particular we have that F^* on $H^0_c(X, \mathscr{F})$ is induced by $\mathscr{F}(\mathrm{Fr}_{X/X_0})^{-1} = \mathscr{F}(\varphi^{-1} \times_{\mathbf{F}_q} \mathrm{id}_{X_0})$.

2.5. If $\mathrm{Gr}_\ell\, \mathscr{F}|_{X_i}$ is locally constant, then each $\mathrm{Gr}^n_\ell\, \mathscr{F}|_{X_i}$ is, since replacing X first by a connected component and then by an étale cover, we may assume X is connected and $\mathrm{Gr}_\ell\, \mathscr{F}|_{X_i}$ is constant, in which case $\mathrm{Gr}_\ell\, \mathscr{F}|_{X_i} = \underline{\Gamma(X_{\text{ét}}, \mathrm{Gr}_\ell\, \mathscr{F}|_{X_i})}$, but since $\Gamma(X_{\text{ét}}, -)$ commutes with filtered colimits, $\Gamma(X_{\text{ét}}, \mathrm{Gr}_\ell\, \mathscr{F}|_{X_i}) = \bigoplus \Gamma(X_{\text{ét}}, \mathrm{Gr}^n_\ell\, \mathscr{F})$, and as sheafification commutes with colimits we find $\bigoplus \mathrm{Gr}^n_\ell\, \mathscr{F} = \bigoplus \underline{\Gamma(X_{\text{ét}}, \mathrm{Gr}^n_\ell\, \mathscr{F})}$ compatibly with the natural map.

N.B. This proposition continues to hold if X is supposed merely topologically noetherian.

3.1. Le sorite de la notation. It is very important to note that in Deligne's notation, $\mathrm{Tr}(F^*_x, \mathscr{F})$ and $\mathrm{Tr}(F^*, \mathscr{F}_x)$ are traces of possibly different operators on the fiber $\mathscr{F}_x$. Namely, if $\mathscr{F}$ is a $\mathbf{Q}_\ell$-sheaf on X_0 a scheme separated and of finite type over $\mathbf{F}_q$, then $\mathscr{F}_{\bar{x}}$ is a $\mathbf{Q}_\ell$ vector space, for a choice of geometric point $\bar{x}$ centered on a closed point x of X_0. Then F^*_x denotes the Frobenius $F^*_{q^n} = F^{*n}_q$ raised to the power of the residue field extension $n = [\deg k(x) : \mathbf{F}_q]$. This power of Frobenius is the least that fixes each geometric point centered on x, and the notation $\mathrm{Tr}(F^*_x, \mathscr{F})$ means $\mathrm{Tr}(F^*_x, \mathscr{F}_{\bar{x}})$.

On the other hand, if, say, $x \in \mathrm{X}^{\mathrm{F}^n}$ is a geometric point centered on a point of X_0 defined over $\mathbf{F}_{q^n}$, $\mathrm{Tr}(\mathrm{F}^*, \mathscr{F}_x)$ denotes (absolute) q-power Frobenius acting on the fiber. So, unless $x \in \mathrm{X}^{\mathrm{F}}$, $\mathrm{Tr}(\mathrm{F}^*, \mathscr{F}_x)$ and $\mathrm{Tr}(\mathrm{F}_x^*, \mathscr{F})$ are traces of different operators on the same vector space, the latter a power of the other.

3.2. Le sorite des faisceaux localement constants.

The case of locally constant sheaves of sets. Let X be a scheme and $\mathscr{L}$ a locally constant sheaf of sets with finite fibers on X. (With additional assumptions on X, the case of a locally constant sheaf of sets with infinite fibers is reduced to the finite case in the course of the discussion of Weil II **??**.) We know that $\mathscr{L} = h_{\mathrm{U}}$ for some $\mathrm{U} \to \mathrm{X}$ revêtement étale. We know that every revêtement étale of X is étale-locally on X trivial; namely for some $\mathrm{V} \to \mathrm{X}$ étale, $\mathrm{U}_{\mathrm{V}} \sim \coprod \mathrm{V}$. We wish to show that we may take $\mathrm{V} \to \mathrm{X}$ to be a revêtement étale (with no further work, we could then take it to be a *galoisian* revêtement, i.e. a connected torsor for the automorphism group of the fiber, as principal Galois objects in a Galois category form a cofinal system).

First note that if $f : \mathrm{X} \to \mathrm{Y}$ is any morphism of schemes and $\mathrm{V} \to \mathrm{Y}$ is étale, then $f^* h_{\mathrm{V}} \sim h_{\mathrm{V} \times_{\mathrm{Y}} \mathrm{X}}$. To see this, observe

$$\mathrm{Hom}(f^* h_{\mathrm{V}}, \mathscr{G}) = \mathrm{Hom}(h_{\mathrm{V}}, f_* \mathscr{G}) = f_* \mathscr{G}(\mathrm{V}) = \mathscr{G}(\mathrm{V} \times_{\mathrm{Y}} \mathrm{X}) = \mathrm{Hom}(h_{\mathrm{V} \times_{\mathrm{Y}} \mathrm{X}}, \mathscr{G}).$$

Evidently, this argument holds true for any morphism of sites.

So, it will suffice to show that any revêtement étale can be trivialized after base extension by a revêtement étale. To see this, assume X connected and let $\mathrm{U} \to \mathrm{X}$ be a revêtement étale of constant degree d and proceed by recurrence on d, the case $d = 1$ being trivial. (Of course, in the special case that $\mathrm{U} \to \mathrm{X}$ is galoisian with Galois group G, $\mathrm{U} \times_{\mathrm{X}} \mathrm{U} \sim \mathrm{U} \times \mathrm{G}$ is a trivial G-torsor, and we are done.)

As $\mathrm{U} \to \mathrm{X}$ is étale, hence net, and finite, the diagonal morphism $\mathrm{U} \to \mathrm{U} \times_{\mathrm{X}} \mathrm{U}$ is simultaneously an open and closed immersion, hence an isomorphism onto a connected component of $\mathrm{U} \times_{\mathrm{X}} \mathrm{U}$, allowing us to write $\mathrm{U} \times_{\mathrm{X}} \mathrm{U} = \mathrm{U} \coprod \mathrm{Z}$ with $\mathrm{Z} \to \mathrm{U}$ of constant degree $d - 1$. By hypothesis, there exists a revêtement étale $\mathrm{V} \to \mathrm{U}$ such that

$Z \times_U V \sim \coprod_{d-1} V$. Our desired revêtement is then simply the composition $V \to U \to X$:

$$V \times_X U = V \times_U U \times_X U = V \times_U (U \coprod Z) = V \coprod (V \times_U Z) = \coprod_d V.$$

The case of locally constant constructible sheaves. Let Λ be a commutative, noetherian torsion ring. We adapt the above discussion to locally constant constructible (l.c.c.) sheaves of Λ-modules. Let $\mathscr{F}$ be a l.c.c. sheaf of Λ-modules on a connected scheme X. Then the fibers of $\mathscr{F}$ are finite sets and the above discussion yields a revêtement étale $f : V \to X$ with V a (connected) galoisian cover with Galois group H (i.e. V is a H-torsor) such that $f^*\mathscr{F}$ is a constant sheaf. Its constant value $H^0(V, f^*\mathscr{F})$ is a $\Lambda[H]$-module.

The sheaf $f_*\Lambda$ on X, together with the natural action of H, is a rank 1 l.c.c. sheaf of $\Lambda[H]$-modules. Relative to the natural action of H on $f_*f^*\mathscr{F}$, the trace morphism $f_*f^*\mathscr{F} \to \mathscr{F}$ factors by an isomorphism

$$(f_*f^*\mathscr{F})_H \to \mathscr{F},$$

and we have $f_*f^*\mathscr{F} = f_*M = f_*\Lambda \otimes_\Lambda M$ with the diagonal action of H.

The above discussion shows that a l.c.c. sheaf of Λ-modules on a connected scheme X is determined by its restriction to the small étale site of X. Sheaves on the small étale site $\mathscr{U}$ are in turn determined by Grothendieck's Galois theory: fixing a geometric point $\overline{x}$ of X and putting $G := \pi_1(X, \overline{x})$, the functor

$$\begin{aligned} \mathrm{Sh}(\mathscr{U}) &\to \text{finite G-sets} \\ \mathscr{F} &\mapsto \mathscr{F}_{\overline{x}} \end{aligned}$$

admits the inverse

$$\begin{aligned} \text{finite G-sets} &\to \mathrm{Sh}(\mathscr{U}) \\ \mathscr{F}_{\overline{x}} &\mapsto [V \in \mathscr{U} \mapsto \mathrm{Hom}_G(V_{\overline{x}}, \mathscr{F}_{\overline{x}})]. \end{aligned}$$

To verify this, as the torsors are cofinal in a covering of any $V \in \mathscr{U}$, we may cover V by a torsor W with Galois group H and combine the equalizer description of $\mathscr{F}(V)$

$$\mathscr{F}(V) \to \mathscr{F}(W) \rightrightarrows \mathscr{F}(W \times_V W)$$

with the description of such in the case of a Galois torsor; c.f. [**SGA**$4\frac{1}{2}$, I §5].

The discussion in the previous section can be rephrased using the monodromy representation of a l.c.c. sheaf. Namely, let $\mathscr{F}$ be a l.c.c. sheaf of Λ-modules on a connected scheme X pointed by a geometric point $\overline{x}$ as above; $\mathscr{F}$ corresponds to a representation $\pi_1(X, \overline{x}) \to GL(\mathscr{F}_{\overline{x}})$. As the latter is a finite group, the kernel of this representation is of finite index, and as the Galois coverings are cofinal, we can find a Galois cover of X corresponding to a open subgroup contained in the kernel. The sheaf $\mathscr{F}$ becomes constant when restricted to this cover.

The case of lisse sheaves. Let $E \subset \overline{\mathbf{Q}}_\ell$ be an finite extension of $\mathbf{Q}_\ell$ with valuation ring R, integral closure of $\mathbf{Z}_\ell$ in E, $\mathfrak{m}$ the maximal ideal of R. Every $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{G}$ is obtained as $\mathscr{F} \otimes_E \overline{\mathbf{Q}}_\ell$ for some E and some torsion-free (i.e. flat) constructible E-sheaf $\mathscr{F}$. This means that $\mathscr{F} = \text{“}\operatorname{lim\,proj}\text{''}\mathscr{F}_n$, the latter a flat R-sheaf. A lisse R-sheaf has all the $\mathscr{F}_n$ locally constant sheaves of $R/\mathfrak{m}^n$-modules, and for each n, the above discussion shows that the functor ‘fiber at $\overline{x}$’ gives an equivalence of categories between the category of lisse $R/\mathfrak{m}^n$-sheaves and the category of $R/\mathfrak{m}^n$-modules of finite type together with a continuous action of $\pi_1(X, \overline{x})$. Since the $\mathscr{F}_n$ have $\mathscr{F}_n \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \xrightarrow{\sim} \mathscr{F}_{n-1}$, by passing to the limit we get an equivalence between the category of lisse R-sheaves and the category of finite R-modules with continuous action of $\pi_1(X, \overline{x})$.

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2. Quasi-unipotent monodromy

Some notes about Grothendieck's theorem on quasi-unipotent monodromy. We study the arithmetic proof. It uses a proposition proved by Grothendieck in the appendix of Serre and Tate's article *Good Reduction of Abelian Varieties*.

We may assume that K is complete since, following Serre, *Corps Locaux*, II§3 Cor. 4, completing K leaves the decomposition unchanged. Now, we may assume that any matrix in the image of ρ has coefficients in $\mathbf{Z}_\ell$ and is congruent to $1 \mod l^2$ as these are both open conditions, ρ is continuous, and we are free to pass to an open subgroup of $\mathrm{I}(\overline{v})$ by making a finite extension of K.

Note also that $\operatorname{im}\rho$ is a pro-l group since, while $\mathrm{GL}(n, \mathbf{Z}_l)$ is not a pro-l-group, its first congruence subgroup of matrices congruent to $1 \mod l$ is a pro-l-group (c.f., e.g., §5.1 of *Analytic Pro-p Groups* by Dixon, du Sautoy, Mann & Segal). We see therefore that the prime-to-l part of the order of $\mathrm{GL}(n, \mathbf{Z}_l)$ is finite. As the image of a pro-p group under a continuous homomorphism is pro-p, the continuous image of a pro-p group in $\mathrm{GL}(n, \mathbf{Z}_l)$ is finite. As $\operatorname{im}\rho$ is by construction pro-l, the image of a pro-p group in $\operatorname{im}\rho$ is $\{1\}$.

Now, if L is a finite extension of K_l, we wish to show that the polynomial $f(\mathrm{T}) = \mathrm{T}^l - a$ splits in L for any $a \in \mathrm{L}$. If it does not, then as f is separable and L contains all l^{th} roots of unity, $\mathrm{L}(\sqrt[l]{a})$ is the splitting field of f and is Galois. The automorphism of $\mathrm{L}(\sqrt[l]{a})/\mathrm{L}$ sending $\sqrt[l]{a} \mapsto \zeta_l\sqrt[l]{a}$, where ζ_l is a primitive l^{th} root of unity, acts transitively on the roots of f, hence f is irreducible. But K_l is the l-part of the maximal tamely ramified extension of K_{nr}, hence l cannot divide $[\mathrm{L} : \mathrm{K}_l]$. (Recall that K_t is the maximal tamely ramified extension of K, and we have

$$\mathrm{Gal}(\mathrm{K}_t/\mathrm{K}_{nr}) \simeq \prod_{q \neq p} \mathbf{Z}_q(1) \qquad \mathrm{Gal}(\mathrm{K}_t/\mathrm{K}_l) \simeq \prod_{q \neq p,l} \mathbf{Z}_q(1) \qquad \mathrm{Gal}(\mathrm{K}_s/\mathrm{K}_t) \text{ a pro-}p\text{ group}$$

as q runs over primes, so $\mathrm{Gal}(\mathrm{K}_s/\mathrm{K}_l)$ is an extension of a group isomorphic to $\prod_{q\neq p,l} \mathbf{Z}_q$ by a pro-p group, and therefore has no finite quotient of order divisible by l.) This allows one to conclude that l does not divide the order of $\mathrm{Gal}(\mathrm{K}_s/\mathrm{K}_l)$. The order of $\operatorname{im}\rho$ is a power of l as it is a pro-l group.

An alternative way to see that l does not divide the order of $\mathrm{Gal}(\mathrm{K}_s/\mathrm{K}_l)$ that is more faithful to the original proof proceeds by showing directly that for a finite extension L/K_l, every element of L is an l^{th} power. To do this, let $\mathrm{L} = \mathrm{K}_l[t]/a(t)$ for $a(t)$ an irreducible separable polynomial $a(t) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, and suppose t is not an l^{th} power in L. This implies that the polynomial $a_l(t) = a_n x^{ln} + a_{n-1}x^{l(n-1)} + \ldots + a_0$ is irreducible and separable over K_l. Let K'/K be a finite Galois extension containing all l^{th} roots of unity and the a_i, and contained in K_l. The extension $\mathrm{K}'[t]/a_l(t)$ is finite and separable and is contained in a finite Galois extension K'' of K' with l dividing $[\mathrm{K}'' : \mathrm{K}']$, so l divides the ramification index or the residual degree. If the latter, making a finite unramified extension of K' produces a contradition on the irreducibility of $a_l(t)$ over K_l. If the former, replacing K'' by $\mathrm{K}'[\pi^{1/l}] \subset \mathrm{K}_l$ similarly yields a contradiction. Now to see that every finite Galois extension L of K_l cannot have l dividing its degree, note that $\mathrm{Gal}(\mathrm{L}/\mathrm{K}_l)$ contains a cyclic subgroup H of order l, and we claim $\mathrm{L} = \mathrm{L}^{\mathrm{H}}(a^{1/l})$ for some element $a \in \mathrm{L}^{\mathrm{H}}$. Let σ generate H, and let $b \in \mathrm{L} - \mathrm{L}^{\mathrm{H}}$. Then the element

$$c = \sum_{m=1}^{l} \zeta_l^m \sigma^m(b)$$

satisfies $c = \zeta_l \sigma(c)$, where ζ_l is a primitive l^{th} root of unity. So

$$c^l = \prod_{m=1}^{l} \sigma^m(c) \in \mathrm{L}^{\mathrm{H}},$$

and letting $a = c^l$ we find that $\mathrm{L}^{\mathrm{H}}(a^{1/l})$ is a nontrivial subextension of L^{H}, hence must actually coincide with L.

Returning to the proof, the part about s and $s^{\chi(t)}$ being conjugate in $\mathrm{Gal}(\mathrm{K}_l/\mathrm{K})$ for each $t \in \mathrm{Gal}(k_s/k)$ is just a matter of understanding the action of $\mathrm{Gal}(\mathrm{K}_l/\mathrm{K})$ on $\mathbf{Z}_\ell(1)$ (`0BU5`).

If χ had kernel of finite index in $\mathrm{Gal}(k_s/k)$, it would be open and would correspond to a finite extension of k containing all ℓ^{th} roots of unity, which is ruled out by (C_l).

Recall that

$$\mathbf{Z}_l^* = \begin{cases} \mu_{\ell-1} \times (1 - \ell\mathbf{Z}_\ell) & \ell \neq 2 \\ \{\pm 1\} \times (1 + 4\mathbf{Z}_2) & \ell = 2. \end{cases}$$

Recall that the l-adic logarithm $\log z$ converges for $|z-1|_l < 1$ and that the l-adic exponential $\exp z$ converges for $|z|_l < l^{-1/(l-1)}$, and for z in the radius of convergence of exp, $\exp\log(1+z) = 1+z$. As $\rho(s) \equiv 1 \mod l^2$, and $|l^2|_l = l^{-2} < l^{-1/(l-1)}$, the same identities hold for $\rho(s)$.

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3. *La conjecture de Weil. II*

0.5. Let X be a scheme of finite type over a field k. If X is connected, the structure morphism $X \to \mathrm{Spec}(k)$ admits a unique factorization $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ with k'/k finite separable and $X \to \mathrm{Spec}(k')$ geometrically connected. $\rightsquigarrow$ Stacks, tag `04PZ`. Proof uses notion of 'weakly étale k-algebra.'

1.1.2. First of all, to see that if $K \in D^b_c(X, R)$, then $K \otimes^{\mathbf{L}} R/m^n \in D^b_{ctf}(X, R/m^n)$, see Stacks, tag `0942`. Note that $K \otimes^{\mathbf{L}} R/m^n$ can be represented by a bounded complex of flat constructible sheaves by *Rapport*, 4.7. Also recall that the locally constant sheaves form a weak Serre subcategory of the constructible sheaves on a site (`093U`).

Claim a). On the subject of the category $D^b_c(X, R)$, claim a) is that for each i, the projective system of cohomology sheaves $\mathscr{H}^i(K) :=$ "lim proj"$\mathscr{H}^i(K \otimes^{\mathbf{L}} R/m^n)$ of a complex K in $D^b_c(X, R)$ is an R-constructible sheaf. First a trivial statement: of course the reduction modulo m^n of a complex of flat sheaves representing K_{n+1} induces a map on cohomology, but a priori it need not induce an isomorphism $\mathscr{H}^i(K_{n+1}) \otimes R/m^n \to \mathscr{H}^i(K_n)$. For example, in

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z}/\ell^n & \xrightarrow{\ell^{n-1}} & \mathbf{Z}/\ell^n & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{Z}/\ell^{n-1} & \xrightarrow{0} & \mathbf{Z}/\ell^{n-1} & \longrightarrow & 0 \end{array}$$

the first nonzero cohomology in the top row is $\ell\mathbf{Z}/\ell^n \simeq \mathbf{Z}/\ell^{n-1}$, which gets mapped by the down arrow to $\ell\mathbf{Z}/\ell^{n-1} \simeq \mathbf{Z}/\ell^{n-2}$, even though the first nonzero cohomology in the bottom row is all of $\mathbf{Z}/\ell^{n-1}$. Moreover, $\mathbf{Z}/\ell^{n-1} \xrightarrow{\sim} \mathbf{Z}/\ell^{n-1} \otimes_{\mathbf{Z}/\ell^n} \mathbf{Z}/\ell^{n-1}$, so the map on cohomology after reduction mod ℓ^{n-1} is neither injective nor surjective.

It is important to note that in (1.1.1), the term 'R-faisceau constructible' is used to describe all pro-sheaves in the essential image of the functor $\mathscr{F} \mapsto$ "lim proj"$\mathscr{F} \otimes R/m^n$. In Exposé V of SGA 5, Jouanolou studies J-adic projective systems, where J is an ideal in a commutative ring A. All references in this paragraph will be to this exposé unless indicated otherwise. There is a conflict of indexing, in that Deligne in Weil II has K_n annihilated by m^n, while Grothendieck, Jouanolou and Deligne in SGA $4\frac{1}{2}$ have K_n

annihilated by m^{n+1}. As a predictable but no less unfortunate result, both conventions are effectively in force in different parts of these notes. Jouanolou begins with an abelian category $\mathscr{C}$ and forms $\mathrm{P} = \underline{\mathrm{Hom}}(\mathbf{N}^\circ, \mathscr{C})$, the category of projective systems indeed by the ordered set $\mathbf{N}$ of positive integers with values in $\mathscr{C}$. Given an object $\mathrm{X} = (\mathrm{X}_n, u_n)_{n\geq 0}$ of P and an integer $r \geq 0$, $\mathrm{X}[r]$ denotes the projective system $(\mathrm{X}_{n+r}, u_{n+r})_{n\geq 0}$. If r, s are integers satisfying $s \geq r \geq 0$, then iterated application of the transition morphisms u define a morphism $w_{sr} : \mathrm{X}[s] \to \mathrm{X}[r]$. For each integer $r \geq 0$ and X in P, the morphism $w_{r0} : \mathrm{X}[r] \to \mathrm{X}$ is also denoted $\mathrm{V}_{r\mathrm{X}}$. With this notation, X is said to satisfy the condition Mittag-Leffler-Artin-Rees (MLAR) if there exists an integer r such that for each integer $s \geq r$,

$$\mathrm{im}(\mathrm{X}[s] \xrightarrow{w_{s0}} \mathrm{X}) = \mathrm{im}(\mathrm{X}[r] \xrightarrow{w_{r0}} \mathrm{X}).$$

In particular, if there exists an $r \geq 0$ such that the canonical morphism $\mathrm{X}[r] \to \mathrm{X}$ is null, X is said to be AR-null. The full subcategory P_0 of P whose objects are the AR-null projective systems is thick. The quotient category is called the category of projective systems in $\mathscr{C}$ up to translation and notated $\underline{\mathrm{Hom}}_{\mathrm{AR}}(\mathbf{N}^\circ, \mathscr{C})$ or P_{AR}. It is abelian and the quotient functor $p_{\mathrm{AR}} : \mathrm{P} \to \mathrm{P}_{\mathrm{AR}}$ is exact (2.4.4). Equivalently, P_{AR} is obtained from P by a (left or right) calculus of fractions with respect to the set of morphisms $\{\mathrm{V}_{r\mathrm{X}}\}$, as X runs over the set of objects in P and r over the set of integers ≥ 0. Denote this set by AR. Now suppose ρ equips $\mathscr{C}$ with the structure of A-category (1.1) and either that J is of finite type or that $\mathscr{C}$ possesses infinite direct sums.

Definition (3.1.1). An object X of P is called J-adic if the following two conditions are verified.

(i) For every integer $n \geq 0$, $\mathrm{J}^{n+1}\mathrm{X}_n = 0$.
(ii) For every couple (m, n) of integers with $m \geq n \geq 0$, the morphism $\mathrm{A}/\mathrm{J}^{n+1} \otimes_{\mathrm{A}} \mathrm{X}_m \to \mathrm{X}_n$ deduced from the transition morphism $\mathrm{X}_m \to \mathrm{X}_n$ is an isomorphism.

If moreover the components of X are noetherian objects of $\mathscr{C}$, then X is called noetherian J-adic.

The full subcategory of P (resp. of P_{AR}) generated by the J-adic projective systems (resp. the images of the noetherian J-adic systems) is notated $\mathrm{J{-}ad}(\mathscr{C})$ (resp. $\mathrm{J{-}adn}(\mathscr{C})$).

Two objects X and Y of the category P are said to be AR-isomorphic if $p_{\mathrm{AR}}(\mathrm{X})$ and $p_{\mathrm{AR}}(\mathrm{Y})$ are isomorphic in the category P_{AR}.

Definition (3.2.1-3.2.2). An object X of P is called AR-J-adic if it satisfies the following conditions.

(i) $\mathrm{J}^{n+1}\mathrm{X}_n = 0$ for all $n \geq 0$.
(ii) There exists a J-adic projective system Y isomorphic to X in the category P_{AR}.

If moreover the components of X are noetherian objects of $\mathscr{C}$, then X is called noetherian AR-J-adic.

The full subcategory of P (resp. of P_{AR}) generated by the noetherian AR-J-adic projective systems (resp. by their images) is denoted $\mathscr{E}_{\mathscr{C}}$ (resp. AR-J-adn($\mathscr{C}$)).

Proposition (3.2.3). — Let X *be in* P*. Suppose* $\mathrm{J}^{n+1}\mathrm{X}_n = 0$ *for all* $n \geq 0$*. In order for* X *to be AR-*J*-adic, it is necessary and sufficient that it verify the property (MLAR) and that, denoting by* X' *its projective system of universal images, there exist an integer* $r \geq 0$ *such that, for each pair* (m, n) *of integers with* $m \geq n + r$*, the 'transition morphism' below be an isomorphism:*

$$\mathrm{X}'_m/\mathrm{J}^{n+1}\mathrm{X}'_m \longrightarrow \mathrm{X}'_{n+r}/\mathrm{J}^{n+1}\mathrm{X}'_{n+r}.$$

Note that if X verifies (MLAR), it is AR-isomorphic to its projective system of universal images. The hypothesis made on X' implies that the projective system $(\mathrm{X}'_{n+r}/\mathrm{J}^{n+1}\mathrm{X}'_{n+r})_{n\geq 0}$ is J-adic. This projective system is AR-isomorphic to X'. This proves sufficiency.

The restriction of the functor p_{AR} to J−ad or J−adn induces an equivalence

$$p^{\mathrm{n}}_{\mathrm{AR}} : \mathrm{J{-}adn}(\mathscr{C}) \longrightarrow \mathrm{AR{-}J{-}adn}(\mathscr{C}).$$

Suppose X, Y are noetherian J-adic projective systems. Then a morphism $\mathrm{X} \to \mathrm{Y}$ is represented for a certain integer r by a morphism $\mathrm{X}[r] \to \mathrm{Y}$. As $\mathrm{J}^{n+1}\mathrm{Y}_n = 0$ for all $n \geq 0$ and X is J-adic, this morphism is the composition of a morphism $\mathrm{X} \to \mathrm{Y}$ with $\mathrm{X}[r] \to \mathrm{X}$. Hence $p^{\mathrm{n}}_{\mathrm{AR}}$ is full. Moreover, a given morphism $\mathrm{X} \to \mathrm{X}$ goes to zero under

$p^{\mathrm{n}}_{\mathrm{AR}}$ if it goes to zero in the inductive limit

$$\varinjlim_{r} \mathrm{Hom}(\mathrm{X}[r], \mathrm{Y});$$

i.e. if precomposition by $\mathrm{V}_{r\mathrm{X}} : \mathrm{X}[r] \to \mathrm{X}$ is null for some r. Such is not the case when X is J-adic. Thus $p^{\mathrm{n}}_{\mathrm{AR}}$ is faithful, and, as AR-J-adn($\mathscr{C}$) is evidently the essential image, an equivalence.

Proposition (*5.2.1*). — *The category* $\mathscr{E}_{\mathscr{C}}$ *is stable by kernels and cokernels in* P. *In other words,* $\mathscr{E}_{\mathscr{C}}$ *is an abelian category and the inclusion functor* $\mathscr{E}_{\mathscr{C}} \longrightarrow \mathrm{P}$ *is exact.*

Theorem (*5.2.3*). — *The categories* J-*adn(*$\mathscr{C}$*) and AR*-J-*adn(*$\mathscr{C}$*) are abelian and noetherian.*

We have enough to prove the first statement. (5.2.1) implies on the spot that the category AR-J-adn($\mathscr{C}$) is abelian: given an arrow $\mathrm{A} \to \mathrm{B}$ in AR-J-adn($\mathscr{C}$), up to isomorphism of A and B this arrow comes from an arrow in P with kernel and cokernel in $\mathscr{E}_{\mathscr{C}}$; as the functors $\mathscr{E}_{\mathscr{C}} \to \mathrm{P} \to \mathrm{P}_{\mathrm{AR}}$ are exact, the kernel and cokernel lie in AR-J-adn($\mathscr{C}$). Therefore J-adn($\mathscr{C}$) must also be abelian as the two categories are equivalent.

Remark. If A is a noetherian (commutative) ring complete and separated with respect to an ideal J such that A/J is artinian, then the following is true and provides a kind of 'spiritual underpinning' for the category J-adn.

Proposition. — The functor $\varprojlim$ *induces an equivalence between the categories* AR-J-adn(A-mod) *and the category of finite* A-*modules.*

Specializing to the category $\mathrm{Ab}(\mathrm{X})$ of abelian sheaves on $\mathrm{X}_{\text{ét}}$, X a scheme, when X is noetherian, the abelian noetherian sheaves are the abelian constructible sheaves [**SGAA**, IX 2.9]. Let $\mathrm{Abc}(\mathrm{X})$ denote the category of abelian constructible sheaves and $\ell\text{−adc}(\mathrm{X})$ the full subcategory of $\underline{\mathrm{Hom}}(\mathbf{N}^{\circ}, \mathrm{Abc}(\mathrm{X}))$ generated by the constructible ℓ-adic sheaves. Then we have shown that, when X is noetherian,

$$\ell\text{−adc}(\mathrm{X}) = (\ell\mathbf{Z})\text{−ad}(\mathrm{Abc}(\mathrm{X})) = (\ell\mathbf{Z})\text{−adn}(\mathrm{Ab}(\mathrm{X})),$$

the first equality holding with no assumptions on X. In P, given an exact sequence $0 \to \mathrm{X} \to \mathrm{Y} \to \mathrm{Z} \to 0$ with X strict and Y J-adic, Z is J-adic. Let $u : \mathscr{F} \to \mathscr{G}$ be a morphism of ℓ-adic sheaves on a scheme X. We apply the above formalism in the case $\mathrm{A} = \mathrm{R}$, $\mathrm{J} = m$, $\mathscr{C} = \mathrm{Ad}(\mathrm{X})$, $\mathrm{P} = \underline{\mathrm{Hom}}(\mathbf{N}^{\circ}, \mathrm{Ad}(\mathrm{X}))$. Let $\mathscr{F}' := \operatorname{im} u$ and $\mathscr{G}' := \operatorname{coker} u$ computed in the category P. Then the following diagram has exact rows and commutes.

$$\begin{tikzcd}
& \mathrm{R}/m^{n+1} \otimes_{\mathrm{R}} \mathscr{F}_m \arrow[r, "\mathrm{id} \otimes u_m"] \arrow[d, "f_{nm}"] & \mathrm{R}/m^{n+1} \otimes_{\mathrm{R}} \mathscr{G}_m \arrow[r] \arrow[d, "g_{nm}"] & \mathrm{R}/m^{n+1} \otimes_{\mathrm{R}} \mathscr{G}'_m \arrow[r] \arrow[d, "h_{nm}"] & 0 \\
0 \arrow[r] & \mathscr{F}'_n \arrow[r] & \mathscr{G}_n \arrow[r] & \mathscr{G}'_n \arrow[r] & 0
\end{tikzcd}$$

As f_{nm} is an epimorphism and g_{nm} an isomorphism, the snake lemma implies that h_{nm} is an isomorphism; therefore $\mathscr{G}'$ is a cokernel of u in m-adc(X). On a noetherian X, we have the following simple description of $\ker u$. Let $\mathscr{K} := \ker u$ in P; $\mathscr{K}$ is AR-m-adic. Denoting by $\mathscr{K}'$ the system of universal images of $\mathscr{K}$, grâce à (3.2.3) there exists an integer r such that the projective system

$$m_r(\mathscr{K}') := (\mathscr{K}'_{n+r}/\ell^{n+1}\mathscr{K}'_{n+r})_{n\in\mathbf{N}}$$

is m-adic constructible. The composition below is a kernel of u in $m-\mathrm{adc}(\mathrm{X})$:

$$m_r(\mathscr{K}') \longrightarrow \mathscr{K}' \longrightarrow \mathscr{K}.$$

Returning to the setting of Deligne's article, note that the AR-null sheaves become null objects in the category of pro-sheaves on X, as morphisms in that category between two objects $\underline{\mathrm{X}} := (\mathrm{X}_i)_{i\in\mathrm{I}}$, $\underline{\mathrm{Y}} := (\mathrm{Y}_j)_{j\in\mathrm{J}}$ indexed by sets I, J both equipped with filtered preorders are given by

$$\text{Pro Hom}(\underline{\mathrm{X}}, \underline{\mathrm{Y}}) := \varprojlim_j \varinjlim_i (\mathrm{X}_i, \mathrm{Y}_j),$$

and if $\underline{\mathrm{X}}$ is an AR-null object of P, it is clear that the identity morphism goes to zero in Pro Hom($\underline{\mathrm{X}}$,$\underline{\mathrm{X}}$), so by the universal property of P_{AR}, the AR-m-adn(Ab(X)) sheaves are in the essential image of the m-adic constructible sheaves in the category of pro-sheaves.

Returning to claim a), the reduction to the punctual case requires a few words. The equivalence between the categories we have seen above specializes to an equivalence

$\mathbf{Z}_\ell$−fc $\to$ (AR, $\mathbf{Z}_\ell$)−fc, where the latter is really the category AR-m-ad(Abc(X))=AR-m-adn(Ab(X)) [**SGA5**, VI, 1.5.5], so given the projective system $(\mathscr{K}_n)$ where $\mathscr{K}_n = \mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^n)$, letting $j : \mathrm{U} \to \mathrm{X}$ be an open over which $\mathscr{K}_0$ is locally constant, $i : \mathrm{X} - \mathrm{U} \to \mathrm{X}$ the inclusion of the complement, by the stability of the property AR-m-adic in short exact sequences [**SGA5**, V, 3.2.4], in order to conclude that $\mathscr{K}_n$ is an (AR, R)-constructible sheaf suffices to show that the projective systems $(j^* j_! \mathscr{K}_n)$ and $(i^* i_* \mathscr{K}_n)$ are (AR, R)-constructible. This allows us to reduce to the situation where $(\mathscr{K}_n)$ is a projective system of locally constant constructible sheaves, since the functors $i_*, j_!$ send m-adic sheaves to m-adic sheaves and AR-null sheaves to AR-null sheaves (this can be checked pointwise), now use [**SGA5**, V, 2.4.5]. By 'gluing' (c.f. proof of [**SGA5**, VI, 1.5.5]), we can reduce to proving over an open cover trivializing $\mathscr{K}_0$. More concretely, we can cover our space by finitely many opens over which $\mathscr{K}_0$ is constant, and if we can show that $\mathscr{K}_0$ is AR-m-adic over each, then the construction (3.2.3) above allows us to find an integer for each open in our cover; taking the maximum r of these, replacing $(\mathscr{K}_n)$ by its system of universal images $(\mathscr{K}_n')$, and forming $(\mathscr{K}'_{n+r}/m^{n+1}\mathscr{K}'_{n+r})_{n\in\mathbf{N}}$, we have produced an R-constructible sheaf which is AR-isomorphic to the system $(\mathscr{K}_n)$.

In the punctual case, the results of [**SGA5**, XV p. 473] allow us to suppose that we have a projective system $(\mathrm{K}_r)_{r\in\mathbf{N}}$, where K_r is a complex of free R/m^r-modules of finite type, null outside of an interval $[a, b]$ independent of r, with transition morphisms $\mathrm{K}_{r+1} \to \mathrm{K}_r$ isomorphic (as morphisms of complexes) to $\mathrm{K}_{r+1} \to \mathrm{K}_{r+1}/m^{r+1}\mathrm{K}_{r+1}$, for all $r \in \mathbf{N}$. We will show that the projective system of cohomology "lim proj" $\mathrm{H}^i(\mathrm{K}_r)$ is AR-m-adic, which will imply that it is an R-constructible sheaf in Deligne's sense; i.e. that it is isomorphic, as a pro-object, to a bona fide R-constructible sheaf. Abusively, put $\mathrm{K} := \varprojlim_r \mathrm{K}_r$ (before, K refers to the stalk of an object in $\mathrm{D}^b_c(\mathrm{X}, \mathrm{R})$). A key ingredient is EGA 0_{III} 13.2.3, which says that if a projective system of complexes such as (K_r) satisfies the Mittag-Leffler condition, and if the projective system $(\mathrm{H}^{i-1}(\mathrm{K}_r))$ does too, then the canonical map $\mathrm{H}^i(\mathrm{K}) \xrightarrow{\sim} \varprojlim_r \mathrm{H}^i(\mathrm{K}_r)$ is bijective. As in our situation, the K_r are complexes of finite groups and their cohomology modules $\mathrm{H}^i(\mathrm{K}_r)$ are also finite groups, the Mittag-Leffler condition is automatic. Our hypotheses on the complexes K_r imply the existence of isomorphisms $\mathrm{K}_{r+s} \otimes_{\mathrm{R}/m^{r+s}} \mathrm{R}/m^s \xrightarrow{\sim} \mathrm{K}_s$ and the exactness of the sequence

$$0 \to \mathrm{K}_{r+s} \otimes_{\mathrm{R}/m^{r+s}} \mathrm{R}/m^s \xrightarrow{m^r} \mathrm{K}_{r+s} \to \mathrm{K}_{r+s} \otimes_{\mathrm{R}/m^{r+s}} \mathrm{R}/m^r \to 0$$

for $1 \leq r, s$. The projective limit as $s \to \infty$ of the associated cohomology sequence is still exact by the fact that everything is still finite (ML). This projective limit can be broken up into short exact sequences

$$0 \to \mathrm{H}^i(\mathrm{K})/m^r \to \mathrm{H}^i(\mathrm{K}_r) \to \mathrm{H}^{i+1}(\mathrm{K})[m^r] \to 0. \tag{*}$$

Note that $\mathrm{Tor}_1^{\mathrm{R}}(\mathrm{M}, \mathrm{R}/m^r\mathrm{R}) \simeq \mathrm{M}[m^r]$ for M an R-module. Since the modules $\mathrm{H}^{i+1}(\mathrm{K})[m^r]$ stabilize as $r \gg 0$ and the transition morphisms on $(\mathrm{Tor}_1^{\mathrm{R}}(\mathrm{H}^{i+1}(\mathrm{K}), \mathrm{R}/m^r))_{r\in\mathbf{N}}$ are multiplication by m, this projective system is evidently AR-null. As the projective system $(\mathrm{H}^i(\mathrm{K})/m^r)_{r\in\mathbf{N}}$ is evidently m-adic, $(\mathrm{H}^i(\mathrm{K}_r))_{r\in\mathbf{N}}$ is AR-m-adic, and hence its image in the category of pro-sheaves "lim proj" $\mathrm{H}^i(\mathrm{K}_r)$ is isomorphic to the image of the m-adic system "lim proj" $\mathrm{H}^i(\mathrm{K})/m^r$, which shows the former is m-adic in Deligne's sense.

Remark. To see that each transition morphism on the projective system

$$(\mathrm{Tor}_1^{\mathrm{R}}(\mathrm{H}^{i+1}(\mathrm{K}), \mathrm{R}/m^r))_{r\in\mathbf{N}}$$

is multiplication by m, note that by the equivalence $\mathrm{K}^-(\mathscr{P}) \to \mathrm{D}^-(\mathrm{R}-\mathrm{mod})$, where $\mathrm{K}^-(\mathscr{P})$ is the full triangulated subcategory of $\mathrm{K}^-(\mathrm{R}-\mathrm{mod})$ generated by the complexes with projective objects in all degrees, there is, up to homotopy, a unique map of projective resolutions of R/m^r and R/m^{r-1} inducing the desired map $\mathrm{R}/m^r \to \mathrm{R}/m^{r-1}$, namely

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{R} & \xrightarrow{m^r} & \mathrm{R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{R} & \xrightarrow{m^{r-1}} & \mathrm{R} & \longrightarrow & 0. \end{array}$$

Tensoring by $\mathrm{H}^{i+1}(\mathrm{K})$ we find that the map on $\mathrm{Tor}_1^{\mathrm{R}}$ is indeed multiplication by m.

Claim c). Let $f : \mathrm{Y} \to \mathrm{X}$ be an arrow between schemes of finite type over S with S regular of dimension ≤ 1. Claim c) is that about the categories $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$ being stable by the six functors, and that these functors also commute with reduction modulo m^n. This can be broken into 3 claims about the six functors: (1) that they preserve constructibility, (2) that they preserve finite tor-dimension, and (3) that they commute with reduction modulo m^n. The discussion of *Th. finitude* 1.1, 1.5–1.7 covers claims 1 & 2 for the four functors $\mathrm{R}f_*, f^*, \mathrm{R}f_!, \mathrm{R}f^!$, as well as $\mathrm{R}\mathscr{H}om$. Claim 3 for $\mathrm{R}\mathscr{H}om$ is discussed below. The case of $\otimes^{\mathbf{L}}$ is more or less trivial. Claim 3 for the four functors is the presence of isomorphisms like $\mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{R}f_!(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^{n+1}) \xrightarrow{\sim} \mathrm{R}f_!(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^n)$ in $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$. Because of the finitude hypotheses (c.f. note to *Th. finitude* 1.3), we can apply the recipe of *Rapport* 4.12 to construct the arrow in $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$ in the cases of $\mathrm{R}f_*, \mathrm{R}f_!$. To see the arrow is an isomorphism, we can then copy the reasoning of *Rapport* 4.9.1. The case of f^* is trivial, since after replacing $\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^{n+1}$ by a bounded complex of flat sheaves M, we have isomorphisms

$$f^*(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^{n+1}) \otimes^{\mathbf{L}} \mathrm{R}/m^n \xrightarrow{\sim} f^*(\mathrm{M}) \otimes \mathrm{R}/m^n \xrightarrow{\sim} f^*(\mathrm{M} \otimes \mathrm{R}/m^n) \xleftarrow{\sim} f^*(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^n),$$

as tensor product commutes with inductive limits. As for $\mathrm{R}f^!$, to simplify the notation, let K_{n+1} denote $\mathrm{K} \otimes^{\mathbf{L}} \mathrm{R}/m^{n+1}$. We obtain an arrow

$$\mathrm{R}/m^n \otimes^{\mathbf{L}}_{\mathrm{R}/m^{n+1}} \mathrm{R}f^!(\mathrm{K}_{n+1}) \to \mathrm{R}f^!(\mathrm{K}_{n+1} \otimes^{\mathbf{L}} \mathrm{R}/m^n)$$

in $\mathrm{D}^+(\mathrm{Y}, \mathrm{R}/m^n)$ from the adjunction

$$\mathrm{Hom}_{\mathrm{D}^+(\mathrm{X},\mathrm{R}/m^n)}(\mathrm{R}f_!\mathrm{L}, \mathrm{M}) \simeq \mathrm{Hom}_{\mathrm{D}^+(\mathrm{Y},\mathrm{R}/m^n)}(\mathrm{L}, \mathrm{R}f^!\mathrm{M})$$

with $\mathrm{L} = \mathrm{R}/m^n \otimes^{\mathbf{L}}_{\mathrm{R}/m^{n+1}} \mathrm{R}f^!(\mathrm{K}_{n+1})$ and $\mathrm{M} = \mathrm{K}_{n+1} \otimes^{\mathbf{L}} \mathrm{R}/m^n$ in the following way:

$$\begin{aligned}\mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{R}f^!\mathrm{K}_{n+1} \to \mathrm{R}f^!\mathrm{R}f_!(\mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{R}f^!\mathrm{K}_{n+1}) \xrightarrow{\sim}& \\ \xrightarrow{\sim} \mathrm{R}f^!(\mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{R}f_!\mathrm{R}f^!\mathrm{K}_{n+1}) \to \mathrm{R}f^!(\mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{K}_{n+1}).\end{aligned}$$

As claims 1 & 2 have been verified, this actually yields an arrow in $\mathrm{D}^b_{ctf}(\mathrm{Y}, \mathrm{R}/m^n)$. We can localize this morphism with respect to $u : \mathrm{U} \to \mathrm{Y}$ étale, as $\mathrm{R}u^! = u^*$, which we know commutes with reduction modulo m^n, and in this way, replacing Y by U, assume f factors as $\mathrm{U} \xhookrightarrow{i} \mathrm{Z} \xrightarrow{h} \mathrm{X}$ with i a closed immersion and h smooth of relative dimension d. As the above morphisms are natural, and the composition of the unit and counit $\mathrm{R}f^! \to \mathrm{R}f^!\mathrm{R}f_!\mathrm{R}f^! \to \mathrm{R}f^!$ is the identity transformation of $\mathrm{R}f^!$, and $\mathrm{R}f^! = \mathrm{R}h^!\mathrm{R}i^!$ (since the left adjoint i_* of $i^!$ is exact), it suffices to show that $\mathrm{R}h^!$ and $\mathrm{R}i^!$ commute with reduction modulo m^n. The case of $\mathrm{R}h^!$ is trivial since $\mathrm{R}h^!\mathrm{K}_{n+1} = \mathrm{K}_{n+1}(d)[2d]$. Turning to $\mathrm{R}i^!$, let $j : \mathrm{Z} - \mathrm{U} \hookrightarrow \mathrm{X}$ denote the open immersion of the complement of U; we have a commutative diagram with distinguished triangles for rows

$$\begin{tikzcd}
\mathrm{R}/m^n \otimes^{\mathbf{L}} i_*\mathrm{R}i^!\mathrm{K}_{n+1} \arrow[r] & \mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{K}_{n+1} \arrow[r] \arrow[d, equal] & \mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{R}j_*j^*\mathrm{K}_{n+1} \arrow[r, "d"] \arrow[d, "\sim"] & {} \\
i_*\mathrm{R}i^!(\mathrm{K}_{n+1} \otimes^{\mathbf{L}} \mathrm{R}/m^n) \arrow[r] & \mathrm{R}/m^n \otimes^{\mathbf{L}} \mathrm{K}_{n+1} \arrow[r] & \mathrm{R}j_*j^*(\mathrm{K}_{n+1} \otimes^{\mathbf{L}} \mathrm{R}/m^n) \arrow[r, "d"] & {} .
\end{tikzcd}$$

We obtain an isomorphism $\mathrm{R}/m^n \otimes^{\mathbf{L}} i_*\mathrm{R}i^!\mathrm{K}_{n+1} \xrightarrow{\sim} i_*\mathrm{R}i^!(\mathrm{K}_{n+1} \otimes^{\mathbf{L}} \mathrm{R}/m^n)$ from (TR3).

Returning now to $\mathrm{R}\mathscr{H}\!om$, here is a useful lemma which adapts *Th. finitude* 4.6.

Lemma. — Let X *be a noetherian scheme,* Λ *a left noetherian ring, and* $\mathrm{K} \in \mathrm{D}^-(\mathrm{X}, \Lambda)$. *Then the following are equivalent.*

(1) K *is of finite Tor-dimension and the sheaves* $\mathscr{H}^i(\mathrm{K})$ *are locally constant constructible.*
(2) K *is locally isomorphic to a bounded complex of locally constant, flat* Λ*-modules; i.e. there is a finite étale covering* $\{\mathrm{U}_i \to \mathrm{X}\}$ *such that* $\mathrm{K}|_{\mathrm{U}_i}$ *is isomorphic (in* $\mathrm{D}^-(\mathrm{U}_i, \Lambda)$*) to a bounded complex of constant sheaves of projective* Λ*-modules of finite type.*

The argument follows *Th. finitude* 4.5, except now the $\mathscr{H}^i(\mathrm{K})$ are moreover locally constant constructible. The sheaves A and B are defined by the cartesian diagram

$$\begin{tikzcd}
\mathrm{K}^n \arrow[r] & \mathrm{K}^n/\mathrm{im}\, d \arrow[r] & \ker d \\
\mathrm{A} \arrow[r, "u"] \arrow[u] & \mathrm{B} \arrow[r] \arrow[u] & \ker d \arrow[u]
\end{tikzcd}$$

and B is locally free as it sits in the middle of the exact sequence

$$0 \to \mathscr{H}^n(\mathrm{K}) \xrightarrow{(\mathrm{id},0)} \mathrm{B} \to \ker d \to \mathscr{H}^{n+1}(\mathrm{K})$$

where $\ker d$ here denotes the kernel of the differential on the complex K' being constructed and is hence locally constant constructible.

As u is surjective, localizing, we may assume B constant constructible and that u surjects on global sections, defining a morphism $v : \Lambda^{\oplus d} \to \mathrm{A}$ with the property that vu is an epimorphism, and we define $\mathrm{K}'^{n} = \Lambda^{\oplus d}$.

Suppose K is of Tor-dimension $\leq r$. Propagating the above procedure to the left as far as degree $-r-1$, we produce an étale morphism $\mathrm{U} \to \mathrm{X}$, constant constructible sheaves

$$\mathrm{K}'^{-r-1} \to \mathrm{K}'^{-r} \to \mathrm{K}'^{-r+1} \to \cdots$$

with the property that $\mathrm{K}'^{-r}/\mathrm{im}\, d$ is constant constructible and flat. Then, over U, the complex

$$\cdots \to 0 \to \mathrm{K}'^{-r}/\mathrm{im}\, d \to \mathrm{K}'^{-r+1} \to \mathrm{K}'^{-r+2} \to \cdots$$

is quasi-isomorphic to K, and has the desired properties.

Corollary (*Th. finitude 1.7*). — *If* Λ *is moreover commutative and of torsion, and in the situation of Th. finitude, then*

$$\mathrm{R}\,\mathscr{H}\!om : \mathrm{D}^b_{ctf}(\mathrm{X},\Lambda) \times \mathrm{D}^b_{tf}(\mathrm{X},\Lambda) \to \mathrm{D}^b_{tf}(\mathrm{X},\Lambda).$$

Following *Th. finitude* 1.7, as the finite Tor-dimension is stable by $\mathrm{R}\,f_*, \mathrm{R}\,f_!, f^*, \mathrm{R}\,f^!$, devissant the first variable (say, $\mathscr{F}$) relative to a partition of X, and using the adjunction

$$\mathrm{R}\,\mathscr{H}\!om(j_!\mathscr{F},\mathscr{G}) \xleftarrow{\sim} \mathrm{R}\,j_*\,\mathrm{R}\,\mathscr{H}\!om(\mathscr{F},\mathrm{R}\,j^!\mathscr{G})$$

for $j : \mathrm{Y} \to \mathrm{X}$ the inclusion of a locally closed subscheme (c.f. SGA 4, IX 2.5 & XVIII §3.1), one reduces to the situation where the $\mathscr{H}^i(\mathscr{F})$ are locally constant. Localizing, we can replace $\mathscr{F}$ by a bounded complex of constant sheaves of projective Λ-modules of finite type. As in this case $\mathscr{H}\!om$ computes pointwise, we can compute $\mathrm{R}\,\mathscr{H}\!om$ with respect to such a complex. Finally, if N and M are Λ-modules, N projective and M of Tor-dimension $\leq r$, then $\mathrm{Hom}(\mathrm{N},\mathrm{M})$ is of Tor-dimension $\leq r$, which can be seen after replacing M by a complex of flat modules 0 to the left of $-r$ and writing N as a direct summand of a free module.

Returning to the setting of the paper, with R the ring of integers of a finite extension of $\mathbf{Q}_p$, m its maximal ideal, let's assume that reduction mod m^n commutes with the four operations $\mathrm{R}\,f_*$, f^*, $\mathrm{R}\,f_!$, and $\mathrm{R}\,f^!$, and show that for $\mathscr{F},\mathscr{G}\in \mathrm{D}^b_c(\mathrm{X},\mathrm{R})$,

$$\begin{aligned}\mathrm{R}\,\mathscr{H}\!om(\mathscr{F}\otimes^{\mathbf{L}}\mathrm{R}/m^{n+1},\mathscr{G}\otimes^{\mathbf{L}}\mathrm{R}/m^{n+1})\otimes^{\mathbf{L}}_{\mathrm{R}/m^{n+1}}\mathrm{R}/m^n\\ =\mathrm{R}\,\mathscr{H}\!om(\mathscr{F}\otimes^{\mathbf{L}}\mathrm{R}/m^{n},\mathscr{G}\otimes^{\mathbf{L}}\mathrm{R}/m^{n})\qquad \text{in } \mathrm{D}^b_{ctf}(\mathrm{X},\mathrm{R}/m^n).\end{aligned}\quad (\dagger)$$

Devissant $\mathscr{F}\otimes^{\mathbf{L}}\mathrm{R}/m$ with respect to a partition of X, we may assume that its cohomology sheaves are locally constant, and therefore the same is true of $\mathscr{F}\otimes^{\mathbf{L}}\mathrm{R}/m^{n+1}$ by considering the m-adic filtration on a finite complex of flat sheaves representing it. Localizing, we may replace $\mathscr{F}\otimes^{\mathbf{L}}\mathrm{R}/m^{n+1}$ with a bounded complex N^* of free R/m^{n+1}-modules of finite type and compute $\mathrm{R}\,\mathscr{H}\!om$ with respect to N^*, since for $\mathscr{F}$ locally free, $\mathscr{H}\!om(\mathscr{F},\mathscr{G})_x=\mathscr{H}\!om(\mathscr{F}_x,\mathscr{G}_x)$. Now the equality $(\dagger)$ is clear.

Consider the example: X finite type over S, $\mathscr{F}$ and $\mathscr{G}$ two constructible torsion-free R-sheaves. The claim is that the projective system

$$\underline{\mathrm{Ext}}^i(\mathscr{F},\mathscr{G}):=\mathscr{H}^i\,\mathrm{R}\,\mathscr{H}\!om(\mathscr{F},\mathscr{G})=\text{“}\lim\mathrm{proj}\text{”}\,\underline{\mathrm{Ext}}^i_{\mathrm{R}/m^n}(\mathscr{F}\otimes\mathrm{R}/m^n,\mathscr{G}\otimes\mathrm{R}/m^n)$$

forms a constructible R-sheaf. By part (a) of 1.1.2, it suffices to show that $\mathrm{R}\,\mathscr{H}\!om(\mathscr{F},\mathscr{G})\in \mathrm{D}^b_c(\mathrm{X},\mathrm{R})$. By *Th. finitude* 1.6 and the previous corollary, $\mathrm{R}\,\mathscr{H}\!om$ sends $\mathrm{D}^b_{ctf}(\mathrm{X},\mathrm{R}/m^n)\times \mathrm{D}^b_{ctf}(\mathrm{X},\mathrm{R}/m^n)$ into $\mathrm{D}^b_{ctf}(\mathrm{X},\mathrm{R}/m^n)$, so $\mathrm{R}\,\mathscr{H}\!om(\mathscr{F}\otimes\mathrm{R}/m^n,\mathscr{G}\otimes\mathrm{R}/m^n)\in \mathrm{D}^b_{ctf}(\mathrm{X},\mathrm{R}/m^n)$. Finally, by $(\dagger)$,

$$\text{“}\lim\mathrm{proj}\text{”}\,\mathrm{R}\,\mathscr{H}\!om(\mathscr{F}\otimes\mathrm{R}/m^n,\mathscr{G}\otimes\mathrm{R}/m^n)\in \mathrm{D}^-(\mathrm{X},\mathrm{R}).$$

Truncation & Tor-dimension. In part (e), Deligne addresses the truncation operators $\tau_{\le n}$. The issue is that, while a submodule of a flat R-module is flat, a submodule of a flat R/m^n-module need not be. To address this deficiency, Deligne introduces the modified truncation operators $\tau'_{\le n}$, which preserve the finite Tor-dimension. As these properties are of a pointwise nature, we may consider the situation in the category of R-modules, and the categories $\mathrm{D}_{\mathrm{parf}}(\mathrm{R}),\mathrm{D}_{\mathrm{parf}}(\mathrm{R}/m^k)$. Applying Houzel's argument at the end of SGA 5, Exp. XV to the stalk of $\mathrm{K}\in\mathrm{D}^b_c(\mathrm{X},\mathrm{R})$, we may represent $\mathrm{K}_k=\mathrm{K}\otimes^{\mathbf{L}}\mathrm{R}/m^k$ by a bounded complex of free R/m^k-modules and the isomorphisms $\mathrm{K}_{k+1}\otimes^{\mathbf{L}}_{\mathrm{R}/m^{k+1}}\mathrm{R}/m^k\xrightarrow{\sim}\mathrm{K}_k$ by isomorphisms of complexes $\mathrm{K}_{k+1}\otimes\mathrm{R}/m^k\xrightarrow{\sim}\mathrm{K}_k$. Taking the projective limit of these complexes, we obtain a bounded complex of free R modules which we will again, as in the notes to claim a), (abusively) notate K. (To see freeness, note that if r equals the rank of $K_1^i = K^i \otimes R/m$, there exists by Nakayama an exact sequence

$$N \to R^r \to K^i \to 0$$

which after tensoring by R/m^k induces an isomorphism $(R/m^k)^r \xrightarrow{\sim} K_k^i$, showing $N \subset m^k R$ for all k and hence $N = 0$.) As before, by the Mittag-Leffler condition, $H^i(K) = \varprojlim H^i(K_k)$. Therefore, the submodule of $\ker d$ consisting of cycles whose image in $H^n(K_k)$ are in $\mathrm{im}(H^n K \to H^n(K_k))$ is the reduction modulo m^k of a flat R-module, as these cycles coincide with the reduction mod m^k of cycles of K^i, which form a free submodule of K^i. More precisely, in view of the commutative diagram

$$\begin{tikzcd} K^{n-1} \arrow[r, "d^{n-1}"] \arrow[d] & K^n \arrow[r, "d^n"] \arrow[d] & K^{n+1} \arrow[d] \\ K^{n-1} \otimes R/m^n \arrow[r, "d^{n-1}\otimes \mathrm{id}"] & K^n \otimes R/m^k \arrow[r, "d^n \otimes \mathrm{id}"] & K^{n+1} \otimes R/m^k \end{tikzcd}$$

the submodule of $\ker(d^i \otimes \mathrm{id})$ consisting of cycles whose image in H^i lie in $\mathrm{im}(H^i(K) \to H^i(K \otimes R/m^k))$ is the submodule $(\ker d^n) \otimes R/m^k + \mathrm{im}(d^{n-1} \otimes \mathrm{id}) \subset K^n \otimes R/m^k$. As $\mathrm{im}(d^{n-1} \otimes \mathrm{id}) \subset (\ker d^n) \otimes R/m^k$, confirming the above description and recognizing $\tau'_{\leq n} K_k$ as a bounded complex of free R/m^k modules. It may be that $\tau'_{\leq n} K_k$ is no longer quasi-isomorphic to K_k, but it is clear by construction that the pro-sheaf $H^n(K_k)$ is not affected by the operator $\tau'_{\leq n}$; more precisely, by the exact sequence (*) of claim a), $H^n(\tau'_{\leq n} K_k) \simeq H^n(K) \otimes_R R/m^k$ and hence both are AR-isomorphic to $H^n(K_k)$.

1.1.3. Extension of scalars from R to E for a constructible R-sheaf $\mathscr{F}$ is more or less straightforward: letting $E = \mathbf{Q}_\ell, R = \mathbf{Z}_\ell$, the idea is that if is $\mathbf{Z}_\ell$-constructible, the torsion subsheaves $\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathscr{F}, \mathbf{Z}/\ell^n) = \ker \ell^n$ stabilize for some n, and then multiplication by ℓ on $\mathscr{F}$ means multiplying $\mathscr{F} \otimes \mathbf{Z}/\ell^m$ by ℓ^n for each m to produce an AR-ℓ-adic sheaf, and then forming the associated ℓ-adic sheaf, to produce the exact sequence of ℓ-adic sheaves

$$0 \to \mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathscr{F}, \mathbf{Z}/\ell^n) \to \mathscr{F} \xrightarrow{\ell^n} \mathscr{F}/\ker \ell^n \to 0.$$

The story in $D_c^b(X, \mathbf{Z}_\ell)$ works more or less the same way. For an object K in this category, we represent $K_m = K \otimes^{\mathbf{L}} \mathbf{Z}/\ell^m$ by a complex of $\mathbf{Z}/\ell^m$-flat sheaves (i.e. with free stalks); multiplication by ℓ^n means multiplying K_m by ℓ^n. This induces multiplication by ℓ on the cohomology, and we're back in the previous situation, since the cohomology sheaves form AR-ℓ-adic systems. More generally, if a is in $\mathbf{Z}_\ell$, letting a_m denote the image of a in $\mathbf{Z}/\ell^m$, the commutativity of the diagram

$$\begin{tikzcd} \mathrm{K}_{m+1} \arrow[r, "a_{m+1}"] \arrow[d, "\otimes \mathbf{Z}/\ell^m"] & \mathrm{K}_{m+1} \arrow[d, "\otimes \mathbf{Z}/\ell^m"] \\ \mathrm{K}_m \arrow[r, "a_m"] & \mathrm{K}_m \end{tikzcd}$$

shows that a induces an endomorphism of $\mathrm{D}^b_c(\mathrm{X}, \mathbf{Z}_\ell)$.

1.1.7. A representation $\mathbf{Z} \to \mathrm{GL}(\mathrm{V})$, sending n to F^n, where V is a $\mathbf{Q}_\ell$-vector space of dimension n, is continuous if and only if the eigenvalues of F are ℓ-adic units.

To see sufficiency, note that we can choose a basis for V so that the morphism $\mathbf{Z} \to \mathrm{GL}(\mathrm{V})$ factors through $\mathrm{GL}(\mathbf{Z}_\ell^n) = \varprojlim \mathrm{GL}((\mathbf{Z}/\ell^m)^n)$, a profinite group, and we may extend $\mathbf{Z} \to \mathrm{G}$ to a morphism $\hat{\mathbf{Z}} \to \mathrm{G}$ for any profinite group G by the universal property of profinite completion, which we state and prove now.

The profinite completion of a group H (with respect to normal subgroups of finite index in H) is denoted $\hat{\mathrm{H}}$, so that $\mathrm{H} \to \hat{\mathrm{H}}$ has dense image. The profinite completion $\hat{\mathrm{H}}$ enjoys the universal property that for every profinite group G and continuous homomorphism $\mathrm{H} \to \mathrm{G}$, there is a unique homomorphism $\hat{\mathrm{H}} \to \mathrm{G}$ making the diagram

$$\begin{tikzcd} \mathrm{H} \arrow[r] \arrow[d] & \hat{\mathrm{H}} \arrow[dl] \\ \mathrm{G} & \end{tikzcd}$$

commute.

To see this, simply use the description of G as $\varprojlim \mathrm{G}/\mathrm{N}$ as N ranges over open normal subgroups of G. The preimage M of N in H is an open normal subgroup of finite index, as G/N is finite. Therefore $\mathrm{H} \to \mathrm{G}/\mathrm{N}$ factors through H/M, and to give a continuous morphism from $\hat{\mathrm{H}}$ to G it suffices to give compatible continuous maps $\hat{\mathrm{H}} \to \mathrm{G}/\mathrm{N}$. Continuity is assured by the above remark; compatibility is assured by the map $\mathrm{H} \to \mathrm{G}$, which determines the maps $\hat{\mathrm{H}} \to \mathrm{G}/\mathrm{N}$.

Returning to (1.1.7), to see necessity, we assume we have found a continuous extension ρ.

$$\begin{tikzcd} \mathbf{Z} \arrow[rr] \arrow[dr] & & \mathrm{GL}(\mathrm{V}) \\ & \hat{\mathbf{Z}} \arrow[ur, dashed, "\rho"] & \end{tikzcd}$$

The image $\rho(\hat{\mathbf{Z}})$ is compact, so the set $\rho(\hat{\mathbf{Z}})\mathbf{Z}_\ell^n$ is compact, for $\mathbf{Z}_\ell^n$ a $\mathbf{Z}_\ell$-lattice in V, so $\rho(\hat{\mathbf{Z}})\mathbf{Z}_\ell^n \subset \frac{1}{\ell^m}\mathbf{Z}_\ell^n$ for some m. Letting L denote the $\mathbf{Z}_\ell$-span of $\rho(\hat{\mathbf{Z}})\mathbf{Z}_\ell^n$, L is a $\mathbf{Z}_\ell$-submodule of $\frac{1}{\ell^m}\mathbf{Z}_\ell^n$, hence free (of rank n). This recognizes $\mathrm{F} \in \rho(\hat{\mathbf{Z}})$ as an element of $\mathrm{Aut}(\mathrm{L})$, so the eigenvalues of F are ℓ-adic units indeed.

1.1.8. The second paragraph continues the first; i.e. the point y remains in force, and if $\overline{y}$ denotes a geometric point centered on y, the point y determines a unique element of $\pi_1(\mathrm{X}, \overline{y})$; viz. the image of F under $\mathrm{Gal}(\overline{k}/k) \to \pi_1(\mathrm{X}, \overline{y})$. The groups $\pi_1(\mathrm{X}, \overline{x})$ and $\pi_1(\mathrm{X}, \overline{y})$ are isomorphic, but not canonically: one must make a choice of path γ from $\overline{y}$ to $\overline{x}$. Two different choices give rise to two different maps, which differ by conjugation by an element of $\pi_1(\mathrm{X}, \overline{x})$. This is why F_y is determined canonically in $\pi_1(\mathrm{X}, \overline{x})$ only up to conjugation. See the introduction of Stix, *Rational Points and Arithmetic of Fundamental Groups*.

1.1.13. As discussed in the note to 1.1.8, F_y is determined canonically in $\pi_1(\mathrm{X}_0, \overline{x})$ only up to conjugation in $\pi_1(\mathrm{X}_0, \overline{x})$. Any element of its conjugacy class $[\mathrm{F}_y]$ also belongs to $\mathrm{W}(\mathrm{X}_0, \overline{x})$, but in order for the conjugacy class of F_y to be well-defined in $\mathrm{W}(\mathrm{X}_0, \overline{x})$, all the different choices of F_y should be in the same $\mathrm{W}(\mathrm{X}_0, \overline{x})$-conjugacy class; i.e. if the set of orbits for the action of $\mathrm{W}(\mathrm{X}_0, \overline{x})$ on $[\mathrm{F}_y]$ by conjugation is denoted by $\mathcal{O}$, $|\mathcal{O}|$ should equal 1. As $\mathrm{W}(\mathrm{X}_0, \overline{x})$ is a normal subgroup of $\pi_1(\mathrm{X}_0, \overline{x})$, the latter group acts on $\mathcal{O}$ by conjugation. Evidently this action factors through the quotient by geometric monodromy, giving an action of $\mathrm{Gal}(\overline{k}, \mathbf{F}_q)$ on $\mathcal{O}$ that factors through the quotient by the image of the centralizer (in $\pi_1(\mathrm{X}_0, \overline{x})$) of F_y (any two choices of F_y are conjugate by an element of $\pi_1(\mathrm{X}_0, \overline{x})$, and their centralizers are conjugate by this same element, so the image of their centralizers is the same in $\mathrm{Gal}(\overline{k}/\mathbf{F}_q)$). This image is an open subgroup and $\pi_1(X_0, \bar{x})$ acts via the quotient $\hat{\mathbf{Z}}/\deg y.\hat{\mathbf{Z}} = \mathbf{Z}/\deg y.\mathbf{Z}$. As $W(X_0, \bar{x})$ surjects onto this quotient, the action of $\pi_1(X_0, \bar{x})$ on $\mathfrak{S}$ is trivial and $|\mathfrak{S}| = 1$. See Stix, §1.4, §3.3.

1.2.5. (i) Stability under reciprocal image follows since the stalk doesn't change, the cardinality of the residue field (at a closed point) increases by the same power as does the Frobenius.

(ii) Stability under direct image by a finite morphism: by base change one reduces to $f : \operatorname{Spec} \mathbf{F}_{q^n} \to \operatorname{Spec} \mathbf{F}_q$ for some n and a representation V of F^n, where F is the geometric Frobenius on $\operatorname{Spec} \mathbf{F}_q$. Suppose that V is ι-pure of weight w. Then $f_* V = V \otimes_{\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{q^n})} \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$ which is isomorphic to V tensored with the permutation representation of $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$ on $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)/\mathrm{Gal}(\mathbf{F}/\mathbf{F}_{q^n}) \simeq \mathbf{Z}/n\mathbf{Z}$. As F^n acts on this induced representation with all eigenvalues ι-pure of weight w rel. q^n, F acts with all eigenvalues ι-pure of weight w rel. q.

1.2.6. 'On notera que, pour k un corps fini, une representation V de $W(\bar{k}, k)$ est automatiquement ι-mixte.' $\rightsquigarrow$ This follows from the existence of Jordan normal form.

1.2.9. This conjecture is now proved by L. Lafforgue. If $\mathscr{F}_0$ is an irreducible lisse sheaf on a $\mathbf{F}_q$-variety X_0, then after a twist (which will be an ℓ-adic unit when $\mathscr{F}_0$ is an étale sheaf and not merely a Weil sheaf), the maximal exterior power of $\mathscr{F}_0$ is defined by a character of finite order (1.3.6). Lafforgue proves (see Laumon, *La correspondance de Langlands sur les corps de fonctions*, §1.3) that when X_0 is normal, such an irreducible lisse sheaf with maximal exterior power defined by a character of finite order is pure of weight 0. This implies that $\mathscr{F}_0$ was pure before the twist, so that every irreducible lisse sheaf on X_0 is pure, and therefore, every lisse sheaf on X_0 is mixed. Of course if X_0 is not normal, with normalization $f : X_0' \to X_0$, and $\mathscr{F}_0$ is lisse, then $\mathscr{F}_0 \hookrightarrow f_* f^* \mathscr{F}_0$, so $\mathscr{F}_0$ is mixed. This shows that every lisse sheaf on a $\mathbf{F}_q$-variety is mixed, and hence that every constructible sheaf on a $\mathbf{F}_q$-variety is mixed.

1.3.9. (When reading the corollary, recall that a semisimple algebraic group is connected by definition.) We wish to understand why G^{00} is reductive. Note first that the sum of the simple $\pi_1(X, \bar{x})$-modules is W-stable since if $w \in W$ and V is a $\pi_1(X, \overline{x})$-module, then wV is again a $\pi_1(X, \overline{x})$-module since $\pi_1(X, \overline{x})$ is a normal subgroup of W; applying this argument with w^{-1} shows that wV is simple iff V is. Next observe that if $\rho : W(X_0, \overline{x}) \to GL(\mathscr{F}_{\overline{x}})$ is the representation defining $\mathscr{F}_0$, then $\rho(\pi_1(X, \overline{x}))$ and its Zariski closure G^0 have the same invariant subspaces (to see this, form a basis for $\mathscr{F}_{\overline{x}}$ beginning with a basis for an invariant subspace). Therefore we see that G^0 acts semisimply since $W(X_0, \overline{x})$ does by assumption.

Now recall that $R(G^0)$, the radical of G^0, is a connected and solvable normal subgroup of G^0. By the argument above, any normal subgroup of G^0 acts semisimply; combining this with the Lie-Kolchin theorem, we see that $\mathscr{F}_{\overline{x}}$ decomposes as a direct sum of one-dimensional irreducible $R(G^0)$-modules. The unipotent part of $R(G^0)$, which is the unipotent radical $R_u(G^0)$, must therefore act by the identity, and we see that $R_u(G^0) = R_u(G^{00}) = \{1\}$; i.e. that G^{00} is reductive. Note we have proved the following

Lemma. — If V *is a finite-dimensional vector space over an algebraically closed field and* G *is a closed subgroup of* GL(V)*, then* G *is reductive.*

This result appears in [**Milne**, 21.60] as

Proposition. — Let G *be a connected group variety over a perfect field* k*. The following conditions on* G *are equivalent.*

(1) G *is reductive;*
(2) The radical R(G) *of* G *is a torus;*
(3) G *is an almost-direct product of a torus and a semisimple group;*
(4) G *admits a semisimple representation with finite kernel.*

More is true. In fact, for G a connected reductive group, say, over an algebraically closed field k, the maximal central k-torus Z coincides with $(\mathscr{C}G)^\circ$, the connected component of the identity of the center of G, and the multiplication homomorphism $Z \times \mathscr{D}(G) \to G$ is a central isogeny, i.e. an isogeny with central kernel, where $\mathscr{D}(G) = (G, G)$ is the derived subgroup. This implies that $Z \to G/\mathscr{D}(G)$ is a central isogeny. Here, our G is Deligne's G^{00}, our Z is Deligne's T_1, and our $G/\mathscr{D}(G)$ is Deligne's T, as a connected, smooth, reductive, and commutative group is a torus [**Milne**, 19.12], and a quotient of a reductive group over a field of characteristic 0 is reductive.

The set F of characters by which T_1 acts on $\mathscr{F}_{\bar{x}}$ generates $X(T_1)$ since the representation of T_1 is faithful, and, as T_1 is a torus, diagonalizable. Therefore, with the right choice of basis, the representation of T_1 looks like $\mathrm{diag}(\chi_i)$ for characters $\chi_i \in F$. As the representation is faithful, these characters generate the character group $X(T_1)$. (The character group of $\mathrm{diag}(\chi_i)$, which is isomorphic to the character group of T_1, is generated by the χ_i.)

The group $W(X_0, \bar{x})$ acts on G^0 by conjugation. Recall that the neutral component of an algebraic group is a characteristic subgroup, and so is the center. Therefore T_1, which can be described as the neutral component of $Z(G^{00})$, is acted upon by $W(X_0, \bar{x})$. Recall that the functor X which takes an algebraic group to its character group induces a contravariant equivalence from the category of diagonalizable algebraic groups with the finitely generated commutative groups, and as we have seen, $W(X_0, \bar{x})$ acts on $X(T_1)$ by permuting factors, hence through a finite quotient.

We would like to know why the group of outer automorphisms of G^{00} restricting to the identity on T_1 is finite. The group G^{00} admits a maximal split torus T_2 so that (G^{00}, T_2) is a split reductive group. The radical $R(G^{00}) = T_1$ is the largest subgroup of the multiplicative group $Z(G^{00})$, so the quotient $Z(G^{00})/R(G^{00})$ is finite [**Milne**, 19.10]. Recall the definition of isomorphism of root data [**Milne**, 23.2]. An isomorphism φ of split reductive groups defines an isomorphism f of root data, and every isomorphism of root data f arises from a φ, unique up to an inner automorphism [**Milne**, 23.26]. Moreover for a split reductive group (G, T) we have a canonical isomorphism $\mathrm{Out}(G) \simeq \mathrm{Aut}(X, \Phi, \Delta)$, where the latter is automorphisms of based root data [**Milne**, 23.46]. Given such a $\varphi : (G, T) \to (G', T')$, the map f is defined by the formula $f(\chi') = \chi' \circ \varphi|_T$ for $\chi' \in X(T')$ [**Milne**, 23.5]. Suppose φ is now an automorphism of (G^{00}, T_2) and restricts to the identity on the radical T_1. The isomorphism f is *a fortiori* a central isogeny and its action on $\mathbf{Z}\Phi$ (the $\mathbf{Z}$-submodule of $X^*(T_2)$ generated by the roots Φ) preserves the base Δ, hence its action on $\mathbf{Z}\Phi$ amounts to permuting a finite set. On the other hand, the quotient $T_2/Z(G^{00})$ has character group the subgroup $\mathbf{Z}\Phi$ of $X^*(T_2)$ [**Milne**, 21.9], hence the the root lattice $\mathbf{Z}\Phi$ has finite index in $X^*(T_2/T_1)$. As $Z(G^{00})/T_1$ is finite, this is enough to conclude that subgroup of $\mathrm{Aut}(X, \Phi, \Delta)$ corresponding to automorphisms of G^{00} which restrict to the identity on T_1 is finite, hence that the subgroup of $\mathrm{Out}(G^{00})$ consisting of those automorphisms fixing T_1 is also finite.

Now, if w is an element of $W(X_0, \overline{x})$ of degree 1, and $\overline{w}$ the image of w in $GL(\overline{F}_{\overline{x}})$, G is the semi-direct product of $\mathbf{Z}$ by $G^0 = G^{00}$ relative to the action $\mathrm{int}(\overline{w})$ of $\mathbf{Z}$ on G^0. As this action is given by an interior automorphism of G^0, by multiplying w by an element of $\pi(X, \overline{x})$, we make the action of $\mathrm{int}(\overline{w})$ trivial, and recognize $G \simeq G^{00} \times \mathbf{Z}$.

The proof of (1.3.9) follows easily from (1.3.8). Note that once one has reduced to $\mathscr{F}_0$ semisimple, it is easy to see that the radical of G^{00} in this case is trivial, as it is by definition the largest connected solvable normal subgroup variety of G^{00}, hence contained in the connected component of the identity of G^0, so if G^0 is an extension of a finite, hence discrete, group, $R(G^{00})$ lies in the kernel of this extension, namely in the semisimple subgroup, so in fact $R(G^{00}) = \{e\}$, as connected normal subgroup varieties of a semisimple group are semisimple [**Milne**, 21.52].

1.3.10. Note that (iv) should read 'Le centre de G s'envoie sur un sous-groupe d'indice fini de $\mathbf{Z}$.' The crux of the direction $(iv) \Rightarrow (i)$ is that, while G is not *a priori* a linear algebraic group, G/Z, as an extension of a finite group by a linear algebraic group, is.

1.3.12. The central element g acts by a scalar by Schur's lemma.

1.3.13. (i) The claim rests on the following

Lemma. — Let X_0, X'_0 *be normal connected schemes of finite type over a field with generic points* ξ, ξ' *and function fields* $K = k(\xi)$ *and* $K' = k(\xi')$. *Let* Ω, Ω' *be algebraically closed extensions of* K, K'*, defining geometric points* a, a' *of* X_0, X'_0 *centered on* ξ, ξ'*, respectively. If* $f : X'_0 \to X_0$ *is a dominant morphism, then the image of the induced map* $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ *is an open subgroup of finite index.*

Observing that $\pi_1(X'_0, a')$ acts on $(f^*\mathscr{F})_{a'}$ via the map on π_1 in the lemma induced by f, we see that there is a central element $g \in G'$ of positive degree and a morphism $G' \to G$ sending g to a central element of G of positive degree, and the action of g on $\mathscr{F}_{0,a}$ via this map is the same as the action of g on $(f^*\mathscr{F}_0)_{a'}$.

Proof of lemma. — The extensions Ω, Ω' define geometric points a_1, a'_1 of $\mathrm{S} = \mathrm{Spec}(\mathrm{K})$ and $\mathrm{S}' = \mathrm{Spec}(\mathrm{K}')$, respectively. The dominant morphism $f : \mathrm{X}'_0 \to \mathrm{X}_0$ induces an extension of fields $\mathrm{K} \subset \mathrm{K}'$. Then $\pi_1(\mathrm{S}, a_1) \to \pi_1(\mathrm{X}_0, a)$ is surjective [**SGA1**, V 8.2], and after identifying $\pi_1(\mathrm{S}, a_1)$ with $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}, \mathrm{K})$, the kernel is identified with those automorphisms which fix all finite extensions of K in Ω which are unramified over X_0, and likewise for $\pi_1(\mathrm{S}', a'_1) \to \pi_1(\mathrm{X}'_0, a')$. If L is an extension of K unramified over X_0, then $\mathrm{L} \otimes_{\mathrm{K}} \mathrm{K}'$ is an extension of K' unramified over X'_0 [**SGA1**, I 10.4(iii)]. The operation on étale covers of X_0 consisting of taking inverse image along f followed by fiber at a' is a fiber functor for X_0, hence induces a continuous homomorphism of groups $\pi_1(\mathrm{X}'_0, a') \to \pi_1(\mathrm{X}_0, a)$ [**SGA1**, V 6.2]. The action of $\pi_1(\mathrm{X}'_0, a')$ on $(f^*\mathscr{F}_0)_{a'}$ is by restriction with respect to this homomorphism. This homomorphism, in turn, is induced by restriction of $\pi_1(\mathrm{S}', a'_1) \to \pi_1(\mathrm{S}, a)$, since if L is as above, an automorphism $\sigma \in \ker(\pi_1(\mathrm{S}', a'_1) \to \pi_1(\mathrm{X}'_0, a')$ acts on $\mathrm{L} \otimes_{\mathrm{K}} \mathrm{K}'$ by the identity as $\mathrm{L} \otimes_{\mathrm{K}} \mathrm{K}'$ is unramified. As K'/K is finitely generated, $\mathrm{K}' \cap \mathrm{K}^{\mathrm{sep}}$ is a finite extension of K, so the image of $\pi_1(\mathrm{S}', a'_1) \to \pi_1(\mathrm{X}_0, a)$ is an open subgroup of finite index isomorphic to the image of $\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K}' \cap \mathrm{K}^{\mathrm{sep}})$ in $\pi_1(\mathrm{X}_0, a)$. □

(ii) Choose a basis for representations corresponding to $\mathscr{F}_0$ and $\mathscr{G}_0$ so that Frobenius is upper-triangular in both, and then recall the form of the Kronecker (tensor) product of matrices, which has the property that the Kronecker product of upper-triangular matrices is upper-triangular.

(iii) The claim rests on two observations. The first is that if $\mathscr{F}_0$ is defined by a representation V of G, the eigenvalues of any $g \in \mathrm{G}$ coincide with the eigenvalues of g acting on the semi-simplification of V with respect to any Jordan-Hölder series. To see this, choose a basis for each graded piece so that g is upper-triangular, and then order a lift of these bases according to the filtration, beginning with the smallest piece. The second observation is that if we begin with an ordered basis (a_i) for V with respect to which g is upper triangular, then a basis B for $\bigwedge^a \mathrm{V}$ consisting of a-forms in the a_i can be found. If the function w takes an a-form in the a_i and outputs the sum of the subscripts which appear (so $w(a_1 \wedge a_3 \wedge a_4) = 8$), then g is upper-triangular with respect to any ordering of B which respects the total order w. The claim follows.

1.3.14. It suffices to show that the image of $W(X_0, x)$ in $GL(r, E)$ is bounded by the argument of (1.1.7), which we repeat now. We lose nothing by supposing $E = \mathbf{Q}_\ell$, in which case the image W of $W(X_0, x)$ in $GL(r, E)$ is bounded if it is contained in $\frac{1}{\ell}\,GL(r, \mathbf{Z}_\ell)$. Applying W to $\mathbf{Z}_\ell^r$ and taking the $\mathbf{Z}_\ell$ span, we get a free $\mathbf{Z}_\ell$-submodule of $\frac{1}{\ell}\mathbf{Z}_\ell^r$ of rank r, on which W acts by automorphisms. This recognizes W as isomorphic to a subgroup of $GL(r, \mathbf{Z}_\ell)$, a profinite group to which it is easy to extend a map $W(X_0, x) \to GL(r, \mathbf{Z}_\ell)$ to a map from the completion $\pi_1(X_0, x) \to GL(r, \mathbf{Z}_\ell)$.

To see that $\rho(W_1^0) \subset G^{00}$ is compact and Zariski dense, observe that W_1^0 is a closed subgroup of $\pi_1(X, \overline{x})$, hence a profinite group, and G^0 is by definition the Zariski closure of the image of $\pi_1(X, \overline{x})$. In particular, the inverse image of G^{00} is Zariski dense in G^{00}.

1.3.15. Relevant sources are Bourbaki, *Lie Groups and Lie Algebras* II, §7, Demazure and Gabriel, *Groupes Algébriques*, II, §6, [**Milne**, 10, 14d]. Bourbaki explains how to extend the logarithm to the union of all compacta. You need to know that for all compact $G \subset H(E)$, $x \in G$, and neighborhood about e, there is a strictly increasing sequence of integers (n_i) such that $x^{n_i} \in V$, which allows one to extend the logarithm by Deligne's formula. Bourbaki also explains that there is an open subgroup V of e in $H(E)$ such that log is an analytic isomorphism of V onto an open subgroup of $\operatorname{Lie} H$, with inverse exp. It follows that L^1, the E-linear span of $\log K$, coincides with the E-linear span of $\log(K \cap V)$. We have for $X \in \log H$ that $\exp(nX) = \exp(X)^n$, and $\log(g^n) = n \log(g)$ for any g where log is defined.

For $g \in H(E)$, $X \in \operatorname{Lie} H$,

$$g \cdot \exp(X) \cdot g^{-1} = \exp(\operatorname{Ad}(g)(X)), \tag{†}$$

whenever these expressions converge. Taking $X \in \log(K \cap V)$ and $g \in V \cap K$, we find that the left side is in $K \cap V$. Moreover there is an $n \in \mathbf{Z}$ such that both $n \operatorname{Ad}(g)(X)$ and nX lie in $\log V$. We find that $\operatorname{Ad}(g)(nX) \in K \cap V$, therefore that for $g \in V \cap K$, $\operatorname{Ad}(g)$ preserves L^1, therefore L^1 is also preserved by V, which is the Zariski closure of $V \cap K$ in K.

Therefore the adjoint representation

$$\operatorname{Ad} : H \to GL(\operatorname{Lie} H)$$

factors through the algebraic subgroup $\mathrm{F} \subset \mathrm{GL}(\mathrm{Lie\,H})$ which fixes the L^1. Applying the functor Lie, one finds that

$$\mathrm{ad} : \mathrm{Lie\,H} \longrightarrow \mathfrak{gl}(\mathrm{Lie\,H})$$

factors through Lie F. As ad induces the bracket on Lie H, it follows that L^1 is an ideal in Lie H. As H is semisimple over a field of characteristic 0, Lie subalgebras of Lie H are in bijection with connected algebraic subgroups of H (c.f. e.g. Demazure and Gabriel, *Groupes Algebriques*, II §6 2.4 & 2.7). As the exponential is functorial (c.f. *ibid*, 3.4), the algebraic subgroup corresponding to $\mathfrak{k}$ contains K, hence must equal H by density, hence $\mathrm{L}^1 = \mathrm{Lie\,H}$.

Let N denote the normalizer of K and $g \in \mathrm{N}$. Let $\mathrm{X} \in \mathrm{K}$. Then there is an integer n such that both $\mathrm{X}^n \in \mathrm{V}$ (hence $n \log \mathrm{X} \in \log \mathrm{V}$) and $n\,\mathrm{Ad}(g)(\log \mathrm{X}) \in \log \mathrm{V}$. Applying (†) to $\log(\mathrm{X}^n) = n \log \mathrm{X}$ we find that $\exp(\mathrm{Ad}(g)(n \log \mathrm{X})) = g \cdot \mathrm{X}^n \cdot g^{-1} \in \mathrm{V} \cap \mathrm{K}$ so that $\log(g \cdot \mathrm{X} \cdot g^{-1}) = \mathrm{Ad}(g)(\log \mathrm{X})$ and we find $\mathrm{Ad}(g)$ preserves the set $\log \mathrm{K}$ and *a fortiori* L^0.

The morphism Ad factors as a quotient $\mathrm{H} \twoheadrightarrow \mathrm{H}/\mathrm{Z}(\mathrm{H})$ followed by a closed immersion. As H is semisimple, Z(H) is finite, hence the quotient $\mathrm{H} \twoheadrightarrow \mathrm{H}/\mathrm{Z}(\mathrm{H})$ is finite (*a fortiori* proper) [**Milne**, 21.7, 7.15, 5.39].

The subgroup $\mathrm{K} \subset \mathrm{H}(\mathrm{E})$ is a compact subset of a complete metric space, hence closed for the topology induced by the non-archimedean metric on E. Hence K is a closed Lie subgroup of H(E) for that metric, and hence N is a closed subgroup of H(E) with respect to the metric induced by the one on E. As L^0 is compact and isomorphic to an $\mathscr{O}_{\mathrm{E}}$-lattice in Lie H, its automorphism group $\mathrm{Aut\,L}^0$ is compact; as Ad is proper, $\mathrm{Ad}^{-1}(\mathrm{Aut\,L}^1) \subset \mathrm{H}(\mathrm{E})$ is compact, and $\mathrm{N} \subset \mathrm{Ad}^{-1}(\mathrm{Aut\,L}^0)$ is a closed subgroup, hence also compact.

1.4.1. (*b*) See Weil I, (2.9).

1.4.2. Let $\overline{x}$ be a geometric point of X; as X_0 is absolutely irreducible, X is connected. The pullback of lisse sheaves along the morphism $\mathrm{X} \longrightarrow \mathrm{X}_0$ identifies with the restriction of representations along the continuous homomorphism $\pi_1(\mathrm{X}, \overline{x}) \longrightarrow \mathrm{W}(\mathrm{X}_0, \overline{x})$, and likewise the pullback of lisse sheaves along the structural morphism $X_0 \to \mathrm{Spec}(\mathbf{F}_q)$ with restriction along $W(X_0,\bar{x}) \to \mathbf{Z}$. Given a lisse sheaf $\mathscr{F}$ on X_0 with monodromy representation V, the largest subsheaf (resp. quotient sheaf) becoming constant on X is obtained by taking invariants (resp. coinvariants) of V with respect to $\pi_1(X,\bar{x})$. Both $V^{\pi_1(X,\bar{x})}$ and $V_{\pi_1(X,\bar{x})}$ carry natural actions of $\mathbf{Z}$ which induces lisse sheaves F_0', F_0'' on $\mathrm{Spec}(\mathbf{F}_q)$ with inverse images $V^{\pi_1(X,\bar{x})}$ and $V_{\pi_1(X,\bar{x})}$, respectively. (The exact sequence $0 \to \pi_1(X,\bar{x}) \to W(X_0,\bar{x}) \to \mathbf{Z}$ identifies those lisse sheaves invariant under geometric monodromy with the inverse image of sheaves on $\mathrm{Spec}(\mathbf{F}_q)$.)

1.4.3. The point is that on the one hand, the constituents of the sheaves F', F'' are among the constituents of $\mathscr{F}_0$, on the other hand as representations of $W(X_0,\bar{x})$, F', F'' are invariant for geometric monodromy, so they have one-dimensional constituents which are determined once Frobenius is put in Jordan normal form. Therefore the eigenvalues of Frobenius on F' and F'' appear among the determinental weights for $\mathscr{F}_0$, and, in consideration of (1.4.2), up to a twist the same is true of eigenvalues of Frobenius on $H^0(X,\mathscr{F}), H^0_c(X,\mathscr{F})$, and $H^2_c(X,\mathscr{F})$.

1.4.6. See Ahlfors, *Complex Analysis*, Ch. 5 §2.2 for a characteristically elegant review of the convergence properties of infinite products, which elucidates the equivalence of the absolute convergence of Deligne's Euler product with that of of his geometric series.

1.5.1. Perhaps the only thing to remark is that if a lisse sheaf $\mathscr{F}$ on X is ι-real, then all of its exterior powers are, too: choosing a basis for $\mathscr{F}_x$ with respect to which F_x is upper-triangular, the resulting canonical basis for $\bigwedge \mathscr{F}$ can be ordered so that F_x remains upper triangular (1.3.13 iii), which makes it easy to see that $\iota \det(1 - F_x t, \bigwedge \mathscr{F})$ has coefficients which are symmetric polynomials in the eigenvalues of F_x. As the coefficients of $\iota \det(1 - F_x t, \mathscr{F})$ are the elementary symmetric polynomials in these eigenvalues, and are real, the coefficients of $\iota \det(1 - F_x t, \bigwedge \mathscr{F})$ are real too.

1.6.11. To see the Clebsch-Gordon decomposition (1.6.11.2), let

$$\mathrm{H} = du\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the notation of (1.6.8), and let

$$\chi_d(\lambda) = \mathrm{Tr}(e^{\lambda \mathrm{H}}) = \sum_{\substack{j=-d \\ j \equiv d (2)}}^{d} \lambda^j \tag{†}$$

as a function of $\lambda \in k$. These characters transform additively under direct sum and multiplicatively under tensor product, so finding the decomposition of $\mathrm{S}_d \otimes \mathrm{S}_{d'}$ is the same as finding the additive decomposition of

$$(\lambda^{-d} + \lambda^{-d+2} + \cdots + \lambda^d)(\lambda^{-d'} + \lambda^{-d'+2} + \cdots + \lambda^{d'})$$

into sums of the form (†). In this case the decomposition is into $d' + 1$ sums of the form $\chi_j(\lambda)$ for $j \in \mathrm{P}(d, d')$.

1.6.13. It is asserted that the inclusion $\subset$ of assertion 2) results from the fact that the image of $\mathrm{N}^i\mathrm{M}_i$ in $\mathrm{Gr}_0^{\mathrm{W}}(\mathrm{V})$ is $\mathrm{N}^i\mathrm{M}_i(\mathrm{Gr}_0^{\mathrm{W}}\mathrm{V}) = \mathrm{M}_i(\mathrm{Gr}_0^{\mathrm{W}}\mathrm{V})$. To see this, recall that in the construction (1.6.1), N^{d-1} sends $\ker \mathrm{N}^d/\mathrm{im}\,\mathrm{N}^d$ onto $\mathrm{M}_{-d+1}/\mathrm{im}\,\mathrm{N}^d$, hence sends M_{d-1} onto M_{-d+1}, and proceed inductively.

The inequality $k - 2i - 2 \geq k \geq 2j - k$ should read $k - 2i - 2 \geq -k \geq 2j - k$ at the end of the discussion of 3). Note $\mathrm{N}^{k-j} : \mathrm{Gr}_k^{\mathrm{M}}\mathrm{G} \to \mathrm{Gr}_{2j-k}^{\mathrm{M}}\mathrm{G}$.

1.6.14. It seems as though (1.6.14.3) should read

$$\mathrm{Gr}_i^{\mathrm{M}}\mathrm{V} \simeq \bigoplus_{\substack{j \geq |i| \\ j \equiv i\ (2)}} \mathrm{P}_{-j}\left(\frac{-i-j}{2}\right).$$

For, when $i \leq 0$, N induces an isomorphism of $\mathrm{Gr}_i^{\mathrm{M}}\mathrm{V}/\mathrm{P}_i$ onto $\mathrm{Gr}_{i-2}^{\mathrm{M}}\mathrm{V}$. Scaling $\mathrm{N} \mapsto \lambda\mathrm{N}$ also multiplies this isomorphism by λ, so we need to twist by $\otimes\mathrm{N}^{-1}$. Similarly, the isomorphism $\mathrm{N}^i : \mathrm{Gr}_i^{\mathrm{M}}\mathrm{V} \xrightarrow{\sim} \mathrm{Gr}_{-i}^{\mathrm{M}}\mathrm{V}$ scales by λ^i so we need to twist by $-i$.

With this modification, the isomorphism

$$\mathrm{P}_{-j} \simeq \bigoplus_{j\in \mathrm{P}(j',j'')} \mathrm{P}'_{-j} \otimes \mathrm{P}''_{-j}\left(\frac{j-j'-j''}{2}\right)$$

is justifiable by passing through a suitable graded piece

$$\mathrm{P}_{-j}\left(\frac{-i-j}{2}\right) \to \mathrm{Gr}^{\mathrm{M}}_i \mathrm{V} \leftarrow \bigoplus_{j\in \mathrm{P}(j',j'')} \mathrm{P}'_{-j} \otimes \mathrm{P}''_{-j}\left(\frac{-i-j'-j''}{2}\right).$$

As for the isomorphism

$$\mathrm{P}_{-j}(\mathrm{V}^*) \simeq \mathrm{P}_{-j}(\mathrm{V})^*(j),$$

consider $\mathrm{V} = \mathrm{S}_j$ identified with the representation of $\mathrm{SL}(2)$ on homogeneous polynomials in variables x and y of degree j. Suppose $\mathrm{P}_{-j}(\mathrm{V})$ is generated by the vector y^j; then $\mathrm{P}_{-j}(\mathrm{V})^*$ is generated by the covector on y^j and $\mathrm{P}_{-j}(\mathrm{V}^*)$ is the covector on x^j. The map $\mathrm{P}_{-j}(\mathrm{V})^* \to \mathrm{P}_{-j}(\mathrm{V}^*)$ is obtained by precomposing by N^j. Upon scaling $\mathrm{N} \mapsto \lambda\mathrm{N}$, this map is scaled by λ^j. Therefore the canonical isomorphism is obtained by twisting by N^{-j}:

$$\mathrm{P}_{-j}(\mathrm{V})^* \to \mathrm{P}_{-j}(\mathrm{V}^*)(-j).$$

1.7.2. Note that in this case V is a $\overline{\mathbf{Q}}_\ell$-vector space. The claim about the existence of $\overline{\rho}$ satisfying

$$\rho(\sigma) = \exp(\overline{\rho} t_\ell(\sigma)) \quad \text{for} \quad \sigma \in \mathrm{I}_1 \subset \mathrm{I}$$

in particular implies that $\mathrm{P} \cap \mathrm{I}_1$ acts trivially on V. Let's recall why this is the case. As $p \neq \ell$, $\rho(\mathrm{P})$ is finite. As $\overline{\mathbf{Q}}_\ell$ has characteristic zero, by Maschke's theorem, the action of P on V is semisimple. As $\overline{\mathbf{Q}}_\ell$ is algebraically closed and V is finite-dimensional, this implies that each element in V is diagonalizable. A diagonalizable element of $\mathrm{GL}(\mathrm{V})$ is unipotent iff it is the identity. This proves our claim. (Recall also the Jordan-Chevalley decomposition.)

1.7.3. 'Il commute à l'action de $W(\overline{K}, K)$' $\rightsquigarrow$ When you let $W(\overline{K}, K)$ act on V via ρ and on $\overline{\mathbf{Q}}_\ell(n)$ via $\mathrm{Gal}(\overline{k}, k)$, the statement is that if $\tau \in W(\overline{K}, K)$,

$$\tau N \tau^{-1} = N : V(1) \to V.$$

Let $\lambda \in \overline{\mathbf{Q}}_\ell(1)$ and $n := \deg \tau$. As $\tau^{-1}\lambda = q^n \lambda \tau^{-1}$, the statement is equivalent to the statement

$$\tau N \lambda \tau^{-1} = q^{-n} N\lambda.$$

$\mathrm{Gal}(\overline{K}, K)$ acts on the inertia character t by the formula

$$t(\tau\sigma\tau^{-1}) = \tau\, t\sigma = q^{-n}\, t\sigma$$

where in the middle τ acts through $\mathrm{Gal}(\overline{k}, k)$, so if we assume I_1 is a normal subgroup of I and take $\sigma \in I_1$,

$$\exp(\rho(\tau)\, N\, t_\ell(\sigma)\, \rho(\tau^{-1})) = \rho(\tau\sigma\tau^{-1}) = \exp(N\, t_\ell(\tau\sigma\tau^{-1})) = \exp(N\, q^{-n}\, t_\ell(\sigma)).$$

Note that the above is written in the setting of a finite residue field with q elements, but holds for any residue field since all that is used is the compatibility between the action of $\mathrm{Gal}(\overline{K}, K)$ on $\overline{\mathbf{Q}}_\ell(1)$ and on the inertia character: in both cases $\mathrm{Gal}(\overline{K}, K)$ acts via $\mathrm{Gal}(\overline{k}, k)$.

1.7.4. As discussed in the note to 1.7.2, the only semisimple and unipotent element of $GL(n, \overline{\mathbf{Q}}_\ell)$ is the identity. The semisimple representation of $W(\overline{K}, K)$ therefore factors through the quotient by a normal closed subgroup $I' \supset I_1$. The conjugation action of $W(\overline{K}, K)$ on I/I' is has a kernel U of finite index, as a finite group has only finitely many automorphisms. So for some m_1, $F'^{m_1}, F''^{m_1} \in U$. $F''^{m_1} = aF'^{m_1}$ for some $a \in I$, but now F''^{m_1} commutes with $a \mod I'$, so if $a^{m_2} = 1 \mod I'$, then $F''^{m_1 m_2} = a^{m_2} F'^{m_1 m_2} = F'^{m_1 m_2} \mod I'$, so we may take $n = m_1 m_2$.

1.7.5. 'On a $NM'_i \subset M'_{i-2}$' $\rightsquigarrow$ this is because of the commutativity of (1.7.3). Namely, $F'N = NF'$ and if $\lambda \in \overline{\mathbf{Q}}_\ell(1)$, $F'N\lambda = q^{-1}N\lambda F'$. Therefore $N\lambda$ sends M'_i into M'_{i-1} and $N : M'_i(1) \subset M'_{i-2}$.

If $M' = M'' =: M$, then both F' and F'' preserve M, so $(F')^{-1}F''$ does, and every element of I is thus obtained, and every element of $W(\overline{K}/K)$ can be obtained as an element of I times a power of F' (or F'', etc.), so M is stable by $W(\overline{K}, K)$.

While proving that M' is independent of F', Deligne introduces $\exp(\lambda N)$ for $\lambda \in \overline{\mathbf{Q}}_\ell(-1)$. Let me instead deduce from the fact $F'^n = F''^n \mod I_1$ that for an appropriate choice of $\lambda \in \overline{\mathbf{Q}}_\ell(1)$,

$$F''^n = \exp(N\lambda)F'^n.$$

If $\mu := \lambda/(1 - q^{-n})$, we have

$$\exp(N\lambda)F'^n = \exp(N\mu)F'^n \exp(N\mu)^{-1},$$

since if we expand $F'^n \exp(-N\mu)$ as a series

$$F'^n \left(1 - N\mu + \frac{(-N\mu)^2}{2!} + \cdots\right),$$

we have that N commutes with F'^n and that $F'^n \mu^i = q^{-ni}\mu^i$, so that

$$\exp(N\mu)F'^n \exp(N\mu)^{-1} = \exp(N\mu)\exp(-q^{-n}N\mu)F'^n = \exp(N\lambda)F'^n.$$

This identity shows that $\exp(N\mu)$ sends M'_i into M''_i, for the reason given that it sends generalized eigenspaces for F'^n to those for F''^n. As $(N\mu)M'_i \subset M'_{i-2}$,

$$\exp(N\mu) = 1 + N\mu + \frac{(N\mu)^2}{2!} + \cdots$$

also sends M'_i into M'_i, so $M''_i \subset M'_i$.

1.7.7. Let F' be as in (1.7.5), let $\alpha \in \overline{\mathbf{Q}}_\ell^*$, and let V'^α equal the sum of the generalized eigenspaces of F' acting on V with eigenvalue in the class of $\alpha \mod$ roots of 1. Then V'^α is independent of lift and $N\lambda : V'^\alpha \to V'^{\alpha/q}$, and

$$V^\alpha = \sum_{i \in \mathbf{Z}} V'^{\alpha q^i},$$

is stable under $\exp(N\lambda)$ for $\lambda \in \overline{\mathbf{Q}}_\ell(1)$ and hence under the action of I_1. By the argument of (1.7.5), V^α is therefore stable under $W(\overline{K}, K)$.

1.7.8. The construction is somewhat reminiscent of the construction of nearby cycles. One sees [**SGA1**, XIII 5.1] that the scheme X_n is again regular. Taking the sheaf of sets $\mathscr{F}$ to be locally constant constructible, the lemma of Abhyankar that is used is [**SGA1**, XIII 5.2]. The point is that over E, X_n is totally ramified (defined by adjoining nilpotents T_i so that $T_i^n = 0$), so that a sheaf on $\pi^{-1}(E)$ is the same as a sheaf on E, and so that the action of μ_n^I on X_n fixes $\pi^{-1}(E)$ pointwise while permuting the inductive system of étale neighborhoods of any given point of $\pi^{-1}(E)$; in this way μ_n^I acts on $\mathscr{F}[E]$ by transport of structure.

(As X_n is a regular scheme, by Zariski-Nagata purity of the branch locus (`0BMB`), if $\overline{\mathscr{F}}$ extends over the generic points of each of the divisors D_i, it extends as a locally constant sheaf over all of X_n. As t_i is a local equation for D_i, there is an n so that the inverse image of $\mathscr{F}$ onto the étale cover of $X - D_i$ defined by adjoining $t_i^{1/n}$ extends to a locally constant sheaf on all of $X[t_i^{1/n}]$.)

1.7.11. This paragraph extends the tame ramification theory of a dvr to a henselian local ring that is the henselization of X at a generic point of E. It is clear how a choice of t_i and of $t_i^{1/n}$ splits the extension of profinite groups: if K' denotes the extension of K generated by these chosen elements, then $K_2 = K_1K'$, and K' is linearly disjoint from K_1 (although K' is not in general normal). Therefore via the splitting $\mathrm{Gal}(k_1/k) \hookrightarrow \pi_1^{\mathrm{mod}}(X - D, \mathrm{Spec}(\overline{K}))$ one obtains a sheaf on $\mathrm{Spec}(k)$. Of course F carries an action of all of $\mathrm{Gal}(K_2/K)$, and therefore of $\mathbf{Z}_L(1)^I$.

That $\mathrm{Gal}(K_1/K) = \mathrm{Gal}(k_1/k)$ is the usual property that change of base point gives isomorphic fiber functors [**SGA1**, V 7]. That is to say, $X_1 = \mathrm{Spec}\, R_1$ is the projective limit of all étale covers of X, and is a fundamental pro-object for the Galois category of revêtements étales of X [**SGA1**, V 4.1], so that whether F is the fiber functor corresponding to a geometric point $*$ centered on the closed or generic point of X, if $X' \to X$ is a revêtement étale, $F(X') \xleftarrow{\sim} \mathrm{Hom}_X(X_1, X')$, and $\pi_1(X, *)$ acts continuously on the latter set. As $F(X')$ is the set of geometric points of X' above $*$, it corresponds to an étale extension of the generic point or closed point of X, respectively. Then the statement that the groups $\pi_1(X, *)$ are isomorphic for these two different choices of $*$ is simply the statement $\mathrm{Gal}(K_1/K) \simeq \mathrm{Gal}(k_1/k)$.

'On déduit donc une action de $\mathrm{Gal}(k_1/k)$ sur F, et une action $\mathrm{Gal}(k_1/k)$-equivariante de $\hat{\mathbf{Z}}_{\mathbf{L}}(1)^{\mathrm{I}}$' $\rightsquigarrow$ the splitting automatically gives an action of $\mathrm{Gal}(k_1/k)$ on F. The matter of a $\mathrm{Gal}(k_1/k)$-equivariant action of $\hat{\mathbf{Z}}_{\mathbf{L}}(1)^{\mathrm{I}}$ involves the subtlety that the group scheme μ_n itself carries an action of $\mathrm{Gal}(k_1/k)$ deduced from the left action of $\mathrm{Gal}(k_1/k)$ on $\hat{\mathbf{Z}}_{\mathbf{L}}(1)^{\mathrm{I}}$. Considering $\hat{\mathbf{Z}}_{\mathbf{L}}(1)^{\mathrm{I}}$ and $\mathrm{Gal}(k_1/k)$ as subgroups of $\mathrm{Gal}(\mathrm{K}_2/\mathrm{K})$, this action is written as conjugation: if $\varsigma \in \mathrm{Gal}(k_1/k)$, $a \in \hat{\mathbf{Z}}_{\mathbf{L}}(1)$, then the action is $a \mapsto \varsigma a \varsigma^{-1}$ (`0BU5`). Then a $\mathrm{Gal}(k_1/k)$-equivariant action of $\hat{\mathbf{Z}}_{\mathbf{L}}(1)^{\mathrm{I}}$ is an action where $\varsigma \cdot a$ acts the same as $\varsigma a \varsigma^{-1} \cdot \varsigma$.

The point is, just as in (1.7.3), the action of $\mathbf{Z}_\ell(1)$ does not commute naïvely with $\mathrm{Gal}(k_1/k)$: it commutes up to the twist coming from the action of $\mathrm{Gal}(k_1/k)$ on $\mathbf{Z}_\ell(1)$. However, taking the logarithm of unipotent monodromy N, then N *does* commute 'on the nose' with $\mathrm{Gal}(k_1/k)$, as is explained in the note to (1.7.3).

Remark. When X_0 in (1.7.8) is a smooth curve and $\mathrm{D}_0 \subset \mathrm{X}_0$ is a point, the Galoisian interpretation (1.7.11) of the construction (1.7.8) is identical to the situation of (1.7.1)–(1.7.5) for the henselization of X_0 at D_0 after quotienting by wild inertia.

1.7.12.1. Provided that the field k satisfies the condition that no finite extension contains all roots of unity a power of l (as is true when X is finitely generated over $\mathbf{Z}$, as in that case k is finitely-generated over $\mathbf{Q}$ or $\mathbf{F}_p$), the (arithmetic) proof of Grothendieck's theorem on quasi-unipotent monodromy gives that $\mathbf{Z}_{\mathbf{L}}(1)$ acts quasi-unipotently. When D is lisse, the discussion of (1.7.11) reduces to the usual one for the X a henselian trait, in which case the conclusion that $\mathbf{Z}_{\mathbf{L}}(1)$ (and indeed all of the inertia) acts quasi-unipotently is the statement of Grothendieck's theorem.

1.8.1. The inclusion $\overset{k}{\otimes} j_*\mathscr{F}_0 \subset j_*\overset{k}{\otimes}\mathscr{F}_0$ comes about by considering that étale locally about a point $s \in \mathrm{S}_0$ neither sheaf may be locally free, as restricting étale neighborhoods of s to U_0 may not be enough to trivialize either sheaf; in other words, a trivialization may ramify when extended to X_0. Let $\mathrm{V} \to \mathrm{X}_0$ be an étale neighborhood of s. Then the inclusion above simply reflects the fact that sections of $\overset{k}{\otimes}\mathscr{F}_0$ over $\mathrm{V}|_{\mathrm{U}_0} := \mathrm{V} \times_{\mathrm{X}_0} \mathrm{U}_0$ include those coming from the tensor product of k sections in $\mathscr{F}_0(\mathrm{V}|_{\mathrm{U}_0})$, but might include more besides.

Here is an example: let $X = \operatorname{Spec}\mathbf{R}$ and let $\mathscr{F}$ be the locally free sheaf on X represented by $\operatorname{Spec}\mathrm{R}[x]/(x^3-1)$; this is the sheaf μ_3 of third roots of unity and it is a locally free sheaf of $\mathbf{Z}/3$-modules of rank 1. Then $\mathscr{F}(\mathbf{R}) = \{1\}$ and if ζ is a primitive 3rd root of unity, $\mathscr{F}(\mathbf{C}) = \{1, \zeta, \overline{\zeta}\}$. $\operatorname{Gal}(\mathbf{C}/\mathbf{R}) \simeq \mathbf{Z}/2$ and $\mathscr{F}$ corresponds to the representation V of $\mathbf{Z}/2$ given by complex conjugation on the $\mathbf{Z}/3$-module $\{1, \zeta, \overline{\zeta}\}$. Then $\mathscr{F} \otimes_{\mathbf{Z}/3} \mathscr{F}$ corresponds to the tensor representation $\mathrm{V} \otimes_{\mathbf{Z}/3} \mathrm{V}$. Its sections over $\mathbf{R}$ are its $\mathbf{Z}/2$-invariant sections of $\mathrm{V} \otimes \mathrm{V}$. These are $\{1 \otimes 1, \zeta \otimes \overline{\zeta}\}$.

1.8.4. The fiber of $j_*\mathscr{F}_0$ at $\overline{s}$ can be computed by taking first the inverse image of $j_*\mathscr{F}$ to $\operatorname{Spec}\mathscr{O}_{\mathrm{X}_0,s}$, the local ring of X_0 at s, and then taking the colimit along all étale ring maps $\mathscr{O}_{\mathrm{X}_0,s} \to \mathrm{U}$, these being equivalent to finite separable extensions of $k(\eta)$ which are non-ramified over s. So in the end we are computing the colimit of sections of the inverse image of $\mathscr{F}_0$ along $\eta \to \mathrm{X}_0$ over finite separable field extensions of $k(\eta)$ fixed by I; this is nothing other than $\mathscr{F}_{0\overline{\eta}}^{\mathrm{I}}$.

The last line of the proof references (1.6.14.3), which has been corrected in the note (1.6.14) above.

1.8.5. The idea is that the global existence of the local monodromy filtration compatible with W (the filtration of our lisse sheaf $\mathscr{F}_0$ by lisse subsheaves with $\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0)$ punctually ι-pure of weight i) is not assured by (1.6.13); we just know that if it exists, it's unique. (We do know, of course, that N preserves W, as W is a filtration by subsheaves, hence once restricted to the geometric generic fiber, by $\mathrm{W}(\overline{\eta}/\eta)$-subrepresentations.) On the other hand, (1.6.1) guarantees the existence of the local monodromy filtration M′ on each graded piece $\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0)$ so that $\mathrm{NM}'_i \subset \mathrm{M}'_{i-2}$ and so that N^k induces an isomorphism

$$\mathrm{Gr}_k^{\mathrm{M}'}\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0) \xrightarrow{\sim} \mathrm{Gr}_{-k}^{\mathrm{M}'}\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0)$$

for each k. Then (1.8.4) shows that the $\mathrm{Gr}_j^{\mathrm{M}'}\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0)$ is ι-pure of weight $i+j$, so by the uniqueness of (1.7.5), the local monodromy filtration on $Gr_i^{\mathrm{W}}(\mathscr{F}_0)$ coincides with the weight filtration M on $Gr_i^{\mathrm{W}}(\mathscr{F}_0)$ shifted by i; i.e. $\mathrm{M}_{i+k}\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0) = \mathrm{M}'_k\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0)$ and

$$\mathrm{Gr}_{i+k}^{\mathrm{M}}\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0) \xrightarrow{\sim} \mathrm{Gr}_{i-k}^{\mathrm{M}}\mathrm{Gr}_i^{\mathrm{W}}(\mathscr{F}_0). \tag{†}$$

Now let an increasing filtration M on all of $\mathscr{F}_0$ be defined by the weight filtration (1.7.5); we can do so because the ι-weights of $\mathscr{F}_0$ are all integers. The filtration M satisfies $\mathrm{NM}_i \subset \mathrm{M}_{i-2}$ and satisfies (†). Therefore it satisfies the requirements of (1.6.13). Summing up, we have shown that the filtration of local monodromy rel. W of (1.6.13) exists (it is therefore unique), and we have shown that it coincides with the weight filtration (1.7.5).

Remark. Both the local monodromy filtration and the weight filtration involve a choice of point $s \in \mathrm{S}$, but end up defining a filtration on the fiber $\mathscr{F}_{\overline{\eta}}$.

1.8.7. (Recall (1.7.11) and its note, especially the remark at the end.) Note that the filtration W on $\mathscr{F}_0$ induces a filtration, also called W, on $\mathscr{F}_0[\mathrm{D}_0]$, with graded pieces $(\mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}_0)[\mathrm{D}_0]$. For each i, $\mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}_0$ is punctually ι-pure of weight i. The filtration of local monodromy around D of $\mathscr{G}_i := (\mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}_0)[\mathrm{D}_0]$ exists by (1.6.1) & (1.7.8); call this filtration M′. The filtration M′ can be seen in the setting of (1.7.11) by applying (1.6.1) to the logarithm of unipotent monodromy (1.7.12.1) acting on the representation of $\mathrm{W}^{\mathrm{mod}}(\mathrm{X}-\mathrm{D}, \mathrm{Spec}(\overline{\mathrm{K}}))$ corresponding to $\mathscr{G}_i$; in particular, the $\mathrm{Gr}_j^{\mathrm{M}'} \mathscr{G}_i$ are lisse. Let $\mathrm{C}_0 \subset \mathrm{X}_0$ be a smooth curve meeting D_0 transversally. As the function t continues to cut out D_0, if $\overline{s} \in \mathrm{C}$, with image s in $\mathrm{C}_0 \cap \mathrm{D}_0$, the action of $\mathbf{Z}_\ell(1)$ on the fiber $(\mathscr{G}_i)_{\overline{s}}$ from (1.7.8) is the same as the action of the maximal pro-ℓ quotient of the local inertia of the henselian trait $\mathrm{C}_{0(s)}$ on the representation of $\mathrm{W}(\overline{\eta}/\eta)$ obtained by restricting $\mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}_0$ to $\mathrm{C}_0 \cap \mathrm{U}_0$, with notation as in (1.8.3). Therefore the logarithm of unipotent monodromy on $\mathscr{G}_i$ restricts to the logarithm of unipotent monodromy for the representation of $\mathrm{W}(\overline{\eta}/\eta)$ corresponding to $(\mathscr{G}_i)_{\overline{s}}$ as in (1.8.3), so that $(\mathrm{Gr}_j^{\mathrm{M}'} \mathscr{G}_i)_{\overline{s}}$ is ι-pure of weight $i+j$ by (1.8.4); varying s_0 (and C_0) over every point of $|\mathrm{D}_0|$, we conclude that $\mathrm{Gr}_j^{\mathrm{M}'} \mathscr{G}_i$ is punctually ι-pure of the same weight.

Let F_s be the Frobenius at s. We follow the notation of (1.7.11): $k =$ the residue field of the henselization of X_0 at the generic point of D_0, k_1 a separable closure. Via $s \to \mathrm{D}_0$, F_s is introduced into the group $\pi_1(\mathrm{D}_0, \overline{s})$ and determines a conjugation class in $\pi_1(\mathrm{D}_0, \mathrm{Spec}(k_1))$. As $\mathrm{Gal}(k_1/k) \twoheadrightarrow \pi_1(\mathrm{D}_0, \overline{\eta})$ surjects, we get a conjugation class in $\mathrm{Gal}(k_1/k)$. By (1.7.12.2), we can speak without ambiguity of the ι-weights of F_s on an ℓ-adic representation of $\mathrm{W}^{\mathrm{mod}}(\mathrm{X}_0-\mathrm{D}_0, \mathrm{Spec}(\overline{\mathrm{K}})) := \mathrm{W}$. (Here, as k is not finite, W is defined as the inverse image of $W(D_0, \mathrm{Spec}(k_1))$ in $\pi_1^{\mathrm{mod}}(X_0 - D_0, \mathrm{Spec}(\overline{K}))$ via

$$\pi_1^{\mathrm{mod}}(X_0 - D_0, \mathrm{Spec}(\overline{K})) \twoheadrightarrow \mathrm{Gal}(k_1/k) \twoheadrightarrow \pi_1(D_0, \mathrm{Spec}(k_1)).)$$

As $\mathscr{F}_0[D_0]$ is a successive extension of lisse sheaves punctually ι-pure of integer weight, *a fortiori* the ι-weights with respect to F_s of $\mathscr{F}_0[D_0]$ are integers. If V denotes the representation of W corresponding to $\mathscr{F}_0[D_0]$, (1.7.12.3) gives that V admits a weight filtration M (with respect to F_s) stable by W that is characterized by the property that $\mathrm{Gr}_i^M V$ is ι-pure of weight i with respect to F_s. The weight filtration also has the property that $NM_i(1) \subset M_{i-2}$. On $\mathscr{G}_i$, M'_{j-i} has the property that characterizes M, so $M_{i+j}\mathscr{G}_i = M'_j\mathscr{G}_i$, and hence

$$N^k : \mathrm{Gr}_{i+k}^M \mathrm{Gr}_i^W(\mathscr{F}_0[D_0]) = \mathrm{Gr}_k^{M'} \mathscr{G}_i \xrightarrow{\sim} \mathrm{Gr}_{-k}^{M'} \mathscr{G}_i = \mathrm{Gr}_{i-k}^M \mathrm{Gr}_i^W(\mathscr{F}_0[D_0]).$$

This shows that M is the filtration of local monodromy around D of $\mathscr{F}_0[D_0]$, rel. W. As $\mathrm{Gr}_i^M \mathscr{F}_0[D_0]$ is a successive extension of $\mathrm{Gr}_j^W \mathrm{Gr}_i^M(\mathscr{F}_0[D_0]) = \mathrm{Gr}_i^M \mathrm{Gr}_j^W(\mathscr{F}_0[D_0])$, which are lisse and punctually ι-pure of weight i, $\mathrm{Gr}_i^M \mathscr{F}_0[D_0]$ is lisse and punctually ι-pure of weight i.

1.8.8. With regards to remark 2), twist the sheaf $\mathscr{F}_0 \rightsquigarrow \mathscr{F}_0^{(b)}$ so that it has weight 0 (following (1.2.7), $b = p^{-\beta}$). Then apply (1.8.7) with the trivial filtration W to see that $\mathrm{Gr}_i^M(j_*\mathscr{F}_0^{(b)}|D_0)$ is punctually ι-pure of weight i. Twist back to conclude that $\mathrm{Gr}_i^M(j_*\mathscr{F}_0|D_0)$ is punctually ι-pure of weight $\beta + i$.

1.8.9. In c), $j_*\mathscr{F}_0 \hookrightarrow \varepsilon_* j_* \varepsilon^* \mathscr{F}_0$ follows, after writing $\varepsilon_* j_* \varepsilon^* \mathscr{F}_0 = j_* \varepsilon_* \varepsilon^* \mathscr{F}_0$, from the observation that $\mathscr{F}_0 \hookrightarrow \varepsilon_*\varepsilon^*\mathscr{F}_0$ injects since on stalks, a finite extension of a henselian ring splits as a product of henselian rings; i.e. the adjunction morphism corresponds to the inclusion along the diagonal

$$\mathscr{F}_x \hookrightarrow \prod_{\varepsilon^{-1}x} \mathscr{F}_x.$$

In d), reduce to a constant sheaf, where it is obvious.

In the explanation for e), the strict henselization of X_0 at x is irreducible hence a fortiori the inverse image of any open set is connected. Then, since the fiber product of any étale cover of x with U_0 is an étale neighborhood of z, there is a map $(i^* j_* \mathscr{F}_0)_x \to (\mathscr{F}_0)_z$. As the former can be computed as sections over the inverse image of U_0 in the strict henselization of X_0 at x; since this is a connected scheme, and $\mathscr{F}_0$ is lisse, the arrow is injective. The factorization of this arrow as

$$(i^* j_* \mathscr{F}_0)_x \to (k_* k^* j_* \mathscr{F}_0)_x \to (j_* \mathscr{F}_0)_y \to (\mathscr{F}_0)_x$$

can be explained as follows. After rewriting $k_* k^* j_* \mathscr{F}_0$ as $k_* k^* i^* j_* \mathscr{F}_0$, the first arrow is just adjunction for k, The middle arrow can be rewritten $(k_* k^* i^* j_* \mathscr{F}_0)_x \to (i^* j_* \mathscr{F}_0)_y$ so that it is a statement about sheaves on F_0. Take an étale neighborhood W_0 of x in F_0; then $W_0 \times_{F_0} V_0$ is an étale neighborhood of y. The projective system of étale $W_0' \to F_0$ s.t. $W_0' \times_{F_0} V_0$ admits an arrow to $W_0 \times_{F_0} V_0$ has the property that the projective system W_0' is a subcategory of the projective system of étale neighborhoods of y in F_0. Therefore there is an arrow from the colimit of $i^* j_* \mathscr{F}_0$ applied to the former system to the colimit of $i^* j_* \mathscr{F}_0$ applied to the latter, which is $(i^* j_* \mathscr{F}_0)_y$. This gives an arrow $(k_* k^* i^* j_* F_0)(W_0) \to (i^* j_* \mathscr{F}_0)_y$, functorial in W_0, and hence an arrow $(k_* k^* i^* j_* F_0)_x \to (i^* j_* \mathscr{F}_0)_y$.

The last arrow is in effect the observation that the fiber product of U_0 with an étale neighborhood of y in X_0 is an étale neighborhood of z.

In the proof of (1.8.9), here is how to reduce: using (a) one may suppose that $\mathscr{F}_0$ is punctually ι-pure and of the form $i_! i^* \mathscr{F}_0$ where i is locally closed as in (b) and $i^* \mathscr{F}_0$ is lisse. Using (b), therefore, we may assume $\mathscr{F}_0$ is punctually ι-pure and lisse. Applying (c) to the normalization $\varepsilon : X_0' \to X_0$, we may assume X_0 normal, and, picking a connected component, connected, with U_0 a dense open and $\mathscr{F}_0$ a lisse sheaf on U_0. If we find a revêtement étale $u : U_0' \to U_0$ so that $u^* \mathscr{F}_0$ has no p-monodromy (i.e. if $\overline{x}$ is a geometric point of U_0, $W(U_0, \overline{x})$ acts through its maximal pro-ℓ quotient), then letting X_0' denote the normalization of X_0 in the function field of U_0, we reduce by (c) to the case where $\mathscr{F}_0$ has no p-monodromy, and hence is tamely ramified at any points of codimension 1 of X_0 (in particular, possibly after using (d) to shrink U_0, at the generic points of the Weil divisor $X_0 - U_0$).

To find the desired U_0', as $W(U_0, \overline{x})$ is compact, after replacing $\overline{\mathbf{Q}}_\ell$ by some finite extension E_λ of $\mathbf{Q}_\ell$, the image of U_0' in $\operatorname{Aut} \mathscr{F}_{\overline{x}}$ stabilizes a lattice; if $\mathscr{F}_0$ corresponds to an R-module V (R the ring of integers of E_λ) with continuous action of $W(U_0, \overline{x})$ on the underlying $\mathbf{Z}_\ell$-module (c.f. [**SGA5**, VI 1.4.1]) and is of rank r as $\mathbf{Z}_\ell$-module, then we can find a basis for V as $\mathbf{Q}_\ell$-vector space for which $W(U_0, \overline{x})$ acts via its image in $GL(\mathbf{Z}_\ell, r)$. The congruence subgroup $\Gamma_1 \subset GL(\mathbf{Z}_\ell, r)$ is open for the ℓ-adic topology, and is pro-ℓ. The preimage W_1 of Γ_1 in $W(U_0, \overline{x})$ is therefore open and acts via a pro-ℓ quotient. The closure of W_1 in $\pi_1(U_0, \overline{x})$ is open, and defines U_0'.

To complete the proof, we are almost there, except F_0 is just a Weil divisor, and need not satisfy the smoothness assumption of (1.8.6). The idea is to use e) and recurrence on $\dim U_0$ to shrink X_0 and throw away the bad points of F_0. If we replace X_0 by an open set containing U_0 whose intersection with F_0 is a lisse divisor, then (1.8.8) 2) shows that (1.8.9) is true there. We can find finitely many such open sets with inclusions j_i, the union of which, X_0', intersects F_0 in a dense set V_0. If $j' : U_0 \hookrightarrow X_0'$, then $j'_*\mathscr{F}_0 \hookrightarrow \prod_i j_{i*}\mathscr{F}_0$ so (1.8.9) is proved for j'. By recurrence on dimension we may assume that (1.8.9) holds for k_*. In light of this, applying e) to the lisse sheaf $\mathscr{F}_0$, yields that $i^* j_* \mathscr{F}_0$ satisfies the conclusions of (1.8.9); in effect, $k^* j_* = k^* j'_*$. Finally, $j_*\mathscr{F}_0 \hookrightarrow j'_*\mathscr{F}_0 \times i_* i^* j_* \mathscr{F}_0$ allows us to conclude that j_* satisfies (1.8.9).

1.8.11. A Jordan-Hölder series for $\mathscr{F}_0$ allows us to reduce to $\mathscr{F}_0$ irreducible. The restriction of an irreducible lisse sheaf to a nonempty open U_0 of a normal connected scheme X_0 is still irreducible because if η denotes the generic point of X_0, and $\overline{\eta}$ a geometric point centered on η, we have by [**SGA1**, Exp. V, 8.2]

$$\mathrm{Gal}(\overline{\eta}, \eta) \twoheadrightarrow \pi_1(U_0, \overline{\eta}) \twoheadrightarrow \pi_1(X_0, \overline{\eta}).$$

Now, $\mathscr{F}_0$ is ι-mixed, so admits an ι-pure subsheaf $\mathscr{G}_0$ which is lisse when restricted to some U_0. Therefore $\mathscr{F}_0|U_0$, irreducible yet containing $\mathscr{G}_0|U_0$, must equal $\mathscr{G}_0|U_0$.

1.8.12. Let $f : X_0' \to X_0$ be the normalization morphism. It induces a bijection on irreducible components, and X_0' is a disjoint of normal integral schemes. We make use of the fact, true for any morphism, that if $x_0 \in X_0$, the weights of $\mathscr{F}_0$ at x_0 coincide with the weights of $f^*\mathscr{F}_0$ at every point of the fiber $f^{-1}(x_0)$. If $\mathscr{F}_0$ is ι-pure of weight β at a point x_0, then all the points of X_0' in the fiber over x_0 are also ι-pure of the same weight; therefore (1.8.11) implies that all the points of the irreducible components of X_0'

meeting the fiber of x_0 are ι-pure of weight β, which in turn implies that all the points in the irreducible components of X_0 meeting x_0 are ι-pure of weight β, which shows that the locus of points where $\mathscr{F}_0$ is ι-pure of weight β is closed. This locus is also open because, taking any x_0 in it, there is an open neighborhood of x_0 which meets only the irreducible components of X_0 on which x_0 lies. Then the above construction shows that all the points in this neighborhood are also ι-pure of weight β.

3.2.3. This theorem is obtained via the Fourier transform in §4 of Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*. Jump to the notes accompanying §4 of that article.

3.3.1. f) Follows immediately from base change along any geometric fiber, which by assumption is discrete.

Dévissages (b) and (c) allow one to assume X and Y affine and f (modulo a closed immersion) corresponding to an extension of rings $A \subset B$, where $B = A[b_1, \dots, b_n]$. The maps $A \subset A[a_1] \subset A[a_1, a_2] \subset \cdots \subset B$ have fibers of dimension ≤ 1, and by (d), we may assume f has fibers of dimension ≤ 1, in which case the set of points in X which are isolated in their fibers describes an open locus in X (EGA IV_3 13.1.4). By (b) and (f), one replaces X by the closed complement and assumes f has relative dimension 1.

The existence of a finite purely inseparable extension of K making (α) true follows from the general recipe of the note to (6.1.9) in the special case that S is a point.

The existence of a revêtement étale u as in (β) is obtained as follows: C' is smooth over K, so is a disjoint union of smooth curves. Following the second-to-last paragraph of the note to (1.8.9) (it is a simple matter of remembering that the first congruence subgroup $\Gamma_1 \subset GL(\mathbf{Z}_\ell, r)$ is open and pro-ℓ) one finds revêtements étales of each of the components of C' so that the pullback of $\mathcal{F}$ to each one has no p-monodromy. To see that $\mathcal{F}$ is a direct factor of $u_*u^*\mathcal{F}$: $u_*u^*\mathcal{F}$ is the direct sum of the direct images of the restriction of $u^*\mathcal{F}$ to the various components of D (in bijection with components of C'). We may therefore assume C', D connected, and the revêtement étale u of constant degree d. Then the surjection $u_*u^*\mathcal{F} \twoheadrightarrow \mathcal{F}$ is split by the injection $\mathcal{F} \hookrightarrow \frac{1}{d}u_*u^*\mathcal{F}$ [**SGAA**, IX 5.1]. D embeds into the disjoint union of smooth projective curves, and the complement of D is finite over K, but the residue extension needn't be separable. A purely inseparable extension of K as in the case of (α) remediates this.

To extend to a neighborhood of η: (c) enables noetherian induction on Y, so it suffices to solve the problem over an arbitrarily small Zariski open neighborhood of η. We immediately replace all schemes in question by their reduced subschemes and silently give the reduced subscheme structure to all closed subschemes in what follows. We find a finite purely inseparable field extension $\eta' \to \eta$ making (α) and (β) true over the η'-fiber. Shrinking Y as necessary, we find à la note to (6.1.9) a finite radicial extension $Y' \to Y$ inducing our $\eta' \to \eta$ generically. By (e) we replace Y, η by Y', η', respectively and assume (α) and (β) are true for the η-fiber. As the η-fiber of f is generically smooth, shrinking Y, the complement of the smooth locus of f becomes finite over Y (`02NW`), so that by (b) and (f), we may assume f smooth and $C = C'$, so that X is a disjoint union of its irreducible components. Shrinking Y, we may assume that all the irreducible components of X dominate Y. Picking one irreducible component of X, by (a) we may assume X (and D) are connected. The normalization $u : X' \to X$ of X in the generic point of D coincides with D over η and hence, shrinking Y, becomes étale over X. As $\mathscr{F}$ is a direct factor of $u_* u^* \mathscr{F}$, by (a) we may replace the former by the latter. Replace X by X′, f by $f \circ u$, and $u_* u^* \mathscr{F}$ by $u^* \mathscr{F}$, so that the η-fiber of f is now D. Shrinking Y, it maps to $\operatorname{Spec} \mathbf{Q}$ or $\operatorname{Spec} \mathbf{F}_p$ in $\operatorname{Spec} \mathbf{Z}$. As Y is reduced by assumption, possibly shrinking further, Y becomes smooth over $k := \operatorname{Spec} \mathbf{Q}$ or $\operatorname{Spec} \mathbf{F}_p$. As $f : X \to Y$ is smooth and X is connected, X is also integral.

Suppose for a moment that the embedding $D \hookrightarrow \overline{D} \subset \mathbf{P}^N_\eta$ extends to an Y-morphism $i : X \to \mathbf{P}^N_Y$. Let $\overline{X} := \overline{i(X)}$; as X is irreducible, so is $\overline{X}$. As $i : X \to \overline{X}$ is an open immersion over η, it induces an isomorphism of function fields. As we are in the category of k-varieties, there is an open loci $U_1 \subset X$ (containing the η-fiber), $U_2 \subset \overline{X}$ so that i restricts to an isomorphism $U_1 \xrightarrow{\sim} U_2$ (`0BXN`). Shrinking Y, we may assume $X = U_1$ and $i : X \hookrightarrow \overline{X}$ is an open immersion. By EGA IV_3 13.1.3, every fiber of $\overline{X} \to Y$ is of dimension ≥ 1 and the condition that a point of X lie on an irreducible component of its fiber of dimension > 1 describes a closed locus of $\overline{X}$ not meeting $\overline{X}_\eta = \overline{D}$ As $\overline{D} \to \eta$ is smooth, shrinking Y, $\overline{X} \to Y$ becomes smooth purely of relative dimension 1. As the closed set $\overline{X} - X$ meets $\overline{D}$ in $\overline{D} - D$, shrinking Y, we may suppose the former is the closure of the latter. Shrinking Y, $\overline{\mathrm{X}} - \mathrm{X}$, which is generically finite étale, becomes finite étale (`02NW`), hence equidimensional of dimension equal to $\dim \mathrm{Y}$, hence a divisor in X. If $\overline{\xi}$ denotes a geometric point centered on the generic point ξ of X, the restriction of $\mathscr{F}$ to X_η corresponds by construction to a representation of $\pi_1(\mathrm{X}_\eta, \overline{\xi})$ which factors through its maximal pro-ℓ quotient. As $\pi_1(\mathrm{X}, \overline{\xi})$ is a quotient of $\pi_1(\mathrm{X}_\eta, \overline{\xi})$, $\pi_1(\mathrm{X}, \overline{\xi})$ also acts through its maximal pro-ℓ quotient (Ribes & Zalesskii (3.4.1) (b)).

How to extend the immersion $\mathrm{D} \hookrightarrow \mathbf{P}^{\mathrm{N}}_\eta$, which of course determines $\overline{\mathrm{D}} \hookrightarrow \mathbf{P}^{\mathrm{N}}_\eta$ (`0BXZ`)? The map $\mathrm{D} \hookrightarrow \mathbf{P}^{\mathrm{N}}_\eta$ corresponds uniquely to the data of an invertible sheaf $\mathscr{L}$ on D and global sections $s_0, \dots, s_{\mathrm{N}} \in \Gamma(\mathrm{D}, \mathscr{L})$ which generate $\mathscr{L}$. Clearly $\mathscr{L}$ extends to an invertible sheaf also called $\mathscr{L}$ on a Zariski open $\mathrm{U} \subset \mathrm{X}$ containing $\mathrm{X}_\eta = \mathrm{D}$ (as X is integral, it is just a matter of ensuring that the transition functions extend to sections of $\mathscr{O}^*_{\mathrm{U}}$). Shrinking U, we may assume the sections s_i extend to global sections of $\mathscr{L}$ over U, defining a morphism $\mathscr{O}^{\mathrm{N}+1}_{\mathrm{U}} \to \mathscr{L}$ with coherent cokernel $\mathscr{K}$. As the support of $\mathscr{K}$ is closed and does not meet D, we subtract $\mathrm{U} \cap \operatorname{supp} \mathscr{K}$ from U so that $\mathscr{L}$ is again generated by $s_0, \dots, s_{\mathrm{N}} \in \Gamma(\mathrm{U}, \mathscr{L})$. As $\mathrm{X}_\eta \subset \mathrm{U}$, we shrink Y so that $\mathrm{X} = \mathrm{U}$ and so that Y is affine, in which case the data of $\mathscr{L}$ and the s_i defines a morphism $\mathrm{X} \to \mathbf{P}^{\mathrm{N}}_{\mathrm{Y}}$ restricting to the given immersion of the generic fiber. We're done.

3.4.2. $\mathrm{F}\varphi - \varphi \in \mathrm{Hom}(\mathscr{F}, \mathscr{G}) \rightsquigarrow$ recall that given sheaves $\mathscr{F}_0, \mathscr{G}_0$ on X_0 and $f \in \mathrm{Hom}(\mathscr{F}, \mathscr{G})$, F acts on $f \mapsto \mathrm{F}f$ by delivering the dashed arrow in the diagram

$$\begin{tikzcd} \mathrm{F}^*\mathscr{F} \arrow[r, "\mathrm{F}^* f"] \arrow[d, "\wr"] & \mathrm{F}^*\mathscr{G} \arrow[d, "\wr"] \\ \mathscr{F} \arrow[r, dashed, "\mathrm{F}f"] & \mathscr{G}. \end{tikzcd}$$

Since the epimorphism $\nu : \mathscr{E} \twoheadrightarrow \mathscr{F}$ is the inverse image of a map $\nu_0 : \mathscr{E}_0 \twoheadrightarrow \mathscr{F}_0$, so invariant under Frobenius. To check that $\mathrm{F}\varphi$ is a splitting (which implies $\mathrm{F}\varphi - \varphi \in \mathrm{Hom}(\mathscr{F}, \mathscr{G})$), we have to show that $\nu \circ \mathrm{F}\varphi = \mathrm{id}$. As $\nu = \mathrm{F}\nu$, this is immediate: $\mathrm{F}\nu \circ \mathrm{F}\varphi = \mathrm{F}(\nu \circ \varphi) = \mathrm{F}\,\mathrm{id} = \mathrm{id}$. A different choice of splitting φ' has $\mathrm{F}\varphi' - \varphi'$ differing from $\mathrm{F}\varphi - \varphi$ by some $\mathrm{F}f - f$, hence goes to the same class in $\mathrm{Hom}(\mathscr{F}, \mathscr{G})_{\mathrm{F}}$.

3.4.4. The point is that the same proof as gives (3.4.3), applied to the H^0, i.e. to $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$, shows that the latter has weights $\gamma - \beta + n$, with n an integer ≥ 0. By (3.4.2),

$\mathrm{Ext}^1(\mathscr{F}_0, \mathscr{G}_0)$ is nonzero only if one of $\mathrm{H}^0(\mathrm{X}, \mathscr{H}om(\mathscr{F}, \mathscr{G}))_{\mathrm{F}}$ or $\mathrm{H}^1(\mathrm{X}, \mathscr{H}om(\mathscr{F}, \mathscr{G}))^{\mathrm{F}}$ is; (3.4.3) gives a necessary condition for the latter and the former is zero unless the action of F on $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ has 1 among its eigenvalues, which, as remarked, can only happen if $\beta \equiv \gamma \pmod{\mathbf{Z}}$ and $\beta \geq \gamma$.

3.4.5. In other words, $\mathscr{F}'$ is the socle of $\mathscr{F}$. There is something to prove: that there is a unique maximal such $\mathscr{F}'$, and that it coincides with the (not necessarily direct) sum of all the simple submodules of $\mathscr{F}$. As the Frobenius correspondence $\mathrm{F}^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ sends simple modules to simple modules, it follows that it sends the socle into the socle; as the same is true of its inverse, it identifies the two socles. Since a semisimple module is the sum of its simple submodules, in order to take care of both points it is enough to show that

(*) If M is a sum of simple submodules, then M is a direct sum of simple submodules.

To be clear, if there were two maximal semisimple submodules $\mathrm{M}_1, \mathrm{M}_2 \subset \mathrm{M}$, then (*) shows that $\mathrm{M}_1 + \mathrm{M}_2$ is semisimple, hence $\mathrm{M}_1 = \mathrm{M}_1 + \mathrm{M}_2 = \mathrm{M}_2$. For the second point, (*) implies that the sum of all the simple submodules of M is semisimple; this sum is the largest semisimple module since if there were a larger one it would have an additional simple factor, contradicting the fact that the original contains all simple submodules of M.

To prove (*), in fact you can prove something stronger, which is that if N is any submodule of M, then $\mathrm{M} = \mathrm{N} \oplus \mathrm{N}'$, where N' is a direct sum of simple submodules. Suppose $\mathrm{M} = \mathrm{S}_1 + \ldots + \mathrm{S}_n$ with S_i simple, and let $\mathrm{I} \subset \{1, \ldots, n\}$ be maximal with respect to the property that the sum $\mathrm{W} := \mathrm{N} + \sum_{i \in \mathrm{I}} \mathrm{S}_i$ is direct. If $\mathrm{W} \neq \mathrm{M}$, then there is some $1 \leq j \leq n$ such that $\mathrm{S}_j \not\subset \mathrm{W}$, so $\mathrm{S}_j \cap \mathrm{W} = 0$ and $\mathrm{W} + \mathrm{S}_j = \mathrm{W} \oplus \mathrm{S}_j$, contradicting the maximality of I. Therefore $\mathrm{W} = \mathrm{M}$.

3.4.7. (i) Evidently $\left(\bigoplus_{b' \neq b} \mathscr{F}_0(b')\right) \cap \mathscr{F}_0(b) = 0$, since this sheaf, as a subsheaf of $\mathscr{F}_0(b)$, has punctual ı-weights in b, but $\bigoplus_{b' \neq b} \mathscr{F}_0(b')$ has no nonzero subsheaf with punctual ı-weights in b. (The punctual ı-weights at any point of a subsheaf are of course a subset of the ı-weights of the larger sheaf at that point.)

(ii) To see that $\mathrm{Gr}_i\,\mathscr{F}_0$ is ι-pure of weight i, apply the nine-lemma to the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{W}_{i-1}\mathscr{F}_0 \arrow[r, equal] \arrow[d] & \mathrm{W}_{i-1}\mathscr{F}''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'_0 \arrow[r] \arrow[d, equal] & \mathrm{W}_i\mathscr{F}_0 \arrow[r] \arrow[d] & \mathrm{W}_i\mathscr{F}''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'_0 \arrow[r] \arrow[d] & \mathrm{Gr}_i\,\mathscr{F}_0 \arrow[r] \arrow[d] & \mathrm{Gr}_i\,\mathscr{F}''_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

3.4.8. (i) The point is that provided that if specialization sends $(i^*\mathscr{F}_0)(b)$ into $i^*j_*((j^*\mathscr{F}_0)(b))$, then we get sheaves $\mathscr{F}_0(b)$ which restrict to the given ones over U_0 and F_0, and inclusions and projections. The required properties for the inclusions and projections can be checked punctually over U_0 and F_0, where they have already been established.

Note that in the above argument we assumed that M is a finite sum of simple submodules. To prove (*) in full generality, i.e. $\mathrm{M} = \sum_{j\in \mathrm{J}} \mathrm{S}_j$, one orders the sets of indices $\mathrm{I} \subset \mathrm{J}$ with the property that $\mathrm{N} + \sum_{i\in\mathrm{I}} \mathrm{S}_i = \mathrm{N} \oplus \bigoplus_{i\in\mathrm{I}} \mathrm{S}_i$ by inclusion and picks a maximal one by Zorn's lemma, then proceeds identically.

(ii) 'par descente' $\rightsquigarrow$ [**SGA1**, IX 4.12]: if $f : \mathrm{X}'_0 \to \mathrm{X}_0$ is the normalization morphism, f is a morphism of effective descent for the category of revêtements étales (fibered over the category of schemes). If $\mathrm{pr}_1, \mathrm{pr}_2 : \mathrm{X}'_0 \times_{\mathrm{X}_0} \mathrm{X}'_0 \to \mathrm{X}'_0$ denote the projections and $\mathscr{F}'_0$ denotes the reciprocal image of $\mathscr{F}_0$ on X', then W induces, by pullback, a finite increasing filtration W^a of $\mathrm{pr}_a^*\,\mathscr{F}'_0$ for $a = 1, 2$ by lisse subsheaves with the property that $\mathrm{Gr}_i^{\mathrm{W}^a}(\mathrm{pr}_a^*\,\mathscr{F}'_0) = \mathrm{pr}_a^*(\mathrm{Gr}_i^{\mathrm{W}}\,\mathscr{F}'_0)$ is punctually ι-pure of weight i. As $\mathscr{F}'_0$ comes from $\mathscr{F}_0$ on X_0, $\mathrm{pr}_1^*\,\mathscr{F}'_0 = \mathrm{pr}_2^*\,\mathscr{F}'_0$, and by the uniqueness already established, $\mathrm{W}^1 = \mathrm{W}^2$. Therefore by descent we get a filtration W on $\mathscr{F}_0$ by lisse subsheaves so that

$f^* \mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}_0 = \mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}'_0$ is ι-pure of weight i. As f is finite, this implies that $\mathrm{Gr}_i^{\mathrm{W}} \mathscr{F}_0$ is ι-pure of the same weight.

6.1.6. To see $(*)_0$, shrinking S we can assume all the irreducible components of Y dominate S; as f is dense, $\dim \mathrm{X}_\eta = 0$ means that $\mathrm{Y} \to \mathrm{S}$ is also generically finite. Shrinking S, $\mathrm{Y} \to \mathrm{S}$ becomes moreover finite (`02NW`). The complement $\mathrm{Y} - \mathrm{X}$ is of dimension strictly less than $\dim \mathrm{Y} = \dim \mathrm{S}$, hence its image in S is a proper closed sublocus, and shrinking S, we can make it disappear completely.

6.1.7. 'L'assertion est locale sur Y' $\rightsquigarrow$ Suppose one has a finite open cover $\{\mathrm{Y}_i\}$ of Y and the assertion on each Y_i. Let $\mathrm{Y}'_i, \mathrm{Y}_{1i}$ be as in the assertion for Y_i and shrink S so that $\mathrm{R}f_*\mathscr{F}$ restricts to a mixed complex on each Y'_i. The locus $\mathrm{Y}' := \cup_i \mathrm{Y}'_i$ is open; its complement $\mathrm{Y}_1 := \mathrm{Y} - \mathrm{U}$ has the property that $\mathrm{Y}_1 \cap \mathrm{Y}_i$ is a closed sublocus of Y_{1i}, hence finite over S. Shrinking S, we may assume every irreducible component of Y_1 dominates S, so that the η-fiber of $\mathrm{Y}_1 \to \mathrm{S}$ is the set of generic points of irreducible components of Y_1; in particular, this set is finite. Shrinking S further, the map $\mathrm{Y}_1 \to \mathrm{S}$ becomes finite.

The locus Y' is covered by open loci $\{\mathrm{Y}'_1, \dots, \mathrm{Y}'_n\}$ such that for each i, the restriction of the complex $\mathrm{R}f_*\mathscr{F}$ to Y'_i is mixed. Let us see that therefore the restriction of $\mathrm{R}f_*\mathscr{F}$ to Y' is mixed. Call this complex $\mathscr{G}$ and proceed by induction on n. Let $\mathrm{Y}'_n \xhookrightarrow{j} \mathrm{Y}' \xhookleftarrow{i} \mathrm{Y}' - \mathrm{Y}'_n$. In the distinguished triangle

$$j_! j^* \mathscr{G} \to \mathscr{G} \to i_* i^* \mathscr{G} \to$$

the first term is mixed by assumption and [**WeilII**, 3.3.1], and the third is mixed by the inductive hypothesis applied to $\mathrm{Y}' - \mathrm{Y}'_n$ and the stability of mixedness under reciprocal image and direct image by a finite morphism [**WeilII**, 1.2.5]. Therefore $\mathscr{G}$ is mixed.

By the above reasoning, if $\mathscr{G}$ is mixed when restricted to $\mathrm{pr}_i^{-1}(\mathrm{U}_i)$, it is mixed when restricted to $\cup_i \mathrm{pr}_i^{-1}(\mathrm{U}_i)$. Let Z denote the complement of this open locus in Y. Shrinking S, we may assume every irreducible component of Z dominates S. As the morphism $\mathrm{Z} \to \mathrm{S}$ factors through $\mathbf{A}^1_{\mathrm{S}}$ via the i projections, and $\mathrm{pr}_i(\mathrm{Z})$ misses the generic point of $\mathbf{A}^1_{\mathrm{S}}$ for all i, $\mathrm{Z} \to \mathrm{S}$ becomes finite after shrinking S: for each i, the restriction to Z of the i^{th} coordinate function on $\mathbf{A}^n_{\mathrm{S}}$ satisfies an equation of algebraic dependence over S; shrinking S, one can make these equations monic.

6.1.9. Let's see how to reduce to the case that X has a dense open locus smooth over S:

(1) Shrinking S, we can assume that every irreducible component of X dominates S.
(2) Going irreducible component by irreducible component, we can assume X (and S) are integral with generic points ξ and η, respectively.
(3) There is a finite, purely inseparable extension $k(\eta) \subset k'$ so that $k' \subset k(\xi) \otimes_{k(\eta)} k'$ induces a separable residual extension.
(4) Shrinking S, we can find a finite radicial extension $S' \to S$ inducing the given map $k(\eta) \subset k'$ on the generic fiber and so that S′ is reduced.
(5) It follows that $(X \times_S S')_{\mathrm{red}} \to S'$ is smooth at ξ.

The first two steps require no justification. As S is of finite type over $\mathbf{Z}[1/\ell]$, it is Nagata and in particular universally Japanese (`0335`). Steps (3) and (4) are trivial when S dominates $\mathbf{Z}[1/\ell]$ as k' can be taken to be the trivial extension; therefore for those steps we can and do assume that S is of finite type over $\mathbf{F}_p$.

Step (3): `04KM` shows that there exists a finite, purely inseparable extension $k(\eta) \subset k'$ such that the residue field of $k(\xi) \otimes_{k(\eta)} k'$ includes into a field K′ separably generated (`030O`) over k'. As this residue field is finitely generated over k', it follows (`030X`) that $k' \subset k(\xi) \otimes_{k(\eta)} k'$ is a separable residual extension.

Step (4): Let S'_1 denote the normalization of S in k'. As S is Japanese, $S'_1 \to S$ is finite (`032F`). Shrinking S, $S'_1 \to S$ is moreover flat (`032N`). Therefore $S'_1 \to S$ is radicial (`01S3`), as every geometric fiber has cardinality 1 by EGA IV_3 (15.5.2).[1]

Step (5): As $S' \to S$ and its base extension $X \times_S S' \to X$ are radicial, we can continue to denote their generic points by η and ξ, respectively. To see that $(X \times_S S')_{\mathrm{red}} \to S'$ is smooth at ξ, it suffices to show that the map of generic fibers is smooth at ξ (`01V9`). This follows by `00TV`) since by step (3), the map of generic points is a separable extension. By the definition of being smooth at a point (`01V5`), there are open neighborhoods $U \subset S'$ of η and $V \subset (X \times_S U)_{\mathrm{red}}$ of ξ such that $V \to U$ is smooth. As S′ is homeomorphic to S, the image of U in S is a nonempty open locus. Replacing S by this locus and

[1]An open morphism is open at every point – see EGA IV_3 (14.1.1) for the definition of open at a point.

S' by U, $S' \to S$ is our promised finite radicial extension such that the composition $\emptyset \neq V \hookrightarrow (X \times_S S')_{\mathrm{red}} \to S'$ is smooth.

6.1.10. This point is something of a mystery: if the conventions of (0.1) are in force, which they should be, X is already separated, so that f is separated, and hence admits a compactification, without assuming X is affine. On the other hand, this argument is lifted from [**D**, 2.8] where the schemes are not supposed separated, so I assume that (0.1) is not in force in the statement of (6.1.3) and X is not supposed separated.

In that case, of course the intersection of affines needn't be affine, so if you take an affine cover of X, terms will show up in the associated Leray spectral sequence `072E` that are not affine. One could take an affine hypercovering, but this is unnecessary: if $\{U_i\}$ are finitely many affine opens in X, although $\cap_i U_i$ needn't be affine, the map $\cap_i U_i \to Y$ factors as an open immersion into, say, U_1, followed by $U_1 \to Y$. As Y is affine, the map $U_1 \to Y$ is separated, so the map $\cap_i U_i \to Y$ is separated and admits a compactification. Therefore, the given argument will apply to each of the E_1 terms in the spectral sequence, so that all the sheaves on the E_1 page will be mixed. By the stability of mixedness under subquotients (1.2.5) (iii), the E_∞ terms will then be mixed, and $R^i f_* \mathscr{F}$ will be a successive extension of mixed sheaves, hence itself mixed.

6.1.12. 'La restriction de cette action au groupe d'inertie est automatiquement quasi unipotente.' $\rightsquigarrow$ Let I denote the inertia. It acts on $R^i\Psi(\mathscr{F}')$; as this sheaf is constructible, there is a finite set of geometric points $\{\overline{x}_i\}$ of $X_{\overline{s}}$ so that if a subgroup $I_1 \subset I$ acts unipotently on the stalks of this sheaf at the s_i, I_1 acts unipotently on the sheaf itself. The stalk of this sheaf at any $\overline{x}_i$ is a finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector space, and as S is of finite type over $\mathbf{F}_p$ or $\mathbf{Q}$, no finite extension of $k(s)$ contains all the roots of unity of order a power of ℓ. Therefore the conditions of Grothendieck's quasi-unipotent mondromy theorem [**SGA 7**, I 1.1] are satisfied, and there exists an open subgroup I_i of I such that σ acts unipotently on $R^i\Psi(\mathscr{F}')_{\overline{x}_i}$ for every $\sigma \in I_i$. The subgroup $I_1 := \cap_i I_i$ is open in I, and has the property that every σ in I_1 acts unipotently on the sheaf $R^i\Psi(\mathscr{F}')$.

A brief digression: let V be a finite-dimensional vector space over a finite extension of $\mathbf{Q}_\ell$ admitting a unipotent action of an open subgroup I_1 of I. As I_1 is compact, this action stabilizes a lattice in V, and the image of $P \cap I_1$ in $GL(V)$ is finite, hence torsion.

There is a basis for V for which I_1 acts via upper-triangular matrices with 1 along the diagonal [**Milne**, 14.2], and the latter subgroup is torsion free since $\mathbf{Q}_\ell$ is (c.f. e.g. *Algebra IV: Infinite Groups, Linear Groups* ed. Kostrikin & Shafarevich Part I, Chapter 2, §4.2 p. 74). It follows that $P \cap I_1$ acts trivially on V.

In the previous paragraph, P is the wild inertia, but it could be just as well be replaced by Q, the kernel of the projection to the maximal pro-ℓ subgroup $\mathbf{Z}_\ell(1)$ of I. The image of I_1 in the quotient $I \to I/Q$ is compact, hence closed, hence open. Moreover, I_1 acts on V via I_1/Q as we have seen, and every subgroup of finite index in $\mathbf{Z}_\ell$ is of the form $\ell^n\mathbf{Z}_\ell$ for some $n \in \mathbf{N}$. It follows that if T denotes a generator of $\mathbf{Z}_\ell(1)$, we know the action of I_1 once we know the action of T, and T is projectively unipotent; i.e. T^n acts unipotently on V for some $n > 0$. After replacing $S_{(s)}$ by a trait finite over $S_{(s)}$, we may assume $I = I_1$ and $n = 1$. Then, to know the unipotent action of I on $R^i\Psi(\mathcal{F}')$, it suffices to know the action of T.

All this to explain point (b): namely, if the action of I on $R^i\Psi(\mathcal{F}')$ is unipotent, we can obtain a finite filtration on $R^i\Psi(\mathcal{F}')$ by taking the filtration induced by the nilpotent operator N, the logarithm of monodromy. (As remarked in [**WeilII**, 1.7.3] and elucidated in the note to 1.7.3, N commutes with the action of $\mathrm{Gal}(\overline{\eta}, \eta)$. The filtration is simply the increasing one $F_k := \ker N^k$.) By construction, I acts as the identity on successive quotients of this filtration.

As for (c), the subgroup I_1 found in the first paragraph is an open subgroup of I, not $G := \mathrm{Gal}(\overline{\eta}, \eta)$. Here is a recipe for cooking up an open subgroup $G_1 \subset G$ so that $I \cap G_1 \subset I_1$, as promised by (c). As in the first paragraph, we can go point by point and find an open $G_i \subset G$ so that $I \cap G_i \subset I_i$, and take $G_1 = \cap_i G_i$. The inertia exact sequence

$$1 \to I \to G \to \mathrm{Gal}(\overline{s}, s) \to 1$$

is (non-canonically) split – see Lemma 7.6 of Serre's article[2] in Garibaldi-Merkurjev-Serre, *Cohomological Invariants in Galois Cohomology* and the second paragraph of (1.7.11). Set $V := R^i\Psi(\mathcal{F}')_{\overline{x}_i}$ and let $\rho_i : I \to GL(V)$ denote the monodromy representation at $\overline{x}_i$ and choose as before a basis for V so that $\rho_i(I_i)$ lands in the subgroup $U \subset GL(V)$ of upper-triangular matrices with 1 along the diagonal with respect to that

[2]Also in Brosnan-Reichstein-Vistoli, *Essential dimension of moduli of curves and other algebraic stacks.*

basis; U is closed in GL(V). Let $J_i := \rho_i^{-1}(U)$, which is closed, and, as it contains I_i, also open in I. As before, $J_i \cap P$ acts trivially so that J_i acts via the inertia character t_ℓ of (1.7). As G is a semidirect product of I and $\mathrm{Gal}(\bar{s}, s)$, identifying $\mathrm{Gal}(\bar{s}, s)$ with its image under a choice of splitting, $G = I\,\mathrm{Gal}(\bar{s}, s)$. Set $G_i := J_i\,\mathrm{Gal}(\bar{s}, s)$. To see that G_i is a subgroup, we have to show that for any $\sigma \in \mathrm{Gal}(\bar{s}, s)$, $\sigma J_i \sigma^{-1} = J_i$. Recall that $t_\ell(\sigma J_i \sigma^{-1}) = \sigma t_\ell(J_i)$, where here $\mathrm{Gal}(\bar{s}, s)$ acts on $\mathbf{Z}_\ell(1)$ via the character $\mathrm{Gal}(\bar{s}, s) \to \mathbf{Z}_\ell^\times$ obtained from the natural action of $\mathrm{Gal}(\bar{s}, s)$ on roots of unity. As $t_\ell(J_i)$ is an open subgroup of $\mathbf{Z}_\ell(1)$, it's of the form $\ell^a \mathbf{Z}_\ell(1)$ for some $a \geq 0$. As multiplication by $\mathbf{Z}_\ell^\times$ preserves this subgroup, $\sigma t_\ell(J_i) \subset t_\ell(J_i)$. As ρ_i factors through t_ℓ, this shows that $\rho_i(\sigma J_i \sigma^{-1}) \subset \rho(J_i) \subset U$. By the definition of J_i, $\sigma J_i \sigma^{-1} \subset J_i$, so G_i is a subgroup. It is evidently closed, and, as the finitely many coset representatives for I/J_i are also coset representatives for G/G_i, open. Finally, $G_i \cap I = J_i$, and by definition J_i acts unipotently on $V = R^i\Psi(\mathcal{F}')_{\bar{x}_i}$. We're done.

6.1.13. The correct version of formula (1.6.14.3) (c.f. the note to (1.6.14)) shows that it suffices to show that the P_{-j} are mixed. Formula (1.6.14.2) then shows that it suffices to show that $\ker N$ is mixed, where if T is any generator of the maximal pro-ℓ quotient of I, $N = \log T$. Then $\ker N$ coincides with invariants under T; as I acts unipotently, this is the same as invariants under I.

That the $(R^i\Psi\mathscr{F}')^I$ are quotients of the $u^*R^i v_*(v^*\mathscr{F})$ is the exact sequence (5) of the proof of (3.6.1) in its incarnation

$$0 \to (R^i\Psi\mathscr{F}')_I(-1) \to u^*R^i v_*(v^*\mathscr{F}) \to (R^i\Psi\mathscr{F})^I \to 0.$$

For this, following appendix A of BBD, we have $R\Gamma(I, R\Psi\mathscr{F}') = u^*Rv_*(v^*\mathscr{F})$ (technically the inverse image of the latter by $S_{(\bar{s})} \to S_{(s)}$ with action of $\mathrm{Gal}(\bar{s}/s)$ compatible with the action of Galois on $X_{\bar{s}}$, but this is the same as what is written). The argument of (1)–(5) in the proof of (3.6.1) then gives the above exact sequence; on the matter of invariants by a finite group of order indivisible by ℓ coinciding with coinvariants (acting on a sheaf of $\mathbf{Z}/\ell^n$-, $\mathbf{Z}_\ell$-, $\mathbf{Q}_\ell$- or $\overline{\mathbf{Q}}_\ell$-modules), the point is that you can *average*, giving a map to invariants through which the map to coinvariants factors – see the note to BBD appendix A. This argument shows that $(R\Psi\mathscr{F}')^{I'} = (R\Psi\mathscr{F}')_{I'}$, where I' is the finite group of order prime to ℓ of (3.6.1) (called Q in the appendix A to BBD).

6.2.5. c) As X is a curve, the restriction of $j_*\mathscr{F}$ to X – U depends only on the generic behavior of $\mathscr{F}$, so that for the purpose of calculating the punctual weights of $\mathscr{F}$ at points of X – U we may assume $\mathscr{F}$ is lisse. Then (1.8.8.1) gives that the weights of $j_*\mathscr{F}$ at points of X – U are $\leq n$. As $\mathscr{F}$ is pure, $\mathscr{H}\!\mathit{om}(\mathscr{F}, \overline{\mathbf{Q}}_\ell)$ is mixed of weight $\leq -n$, so again by (1.8.8.1) $j_*\mathscr{H}\!\mathit{om}(\mathscr{F}, \overline{\mathbf{Q}}_\ell)$ is mixed of weight $\leq -n$.

d) Same argument with (1.8.8.1) replaced by (1.8.8.2).

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4. *Faisceaux Pervers*

1.1.11. To verify the anti-commutativity of the 9th square, as the morphism of triangles $(X'', Y'', Z'') \to (X'[1], Y'[1], Z'[1])$ factors as the composition of two morphisms of triangles $(X'', Y'', Z'') \to (A, Y'', Z'[1]) \to (X'[1], Y'[1], Z'[1])$, where the second arrow is the rotation of $(Z', A, Y'') \to (Z', X'[1], Y'[1])$, it suffices to verify that the triangle $(Z', X'[1], Y'[1])$ which appears in this last morphism of triangles has all arrows induced by the arrows of (X', Y', Z') or translates of them (with the same parity). This is not hard to check from the diagram (1). (The stated explanation appears to be an un-explanation.)

1.3.3. Though it is not stated explicitly, it is immediate from the definition $\mathscr{D}^{\leq n} := \mathscr{D}^{\leq 0}[-n]$, $\mathscr{D}^{\geq n} := \mathscr{D}^{\geq 0}[-n]$ that $(\tau_{\leq n} X)[m] = \tau_{\leq n-m}(X[m])$ and $(\tau_{\geq n} X)[m] = \tau_{\geq n-m}(X[m])$. Namely, for X in $\mathscr{D}$ and T in $\mathscr{D}^{\leq n}$, $T = T'[-m]$ for some T' in $\mathscr{D}^{\leq n-m}$, so

$$\begin{aligned} &\mathrm{Hom}(T, (\tau_{\leq n-m}(X[m]))[-m]) = \mathrm{Hom}(T'[-m], (\tau_{\leq n-m}(X[m]))[-m]) \\ &= \mathrm{Hom}(T'[-m], \tau_{\leq n-m}(X[m]))[-m] = \mathrm{Hom}(T', X[m])[-m] = \mathrm{Hom}(T'[-m], X) \\ &= \mathrm{Hom}(T, X). \end{aligned}$$

1.4.2.1. The argument for why the derived functors continue to satisfy the stated adjunctions is as follows (this argument is also found in SGA 4 Exp. XVIII 3.1.4.11). Given F, G an adjoint pair of functors on abelian categories

$$\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$$

where both categories have enough injectives and L is exact. The functors F and G extend to functors $D^+(\mathscr{A}) \rightleftarrows D^+(\mathscr{B})$. Given $K^\cdot \in D^+(\mathscr{A}), L^\cdot \in D^+(\mathscr{B})$, we may assume $L^\cdot$ is a complex of injective objects; we have an isomorphism of triple complexes

$$\mathrm{Hom}^\cdot(F(K^\cdot), L) \overset{\sim}{\longleftrightarrow} \mathrm{Hom}^\cdot(K^\cdot, G(L^\cdot)).$$

As G preserves injectives, taking H^0 of the associated simple complex (calculated with products) finds the desired

$$\mathrm{Hom}_{K(\mathscr{B})}(F(K^\cdot), L^\cdot) \overset{\sim}{\longleftrightarrow} \mathrm{Hom}_{K(\mathscr{A})}(K^\cdot, G(L^\cdot)),$$

where both sides are also Hom in the respective derived categories, since $\mathrm{L}^\bullet$ and $\mathrm{G}(\mathrm{L}^\bullet)$ are complexes of injectives.

1.4.4. The question is, why are the adjoints to the Verdier quotients fully faithful? Let's consider the quotient $\mathrm{Q} : \mathscr{T} \to \mathscr{T}/\mathscr{U}$, where $\mathscr{U}$ is the strictly full coreflective triangulated subcategory of $\mathscr{T}$; $(\mathscr{U}, \mathscr{V})$ form a t-structure on $\mathscr{T}$; $\mathscr{U} = {}^\perp\mathscr{V}$, and $\mathscr{V} = \mathscr{U}^\perp$. Since the embedding $u : \mathscr{U} \to \mathscr{T}$ admits a right adjoint $u_\bullet$ it follows that Q admits a right adjoint $\mathrm{Q}_\bullet$ [**CD**, I 6-5]. There is a natural isomorphism of functors $\mathrm{Q}_\bullet \circ \mathrm{Q} \xrightarrow{\sim} v \circ v^\bullet$, where $v^\bullet$ is the left adjoint to the inclusion $v : \mathscr{V} \to \mathscr{T}$ [**CD**, I 6-6]. The functor $v^\bullet$ is nothing other than $\tau_{\geq 0}$ for the t-structure $(\mathscr{U}, \mathscr{V})$, and therefore, restricted to $\mathscr{V}$, $v \circ v^\bullet = \mathrm{id}|_{\mathscr{V}}$. On the other hand, the functor Q when restricted to $\mathscr{V}$ is fully faithful [**CD**, I 5-3]. Therefore $\mathrm{Q}_\bullet$ restricted to the essential image of $\mathscr{V}$ under Q is fully faithful. This essential image is all of $\mathscr{T}/\mathscr{U}$, since every object X in $\mathscr{T}$ belongs to an exact triangle $(\mathrm{U}, \mathrm{X}, \mathrm{V})$ with U, V objects in $\mathscr{U}, \mathscr{V}$, respectively. The assertion that $v^\bullet$ yields an equivalence $\underline{v}^\bullet : \mathscr{T}/\mathscr{U} \xrightarrow{\sim} \mathscr{V}$ (functor obtained by applying the universal property of $\mathscr{T}/\mathscr{U}$ to $v^\bullet$) is easy since $\mathrm{Q} \circ v : \mathscr{V} \to \mathscr{T}/\mathscr{U}$ is an equivalence, and $\underline{v}^\bullet \circ \mathrm{Q} \circ v = v^\bullet \circ v = \mathrm{id}$. The corresponding statement for $\mathscr{T} \to \mathscr{T}/\mathscr{V}$ follows identically.

1.4.6. In part b), there is the following consideration. Given triangulated categories $\mathscr{T}, \mathscr{T}'$, a thick subcategory $\mathscr{U} \subset \mathscr{T}$ with Verdier quotient $\mathrm{Q} : \mathscr{T} \to \mathscr{T}/\mathscr{U}$, exact functors $\mathrm{F}, \mathrm{G} : \mathscr{T}/\mathscr{U} \to \mathscr{T}'$ and a natural transformation $\overline{\varphi} : \mathrm{F} \circ \mathrm{Q} \to \mathrm{G} \circ \mathrm{Q}$, there is an obvious candidate for a natural transformation $\varphi : \mathrm{F} \to \mathrm{G}$, since $\mathrm{Ob}(\mathscr{T}/\mathscr{U}) = \mathrm{Ob}(\mathscr{T})$. But is it still a natural transformation? Let $f : \mathrm{X} \to \mathrm{Y}$ in $\mathscr{T}/\mathscr{U}$ be represented by $\mathrm{X} \xleftarrow{s} \mathrm{Z} \xrightarrow{a} \mathrm{Y}$, where s is in the saturated multiplicative system of morphisms corresponding to $\mathscr{U}$. The commutative diamond

$$
\begin{tikzcd}
 & & \mathrm{F(Z)} \arrow[dl, "\mathrm{F}(s)"'] \arrow[dr, "\varphi(\mathrm{Z})"] & & \\
 & \mathrm{F(X)} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\varphi(\mathrm{X})"] & & \mathrm{G(Z)} \arrow[dl, "\mathrm{G}(s)"'] \arrow[dr, "\mathrm{G}(a)"] & \\
\mathrm{F(X)} & & \mathrm{G(X)} & & \mathrm{G(Y)}
\end{tikzcd}
$$

shows that $\varphi(\mathrm{X}) \circ \mathrm{G}(f)$ coincides with $\mathrm{F}(\mathrm{X}) \xleftarrow{\mathrm{F}(s)} \mathrm{F}(\mathrm{Z}) \xrightarrow{\mathrm{G}(a)\circ\varphi(\mathrm{Z})} \mathrm{G}(\mathrm{Y})$, but as

$$\begin{array}{ccc} \mathrm{F}(\mathrm{Z}) & \xrightarrow{\mathrm{F}(a)} & \mathrm{F}(\mathrm{Y}) \\ \downarrow{\scriptstyle \varphi(\mathrm{Z})} & & \downarrow{\scriptstyle \varphi(\mathrm{Y})} \\ \mathrm{G}(\mathrm{Z}) & \xrightarrow{\mathrm{G}(a)} & \mathrm{G}(\mathrm{Y}) \end{array}$$

commutes, this morphism is just $\mathrm{F}(\mathrm{X}) \xleftarrow{\mathrm{F}(s)} \mathrm{F}(\mathrm{Z}) \xrightarrow{\varphi(\mathrm{Y})\circ\mathrm{F}(a)} \mathrm{G}(\mathrm{Y}) = \mathrm{F}(f) \circ \varphi(\mathrm{Y})$, which shows that φ is indeed a natural transformation.

1.4.7. In c), by 1.1.9, the morphism $\mathrm{B} \to \mathrm{C}$ is the unique such that completes the morphism of triangles $(\mathrm{B}, j_!j^*\mathrm{X}[1], j_*j^*\mathrm{X}[1]) \to (\mathrm{C}, \mathrm{X}[1], j_*j^*\mathrm{X}[1])$.

1.4.13. The distinguished triangle $(\tau^{\mathrm{F}}_{\leq p}\mathrm{X}, \mathrm{X}, i_*\tau_{>p}i^*\mathrm{X})$ is the distinguished triangle $(\mathrm{A}, \mathrm{Y}, i_*\tau_{>p}i^*\mathrm{Y})$ of 1.4.10, since as remarked, $\mathrm{X} = \mathrm{Y}$ since $\tau_{>0}j^*\mathrm{X} = 0$. To check that $\mathrm{A} = \tau^{\mathrm{F}}_{\leq p}\mathrm{X}$, note that A belongs to $\mathscr{D}^{\leq p}$, as $i^*\mathrm{A} \simeq \tau_{\leq p}i^*\mathrm{X}$, and that if T belongs to $\mathscr{D}^{\leq p}$, then by applying $\mathrm{Hom}(\mathrm{T}, -)$ to the above distinguished triangle and observing that as i_* is t-exact, it commutes with truncation, so $i_*\tau_{>p}i^*\mathrm{X}[-1]$ lies in $\mathscr{D}^{>p+1}$, $\mathrm{Hom}(\mathrm{T}, i_*\tau_{>p}i^*\mathrm{X}) = 0 = \mathrm{Hom}^{-1}(\mathrm{T}, i_*\tau_{>p}i^*\mathrm{X})$, and $\mathrm{Hom}(\mathrm{T}, \mathrm{X}) \simeq \mathrm{Hom}(\mathrm{T}, \mathrm{A})$.

The distinguished triangle $(\tau^{\mathrm{F}}_{\leq p-1}\mathrm{X}, \mathrm{X}, i_*\tau_{>p-1}i^*\mathrm{X})$ and the fact that i_* commutes with truncation establishes $i_*\tau_{>p-1}i^*\mathrm{X}$ as $\tau_{>p-1}\mathrm{X}$ for the t-structure on $\mathscr{D}$; applying $\tau_{\leq p}$ and passing it through the i_* gives the statement about cohomology.

1.4.14. To find the dual statement at the end of the proof, reverse arrows and exchange $j_! \leftrightarrow j_*$, $i^* \leftrightarrow i^!$ to obtain the distinguished triangle $(i_*i^*\mathrm{X}[-1], j_!\mathrm{Y}, \mathrm{X})$, then use (b'), the isomorphism $j_*/j_! \simeq i^!j_![1]$ of 1.4.6.4, (and the note to 1.3.3) to write

$$i_*i^*\mathrm{X}[-1] = i_*(\tau_{\leq p-1}(j_*/j_!)\mathrm{Y})[-1] = i_*\tau_{\leq p}((j_*/j_!)\mathrm{Y}[-1]) = i_*\tau_{\leq p}i^!j_!\mathrm{Y},$$

establishing X as $\tau^{\mathrm{F}}_{\geq p+1}j_!\mathrm{Y}$.

1.4.17.1. A little note: ${}^pi^*\mathrm{X}$ is the largest quotient of X belonging to $\mathscr{C}_{\mathrm{F}}$. First we check that it is a quotient from 1.4.17 (ii). Then, suppose A belongs to $\mathscr{C}_{\mathrm{F}}$ and $\mathrm{X} \twoheadrightarrow \mathrm{A}$;

then ${}^p i^* \mathrm{X} \twoheadrightarrow {}^p i^* \mathrm{A} \xrightarrow{\sim} \mathrm{A}$, as ${}^p i_*$ is fully faithful, so the adjunction morphism ${}^p i^* {}^p i_* \to \mathrm{id}$ is an isomorphism, and ${}^p i^* \mathrm{X}$ is indeed the largest quotient of X in $\mathscr{C}_\mathrm{F}$. Dually for ${}^p i^!$.

1.4.18. A little note about T faithful: as ${}^p j^*$ is an exact functor, if ${}^p j^* f_1 = 0$, this means that ${}^p j^* \operatorname{im}(f_1) = \operatorname{im}({}^p j^* f_1) = 0$, which is to say that $\operatorname{im} f_1$ belongs to $\overline{\mathscr{C}}_\mathrm{F}$.

1.4.23. In the distinguished triangle $(i_* \mathrm{H}^0 i^! j_! \mathrm{B}, \tau^\mathrm{F}_{\geq 0} j_! \mathrm{B}, \tau^\mathrm{F}_{\geq 1} j_! \mathrm{B})$, as $j_! \mathrm{B} = \tau^\mathrm{U}_{\geq 0} j_! \mathrm{B}$, $\tau_{\geq p} = \tau^\mathrm{F}_{\geq p} \tau^\mathrm{U}_{\geq p}$, and $j_!$ is right t-exact, $\tau^\mathrm{F}_{\geq 0} j_! \mathrm{B}$ sits in $\mathscr{C}$. Likewise, i_* is t-exact, so $i_* \mathrm{H}^0 i^! j_! \mathrm{B}$ also sits in $\mathscr{C}$, and from the long exact sequence of H^i one finds that $\tau^\mathrm{F}_{\geq 1} j_! \mathrm{B}$ is in $\mathscr{D}^{[-1,0]}$.

2.1.2. In the discussion '*Si les foncteurs* ${}^\circ i^!_\mathrm{S}$ *sont de dimension cohomologique finie...*' it is claimed that there is a neighborhood of S in which $\mathrm{H}^j \tau_{<a} \mathrm{K}$ is supported on S. To find such a neighborhood, simply discard $\overline{\mathrm{S}} - \mathrm{S}$ and the closure of any stratum which doesn't meet S. The assumption that the closure of each stratum is a union of strata implies that the induced stratification of the resulting neighborhood of S has the property that every stratum contains S in its closure, and therefore $\mathrm{H}^j \tau_{<a} \mathrm{K}$ vanishes on every stratum distinct from S. By construction, S is a closed set in this neighborhood.

As for the isomorphism $\mathrm{H}^i(i^*_\mathrm{S} \tau_{<a} \mathrm{K}) \xleftarrow{\sim} \mathrm{H}^i(i^!_\mathrm{S} \tau_{<a} \mathrm{K})$ for $i < a$, let us replace X by the neighborhood above, in which case the adjunction morphism $\tau_{<a} \mathrm{K} \to i_{\mathrm{S}*} i^*_\mathrm{S} \tau_{<a} \mathrm{K}$ is an isomorphism as ${}^\circ i_{\mathrm{S}*}$ and ${}^\circ i^*_\mathrm{S}$ are exact and the induced morphism on cohomology $\mathrm{H}^j(\tau_{>a} \mathrm{K}) \to i_{\mathrm{S}*} i^*_\mathrm{S} \mathrm{H}^j(\tau_{<a} \mathrm{K})$ is an isomorphism for all j. Therefore by 1.4.1.2, $i^!_\mathrm{S} \tau_{<a} \mathrm{K} \xrightarrow{\sim} i^!_\mathrm{S} i_{\mathrm{S}*} i^*_\mathrm{S} \tau_{<a} \mathrm{K} \xrightarrow{\sim} i^*_\mathrm{S} \tau_{<a} \mathrm{K}$.

2.1.13. To get the desired conclusion from the spectral sequence $\mathrm{R}^p j_* \mathrm{H}^q \mathrm{K} \Rightarrow \mathrm{H}^{p+q} \mathrm{R} j_* \mathrm{K}$, recall that the locally constant constructible sheaves form a weak Serre subcategory of the category of constructible sheaves of $\mathscr{O}$-modules. For $j^!$ when j is a closed immersion, just use that $j_! = j_*$ is exact in the long exact sequence of cohomology for the distinguished triangle $(j_! j^! \mathrm{K}, \mathrm{K}, k_* k^* \mathrm{K})$.

One can deduce that the truncation operators $\tau_{\leq p}$ and $\tau_{\geq p}$ respect $\mathrm{D}_c(\mathrm{X}, \mathrm{R})$ from the proof of 1.4.10 by induction on the number of strata à la 2.1.3.

2.1.14. For the isomorphism $i^!\mathrm{L} \simeq i^*\mathrm{L} \otimes_{\mathbf{Z}} \mathrm{or}[-d]$, combine Proposition 4.3.6 from Dimca, *Sheaves in Topology* with the description of the relative dualizing complex in, e.g., Remark 3.3.5 in Kashiwara-Schapira, *Sheaves on Manifolds*, taking $\mathrm{S} = \{*\}$ there and noting that $f^!\mathrm{A_X}$ is the relative dualizing complex $\omega_{\mathrm{Y/X}}$ (Definition 3.1.16), and the orientation sheaves are self-dual (so that Deligne's or coincides with $\mathrm{or_T} \otimes i^* \mathrm{or_S}$). The 'non-characteristic' hypothesis in Dimca's Proposition is trivial in light of the result that precedes it and the fact that the $\mathrm{H}^m j^!\mathrm{K}$ are locally constant.

Note that by the description of $i^!$ in terms of local cohomology, this gives a statement like, $\mathrm{H}^n_{\mathrm{T}}(\mathrm{S}, \mathscr{F})$ vanishes in degrees less than the codimension of T, for $\mathscr{F}$ a locally free sheaf (with obvious extension to $\mathscr{F}$ a bounded below complex with locally free H^n).

2.1.16. In the remarks about Verdier duality, the fact that the functor D exchanges $j_! \leftrightarrow j_*$, $j^! \leftrightarrow j^*$ follows by taking L to be the dualizing complex in the local formulas of adjunction

$$j_*\mathrm{DK} = j_*\mathrm{R}\mathscr{H}om(\mathrm{K}, j^!\mathrm{L}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(j_!\mathrm{K}, \mathrm{L}) = \mathrm{D}j_!\mathrm{K}$$

$$\mathrm{D}j^*\mathrm{K} = \mathrm{R}\mathscr{H}om(j^*\mathrm{K}, j^!\mathrm{L}) \xrightarrow{\sim} j^!\mathrm{R}\mathscr{H}om(\mathrm{K}, \mathrm{L}) = j^!\mathrm{DK}$$

(the latter may be found, e.g. in SGA 4 Exp. XVIII 3.1.12.2, or Dimca, 3.3.7, but see especially SGA 5 Exp. I 1.12). These formulæ hold for more general morphisms than the inclusion of a locally closed subscheme; the condition is compactifiability.

In *Th. finitude* §4 'Bidualité locale,' Deligne puts Verdier duality for étale cohomology on firm footing in the case of $a : \mathrm{X} \to \mathrm{S}$ a scheme of finite type over S a regular scheme of dimension 0 or 1. If $\mathrm{A} = \mathbf{Z}/n$, $\mathrm{K_S}$ constant sheaf on S with value A, and K in $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{A})$, put $\mathrm{K_X} := \mathrm{R}a^!\mathrm{K_S}$ and $\mathrm{DK} := \mathrm{R}\mathscr{H}om(\mathrm{K}, \mathrm{K_X})$. Then $\mathrm{K_X}$ is dualizing; i.e.

Theorem (*Deligne*). — $\mathrm{K} \xrightarrow{\sim} \mathrm{DDK}$.

This involutivity establishes the stated duality in the formalism, since we may write

$$\mathrm{D}j_*\mathrm{K} = \mathrm{D}j_*\mathrm{DDK} = \mathrm{DD}j_!\mathrm{DK} = j_!\mathrm{DK}.$$

It is essential that the cohomology sheaves be locally constant of finite rank when restricted to each stratum and that R have the proscribed properties so that Poincaré duality holds on each stratum. In fact, the definition of perverse sheaf is engineered expressly so that on each stratum we have Poincaré duality, and this data determines the Verdier dual of the sheaf on the stratified space.

About the formula $\mathrm{H}^i\mathrm{DK} = (\mathrm{H}^{-d-i}\mathrm{K})^\vee \otimes$ or, let's do it instead in the case $\mathrm{R} = \mathbf{Z}/\ell$, in which case the dualizing complex is $\mathbf{Z}/\ell(d)[2d]$ and the formula is

$$\mathrm{H}^i\mathrm{DK} = (\mathrm{H}^{-2d-i}\mathrm{K})^\vee(d). \tag{†}$$

This is simply Poincaré duality (SGAA XVIII 3.2.5). (In light of the note to 2.1 in *Th. finitude*, this is the immediate consequence of the weakly convergent spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{Ext}^p(\mathrm{H}^{-q}(\mathrm{K}), \mathbf{Z}/\ell(d)) \Rightarrow \mathrm{Ext}^{p+q}(\mathrm{K}, \mathbf{Z}/\ell(d))$$

which collapses at the E_2 page.)

The business about exchanging ${}^{p^*}\mathrm{H}^i$ and ${}^p\mathrm{H}^{-i}$ is seen to be true for H^0, then use $\mathrm{H}^i\mathrm{K} = \mathrm{H}^0(\mathrm{K}[i])$ and $\mathrm{D}(\mathrm{K}[n]) = \mathrm{R}\mathscr{H}om(\mathrm{K}[n], -) = \mathrm{R}\mathscr{H}om(\mathrm{K}, -)[-n] = (\mathrm{DK})[-n]$

2.2.2. A word about the 'trivial' implication (ii)⇒(iii). As each S in $\mathscr{T}$ is lisse equidimensional, applying (ii) to each irreducible component we find there exists a Zariski dense open $i : \mathrm{U} \hookrightarrow \mathrm{S}$ such that $i^*\mathrm{H}^i i_\mathrm{S}^*\mathrm{K} = \mathrm{H}^i i^* i_\mathrm{S}^*\mathrm{K}$ (resp. $i^*\mathrm{H}^i i_\mathrm{S}^!\mathrm{K} = \mathrm{H}^i i^! i_\mathrm{S}^!\mathrm{K}$) vanish in degrees $i > p(\mathrm{S})$ (resp. $i < p(\mathrm{S})$); as $\mathrm{H}^i i_\mathrm{S}^*\mathrm{K}$ and $\mathrm{H}^i i_\mathrm{S}^!\mathrm{K}$ are locally constant, this implies they also vanish when their restrictions to U do.

In the second paragraph, '*il reste à montrer que chacune implique que* $\mathrm{H}^i\mathrm{K} = 0$ *pour $i > b$ (resp. $i < a$).*' Without securing this, we would not have that $\tau_{\geq a}\mathrm{K}$ (resp. $\tau_{\leq b}\mathrm{K}$) belongs to D_c^b and the proof in the first paragraph wouldn't apply. In both cases, the verification for $i > b$ is easy as ${}^\circ i^*$ is exact. The verification for (iii) $i < a$ follows the proof of (2.1.2.1) exactly, after $\mathscr{T}$ has been replaced by a finer stratification. The verification for (ii) follows immediately by the noetherian property from the stated claim that for each irreducible subvariety S′ and each $i < a$, there exists a dense open S of S′ such that $\mathrm{H}^i\mathrm{K}$ vanishes on S. To prove this claim we proceed by descending induction on dim S′, the maximal nonvacuous case being easy since such an S′ is an irreducible component of X, and $\mathrm{H}^i\mathrm{K}$ vanishes on the dense open S obtained from (ii) since S is open in X and $i_\mathrm{S}^! = i_\mathrm{S}^*$ is exact. The case of general S′ will follow from the argument of the proof of (2.1.2.1) (see note for 2.1.2) if we can find a neighborhood U of the S obtained from (ii) so that $\mathrm{supp}((\mathrm{H}^j\tau_{<a}\mathrm{K})|_{\mathrm{U}}) \subset \mathrm{S}$. Begin with the irreducible component containing S′; the inductive assumption gives an open set which has either empty or nonempty intersection with S′. If nonempty, then this intersection is the desired open of S′ of the claim. If empty, S′ belongs to the complement. Throw away all irreducible components of the complement that do not contain S′ and apply the same process to the irreducible component of the complement that contains S′. After finitely many steps we are left only with S′, and we throw away S′ − S. The open neighborhood U is the set we are left with after throwing away finitely many closed sets of X, and S is closed in U.

2.2.3. In the discussion of the intermediate extension, the triangle $(\tau_{<t}i^*j_*\mathrm{A}, i^*j_*\mathrm{A}, \tau_{\geq t}i^*j_*\mathrm{A})$ is distinguished, not the one written, and if $\tau_{\geq t}i^*j_*\mathrm{A}$ is in ${}^p\mathrm{D}_c^{\geq 0}$, then indeed ${}^p\tau_{<0}i^*j_*\mathrm{A} \xrightarrow{\sim} \tau_{<t}i^*j_*\mathrm{A}$ and $\tau_{\geq t}i^*j_*\mathrm{A} \simeq {}^p\tau_{\geq 0}i^*j_*\mathrm{A}$; the latter isomorphism establishes an isomorphism ${}^p\tau^{\mathrm{F}}_{<0}j_*\mathrm{A} \xrightarrow{\sim} \tau^{\mathrm{F}}_{<t}j_*\mathrm{A}$, which differs by one character from what is written.

2.2.8. The discussion of $\mathrm{R}\mathscr{H}om$ differs from what is proved in 2.1.20 in that there is no longer a fixed stratification. Fortunately, if $p \leq a, b \geq q$, there are only finitely many solutions to $i = m - n$ for a fixed i and for $n \leq a$, $m \geq b$. Therefore for each i, we can apply the reasoning of 2.1.20 to a common refinement of only finitely many stratifications.

2.2.10. As remarked in the note to 2.1.16, we will an involutive Verdier duality regardless of whether each stratum is smooth or the sheaf on it is lisse (or has lisse cohomology). The reason we restrict to smooth strata and complexes of sheaves K with lisse cohomology on each stratum is so that we can determine precisely the degrees in which the Verdier dual DK is concentrated due to the simple description of the dualizing complex and the simple computation of $\mathrm{R}\mathscr{H}om$ (c.f. the note to *Th. finitude* 1.6). This is necessary to make Verdier duality a compatible operation vis à vis the t-structures attached to a given perversity function and its dual.

The program outlined in this paragraph achieves both the smoothness of the strata and the lisseté of the cohomology sheaves on each stratum so that the latter property is moreover respected by the six functors (and hence also by Verdier duality). Here is an explication of the smoothness condition (a).

Lemma (*EGA* 0_{IV} *22.5.8 & IV 6.7.6, 6.7.8, Stacks tags* `07EL` *&* `038X`). — *Let* X *be a scheme locally of finite type over a field* k *and* $x \in \mathrm{X}$. *Then the following are equivalent:*

(i) $\mathrm{X} \to \operatorname{Spec} k$ *is smooth at* x.
(ii) X *is geometrically regular at* x, *i.e. for every finite extension* k' *of* k, *the semi-local ring* $(\mathscr{O}_{\mathrm{X}})_x \otimes_k k'$ *is regular.*
(iii) $\mathrm{X} \times_k \overline{k}$ *is regular at every point lying over* x.

The smoothness condition on strata is that over $\overline{k}$, each stratum S, with the reduced subscheme structure, is smooth. The claim is then that on S equidimensional of dimension d, the dualizing complex is given by $\mathbf{Z}/\ell(d)[2d]$. After replacing k by its perfect closure, anodyne operation with respect to the étale topology, we may assume S is of finite type over a perfect field k. The fact that $\mathrm{S} \times_k \overline{k}$ is smooth implies (in light of the lemma and Stacks tag `030U`) that S, with its reduced scheme structure, is smooth. In this case, the fact about the dualizing complex is standard (SGAA Exp. XVIII 3.2.5).

2.2.14. A brief review of Galois cohomology of a finite field k, to recall why the groups $\mathrm{H}^i(\mathrm{Gal}(\overline{k}/k'), \mathbf{Z}/\ell)$ are finite for every finite extension k' of k. We have

$$\mathrm{H}^0 = \mathrm{H}^1 = \mathbf{Z}/\ell$$
$$\mathrm{H}^i = 0 \quad i > 1.$$

The case of H^0 is obvious as it corresponds to taking $\mathrm{G} = \mathrm{Gal}(\overline{k}/k')$-invariants of a trivial G-module. The H^1 is the corollary of a formula given in Serre, *Corps Locaux*, Ch. XIII Prop. 1 (p. 197 in the 1968 édition Hermann). The vanishing in degrees > 1 is because a finite field is C_1 and hence has finite cohomological dimension; now see (1.6) in *Arcata*, SGA $4\frac{1}{2}$. For more details see Serre, *Cohomologie Galoisienne*, Ch. II, §3.

Let $\mathscr{D}_n = \mathrm{D}^b_{ctf}(\mathrm{X}, \mathbf{Z}/\ell^n)$, K, L objects of $\mathscr{D}_n$, $\mathrm{G} = \mathrm{Gal}(\overline{k}, k)$, and $f : \mathrm{X} \to \operatorname{Spec} k$ the structure morphism. Why does the above imply that $\mathrm{Hom}_{\mathscr{D}_n}(\mathrm{K}, \mathrm{L})$ is finite? Deligne's finiteness theorems show that the sheaves $\mathrm{R}\mathscr{H}\!om(\mathrm{K}, \mathrm{L})$ belong to $\mathscr{D}_n$. The

Hochschild-Serre spectral sequence gives

$$E^2_{ij} = H^i(G, R^j\Gamma(X \times_k \overline{k}, R\mathscr{H}om(K,L))) \Rightarrow R^{i+j}\Gamma(X, R\mathscr{H}om(K,L)). \qquad (\dagger)$$

As $R\Gamma(X \times_k \overline{k}, R\mathscr{H}om(K,L))$ coincides with the stalk of the constructible sheaf $Rf_*R\mathscr{H}om(K,L)$ at any geometric point of Spec k, it belongs to $D^+_c(\mathbf{Z}/\ell^n)$ (even D^+_{ctf}; c.f. Th. finitude 1.7). Fix j and let $A := R^j\Gamma(X \times_k \overline{k}, R\mathscr{H}om(K,L))$; it is a finite G-module. Let U denote the kernel of $G \to \operatorname{Aut} A$; it is an open normal subgroup of finite index corresponding to a finite extension k' of k. The Galois group $U = \operatorname{Gal}(\overline{k}, k')$ acts trivially on A and a simple dévissage reducing to the case $\mathbf{Z}/\ell$ shows that the $H^i(U, A)$ are finite. As G/U is a finite group, the spectral sequence (c.f. *Cohomologie Galoisienne* §2.6b)

$$H^p(G/U, H^q(U,A)) \Rightarrow H^{p+q}(G, A)$$

shows that the groups $H^i(G, A)$ are also finite, and therefore that the objects on the E^2 page of (†) are finite $\mathbf{Z}/\ell^n$-modules so that $R^0\Gamma(X, R\mathscr{H}om(K,L)) = \operatorname{Hom}_{\mathscr{D}_n}(K,L)$ is finite.

2.2.16. See note to Weil II (1.1.2). In that paper, the functor $H^i(K)$ is defined as the pro-sheaf which is the projective system defined by the $H^i(K_n)$; the corresponding projective system is AR-isomorphic to an $(\ell\mathbf{Z})$-adic sheaf in the naïve sense. This allows us to upgrade the pointwise exact sequence (*) of those notes to the corresponding sequence of sheaves (2.2.16.1). There, he uses the notation $K \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n$ for K_n, whereas here $\otimes$ is used instead of $\otimes^{\mathbf{L}}$. In the interest of consistency, I will continue with the notation $K \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n$ for K_n. So in this paragraph, we are implicitly in the AR category or in the category of pro-sheaves. (In Weil II, Deligne uses the definition of $\mathbf{Z}_\ell$-sheaf as any pro-sheaf in the essential image of the $\mathbf{Z}_\ell$-sheaves; i.e. the $(\ell\mathbf{Z})$-adic objects of the category of abelian constructible sheaves.) For the business about H^0 inducing an equivalence between $D^b_c(X, \mathbf{Z}_\ell)^{\leq 0} \cap D^b_c(X, \mathbf{Z}_\ell)^{\geq 0}$ and $\mathbf{Z}_\ell$-constructible sheaves, this is simple, since for such a K we can represent $K \otimes^{\mathbf{L}} \mathbf{Z}/\ell$ by a sheaf concentrated in degree 0. As every complex of flat sheaves representing the K_n admits an ℓ-adic filtration with successive quotients quasi-isomorphic to $K \otimes^{\mathbf{L}} \mathbf{Z}/\ell$, we can represent K by a projective system of flat sheaves concentrated in degree 0. In this case, K is a bona fide ℓ-adic sheaf.

As for checking whether K belongs to $\mathrm{D}_c^b(\mathrm{X}, \mathbf{Z}_\ell)^{\leq 0}$, the statement is punctual and we may consider the problem in $\mathrm{D}_{\mathrm{parf}}$. If $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n)$ is null for one n, then $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$ is null by (2.2.16.1), so $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell$ is null. This imples by the exact sequence

$$0 \to \ell\mathrm{H}^i(\mathrm{K})/\ell^n\mathrm{H}^i(\mathrm{K}) \to \mathrm{H}^i(\mathrm{K})/\ell^n\mathrm{H}^i(\mathrm{K}) \to \mathrm{H}^i(\mathrm{K})/\ell\mathrm{H}^i(\mathrm{K}) \to 0$$

and the fact that $\mathrm{H}^i(\mathrm{K})/\ell^{n-1} \twoheadrightarrow \ell\mathrm{H}^i(\mathrm{K})/\ell^n$ surjects that $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$ is null for all n and so K is in $\mathrm{D}_c^b(\mathrm{X}, \mathbf{Z}_\ell)^{\leq 0}$. On the other hand, if K is in $\mathrm{D}_c^b(\mathrm{X}, \mathbf{Z}_\ell)^{\leq 0}$, the exact sequence (2.2.16.1) for $n = 1$ tells us that $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell) = 0$ for $i > 0$. The ℓ-adic filtration on flat complexes representing $\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n$ has successive quotients quasi-isomorphic to $\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell$ and the sequence of cohomology then establishes that $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n)$ is null.

2.2.17. On the equivalent conditions: suppose $\mathrm{K} \otimes^{\mathbf{L}}_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^n$ is in $\mathrm{D}^b_{\mathcal{S},\mathcal{L}}(\mathrm{X}, \mathbf{Z}/\ell^n)$ for one n. In the spirit of Weil II (1.1.2) claim a), observe that as the projective system $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^k)$ is noetherian AR-($\ell\mathbf{Z}$)-adic, the projective system $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^{kn})$ is noetherian AR-($\ell^n\mathbf{Z}$)-adic and AR-isomorphic by (3.2.3) of that section (i.e. [**SGA5**, V, 3.2.3]) to the ($\ell^n\mathbf{Z}$)-adic system $(\mathscr{K}^i_{k+r}/\ell^{kn}\mathscr{K}^i_{k+r})_{k\in\mathbf{N}}$ for some integer $r \geq 0$, where $\mathscr{K}^i_k$ denotes the projective system of universal images of the system $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^{kn})$. As $\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n$ is in $\mathrm{D}^b_{\mathcal{S},\mathcal{L}}(\mathrm{X}, \mathbf{Z}/\ell^n)$, the $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n)$ are locally constant on the strata in $\mathcal{S}$. Taking successive quotients on the ℓ^n-adic filtration on bounded complexes of flat sheaves representing the $\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^{kn}$, we find that the $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^{kn})$ are also locally constant on the strata. Therefore the sheaves $\mathscr{K}^i_k$ are, as well as the sheaves $\mathscr{K}^i_{k+r}/\ell^{(k+1)n}\mathscr{K}^i_{k+r}$. For $k = 0$, the latter sheaf is isomorphic to $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$, so we have shown that $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$ is locally constant on the strata $\mathcal{S}$.

(An equivalent way to argue is again to use the description of [**SGA5**, V, 3.2.3] and just note that when computing the universal image subsheaves of $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^{kn})$, one can restrict to looking at the images of the sheaves $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^{kn}))$, and when finding an r, if r works, then s works for any $s \geq r$, so r can be taken to be a multiple of kn.)

As $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$ is locally constant and includes into $\mathrm{H}^i(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n)$, on each stratum $\mathrm{S} \in \mathcal{S}$ consider a Jordan-Hölder series for both. The constituents of the former are a subset of the constituents of the latter and therefore also belong to $\mathscr{L}(\mathrm{S})$. We have shown that $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$, and therefore $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell$, is $(\mathcal{S}, \mathscr{L})$-constructible.

On the other hand, suppose the $\mathrm{H}^i(\mathrm{K}) \otimes \mathbf{Z}/\ell$ are $(\mathcal{S}, \mathscr{L})$-constructible, and without loss of generality let K be in $\mathrm{D}^b_c(\mathrm{X}, \mathbf{Z}_\ell)^{\leq 0}$. Proceed by recurrence on $-j$; in the case $j = 0$, $\mathrm{H}^1(\mathrm{K}) = 0$ and $\mathrm{H}^0(\mathrm{K}) \otimes \mathbf{Z}/\ell \xrightarrow{\sim} \mathrm{H}^0(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell)$. The ℓ-adic filtration on any bounded complex of flat sheaves representing $\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n$ has successive quotients quasi-isomorphic to $\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell$; taking the long exact sequence of cohomology finds that $\mathrm{H}^0(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$ are $(\mathcal{S}, \mathscr{L})$-constructible for all n. In particular, this implies that each $\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathrm{H}^0(\mathrm{K}), \mathbf{Z}/\ell^n)$ is $(\mathcal{S}, \mathscr{L})$-constructible. To see why, note that the increasing sequence of ℓ-adic subsheaves $\ker \ell^a \subset \mathrm{H}^0(\mathrm{K})$ must stabilize, say at $a = \mathrm{N}$ as $\mathrm{H}^0(\mathrm{K})$ is noetherian. Let $\mathscr{K}$ denote $\ker \ell^{\mathrm{N}}$; it is an ℓ-adic subsheaf of $\mathrm{H}^0(\mathrm{K})$ and is $(\mathcal{S}, \mathscr{L})$-constructible since $\mathrm{H}^0(\mathrm{K}) \otimes \mathbf{Z}/\ell^{\mathrm{N}}$ is. Then

$$\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathrm{H}^0(\mathrm{K}), \mathbf{Z}/\ell^n) \simeq \mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathscr{K}, \mathbf{Z}/\ell^n) \simeq \mathscr{K}[\ell^n]$$

This last sheaf is easily seen to be $(\mathcal{S}, \mathscr{L})$-constructible, as it is locally constant wherever $\mathscr{K}$ is, and as a subsheaf, its constituents on a stratum are a subset of the constituents of $\mathscr{K}$ on that stratum. Since $\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathrm{H}^0(\mathrm{K}), \mathbf{Z}/\ell^n)$ and $\mathrm{H}^{-1}(\mathrm{K}) \otimes \mathbf{Z}/\ell$ are $(\mathcal{S}, \mathscr{L})$-constructible, (2.2.16.1) shows that $\mathrm{H}^{-1}(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell)$ is $(\mathcal{S}, \mathscr{L})$-constructible, hence that $\mathrm{H}^{-1}(\mathrm{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell^n)$ is for all n. The argument above then shows that $\mathrm{H}^{-1}(\mathrm{K}) \otimes \mathbf{Z}/\ell^n$ are $(\mathcal{S}, \mathscr{L})$-constructible for all n, and hence that $\mathrm{Tor}_1^{\mathbf{Z}_\ell}(\mathrm{H}^{-1}(\mathrm{K}), \mathbf{Z}/\ell^n)$ are for all n, etc.

In order to proceed to define the t-structure in imitation of 2.2.10, one needs to extend the six functors to $\mathrm{D}^b_c(\mathrm{X}, \mathbf{Z}_\ell)$. This is trivial because they commute with reduction modulo ℓ^n; see note to Weil II 1.1.2c. Then the claim about K belonging to ${}^p\mathrm{D}^{\leq 0}_c(\mathrm{X}, \mathbf{Z}_\ell)$ iff its reduction modulo ℓ belongs to ${}^p\mathrm{D}^{\leq 0}_c(\mathrm{X}, \mathbf{Z}/\ell)$ is also trivial.

2.2.18. See note to Weil II 1.1.3. Multiplication by ℓ on complexes of flat sheaves representing K in $\mathrm{D}^b_c(\mathrm{X}, \mathbf{Z}_\ell)$ induces a multiplication by ℓ on their cohomology. To see that the image $\mathrm{D}^b_{\mathcal{S},\mathscr{L}}(\mathrm{X}, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell$ consists of those K such that each $\mathrm{H}^i\mathrm{K}$ is the $\mathbf{Q}_\ell \otimes$ of an $(\mathcal{S}, \mathscr{L})$-constructible $\mathbf{Z}_\ell$-sheaf, well, certainly it is contained in it. On the other hand, given K in $\mathrm{D}^b_c(\mathrm{X}, \mathbf{Z}_\ell)$ with each $\mathrm{H}^i\mathrm{K}$ the $\mathbf{Q}_\ell \otimes$ of an $(\mathcal{S}, \mathscr{L})$-constructible $\mathbf{Z}_\ell$-sheaf $\mathscr{F}_i$, then for each of finitely many nonzero i, there exist nonzero $a_i, b_i \in \mathbf{Z}_\ell$

such that $a_i \mathrm{H}^i\mathrm{K} = b\mathscr{F}_i$; $b_i\mathscr{F}_i$ is a subsheaf of $\mathscr{F}_i$ and is also $(\mathcal{S}, \mathscr{L})$-constructible. Then $(\prod_i a_i)\mathrm{K}$ has cohomology sheaves which are $(\mathcal{S}, \mathscr{L})$-constructible, showing the reverse containment.

Turning now to the claim that the forgetful functor ω induces an equivalence

$$\mathrm{D}^b_c(\mathrm{X}, \mathrm{E}_\lambda) \to \{\text{category of objects K of } \mathrm{D}^b_c(\mathrm{X}, \mathbf{Q}_\ell) \text{ equipped with a morphism of } \mathbf{Q}_\ell\text{-algebras } \mathrm{E}_\lambda \to \mathrm{End}(\mathrm{K})\},$$

the essential surjectivity of ω follows from the fact that if K is in $\mathrm{D}^b_c(\mathrm{X}, \mathbf{Q}_\ell)$ and equipped with an action $\phi : \mathrm{E}_\lambda \to \mathrm{End}(\mathrm{K})$, and $\mathrm{K} \otimes_{\mathbf{Q}_\ell} \mathrm{E}_\lambda$ has E_λ acting on itself, then there is an E_λ-equivariant imbedding $\mathrm{K} \to \mathrm{K} \otimes \mathrm{E}_\lambda$ with retraction r. Let α be a primitive element for the extension $\mathrm{E}_\lambda/\mathbf{Q}_\ell$ so that $\mathrm{E}_\lambda \simeq \mathbf{Q}(\alpha)$, and let d denote the degree $[\mathrm{E}_\lambda : \mathbf{Q}_\ell]$. The maps are given by

$$i : \mathrm{K} \to \mathrm{K} \otimes \mathrm{E}_\lambda \qquad\qquad r : \mathrm{K} \otimes \mathrm{E}_\lambda \to \mathrm{K}$$

$$\mathrm{K} \mapsto \frac{1}{d}\sum_{i=0}^{d-1} \phi(\alpha)^i\mathrm{K} \otimes \alpha^{-i} \qquad\qquad \mathrm{K} \otimes a \mapsto \phi(a)\mathrm{K}.$$

This implies that K is indeed a direct factor of $\mathrm{K} \otimes \mathrm{E}_\lambda$; see Neeman, *Triangulated Categories* 1.2.10. This in turn gives an idempotent in $\mathrm{End}(\mathrm{K} \otimes \mathrm{E}_\lambda)$, and if the image of this idempotent is represented in $\mathrm{D}^b_c(\mathrm{X}, \mathrm{E}_\lambda)$, then this implies a splitting of $\mathrm{K} \otimes \mathrm{E}_\lambda$ in $\mathrm{D}^b_c(\mathrm{X}, \mathrm{E}_\lambda)$ which is sent by ω to the direct factor K. A category in which every idempotent splits is called alternatively Cauchy complete, idempotent complete, or Karoubi complete (see SGA 4, I 8.7.8), so we are done if we show that $\mathrm{D}^b_c(\mathrm{X}, \mathrm{E}_\lambda)$ is Karoubi complete. The splitting of an idempotent e in the endomorphism ring of an object in some category is equivalent to the existence of the equalizer $i = \ker(e, \mathrm{id})$ or the coequalizer $r = \mathrm{coker}(e, \mathrm{id})$, and this (co)equalizer, if it exists, is an absolute (co)limit; i.e. it is preserved by every functor (see Proposition 1 of Borceaux and Dejean, *Cauchy Completion in Category Theory*). As $\mathrm{D}^b_c(\mathrm{X}, \mathrm{E}_\lambda)$ is a projective limit of categories $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$ for R the ring of integers in E_λ, it is easily seen that if the categories $\mathscr{D}_n := \mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$ are Karoubi complete, if e is an idempotent in $\mathrm{End}(\mathrm{K} \otimes \mathrm{E})$, its reductions in $\mathscr{D}_n$ are idempotents which split, and as these splittings are absolute (co)limits, they automatically give an object in $\mathrm{D}^b_c(\mathrm{X}, \mathrm{E}_\lambda)$ splitting e. It will suffice to show that $\mathscr{D}_n$ is Karoubi complete. Let's say that a triangulated category has direct sums if it has (arbitrary) categorical direct sums and if the (arbitrary) direct sum of distinguished triangles is distinguished. Bökstedt and Neeman show in *Homotopy limits in triangulated categories* 3.2 that if a triangulated category has direct sums, it is Karoubi complete. (Actually, all that is needed is that countable coproducts of objects in the triangulated category exist; a simple exposition is Neeman, *Triangulated Categories* §1.6, in particular (1.6.8).) The category $\mathrm{D}(\mathrm{X}, \mathrm{R}/m^n)$ is therefore Karoubi complete. An object C of an additive category with arbitrary direct sums is said to be compact if $\mathrm{Hom}(\mathrm{C}, -)$ commutes with arbitrary direct sums. Any direct summand of a compact object C is compact, since a finite colimit of compact objects is compact*. Therefore as $\mathrm{D}(\mathrm{X}, \mathrm{R}/m^n)$ is Karoubi complete, so is the full subcategory generated by compact objects. That $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$ coincides with the subcategory generated by compact objects of $\mathrm{D}(\mathrm{X}, \mathrm{R}/m^n)$ is 6.4.8 of Bhatt-Scholze, *The pro-étale topology for schemes*, after you recall that the objects of $\mathrm{D}^b_{ctf}(\mathrm{X}, \mathrm{R}/m^n)$ can be represented by bounded complexes of R/m^n-flat constructible sheaves (*Rapport* 4.6).

(*) To recognize A, a direct summand of C, as a finite colimit, note that for the retraction $\mathrm{A} \xrightarrow{i} \mathrm{C} \xrightarrow{p} \mathrm{A}$, A is both the equalizer and coequalizer of id_C and ip (by the maps i and p, respectively). As coequalizer, A is a finite colimit; we can therefore write

$$\mathrm{Hom}\Big(\operatorname*{colim}_{\bullet\rightrightarrows\bullet} \mathrm{C}, \bigoplus_{i\in\mathrm{I}} \mathrm{U}_i\Big) = \lim_{\bullet\rightrightarrows\bullet} \mathrm{Hom}\Big(\mathrm{C}, \bigoplus_{i\in\mathrm{I}} \mathrm{U}_i\Big) = \lim_{\bullet\rightrightarrows\bullet} \bigoplus_{i\in\mathrm{I}} \mathrm{Hom}(\mathrm{C}, \mathrm{U}_i),$$

where the last equality is because C is compact; now use that finite limits commute with filtered colimits in Set to write

$$\lim_{\bullet\rightrightarrows\bullet} \bigoplus_{i\in\mathrm{I}} \mathrm{Hom}(\mathrm{C}, \mathrm{U}_i) = \bigoplus_{i\in\mathrm{I}} \lim_{\bullet\rightrightarrows\bullet} \mathrm{Hom}(\mathrm{C}, \mathrm{U}_i) = \bigoplus_{i\in\mathrm{I}} \mathrm{Hom}(\operatorname*{colim}_{\bullet\rightrightarrows\bullet} \mathrm{C}, \mathrm{U}_i).$$

As $\mathrm{A} = \operatorname*{colim}_{\bullet\rightrightarrows\bullet} \mathrm{C}$, this proves A compact.

2.2.19. The fact that $i^! f_* = f_* i^!$ is SGAA XVIII 3.1.12.3. The 'argument habituel d'homotopie' referenced in the last sentence of the proof is referring to, e.g. Th. 5.7.1 in Godement, *Théorie des faisceaux*, but a more accessible reference is Stacks `09UY`. If you have two open coverings $\mathscr{U} = (\mathrm{U}_i)_{i\in\mathrm{I}}$ and $\mathscr{V} = (\mathrm{V}_j)_{j\in\mathrm{J}}$ of a space X, and $\mathscr{U}$ is a refinement of $\mathscr{V}$, so that we can choose a map $\phi : \mathrm{I} \longrightarrow \mathrm{J}$ such that $\mathrm{U}_i \subset \mathrm{V}_{\phi(i)}$ for all $i \in \mathrm{I}$.

This induces a map of Čech complexes $\phi^* : \check{C}^\bullet(\mathscr{V}, \mathscr{F}) \to \check{C}^\bullet(\mathscr{U}, \mathscr{F})$ for any sheaf $\mathscr{F}$ on X. The result then says that if you have a second $\phi' : I \to J$ such that $U_i \subset V_{\phi'(i)}$, the maps ϕ^*, ϕ'^* are homotopic. This instantly implies that if $\mathscr{U}, \mathscr{V}$ are mutual refinements, they have the same Čech cohomology, since in this case we can choose $\phi : I \to J$ and $\psi : J \to I$ satisfying $U_i \subset V_{\phi(i)}$ and $V_j \subset U_{\psi(j)}$; the maps $\phi \circ \psi$ and $\psi \circ \phi$ must then induce the identity on Čech cohomology as they are homotopic to the identity on the chain level. Our case is formally equivalent to this statement for $\check{H}^0$.

In more words: A is defined as an equalizer of two maps

$$\prod_{I} {}^p j_{i*} A_{U_i} \rightrightarrows \prod_{I \times I} {}^p j_{ij*} A_{U_{ij}}$$

where one map d_0^0 is the product over $(i, j) \in I \times I$ of the maps

$$\prod_{I} {}^p j_{i*} A_{U_i} \xrightarrow{\mathrm{pr}_i} {}^p j_{i*} A_{U_i} \to {}^p j_{\bullet j*} A_{U_{\bullet j}},$$

(here the second map is j_{i*} of the unit of the adjunction on U_i if $\bullet = i$ and 0 if $\bullet \neq i$), and the other map d_1^0 is the product over $(i, j) \in I \times I$ of the maps

$$\prod_{I} {}^p j_{i*} A_{U_i} \xrightarrow{\mathrm{pr}_j} {}^p j_{j*} A_{U_j} \to {}^p j_{i\bullet *} A_{U_{i\bullet}}.$$

(the second map is j_{j*} of the unit of the adjunction on U_j if $\bullet = j$ and 0 if $\bullet \neq j$). This recognizes A as $\ker(d_0^0 - d_1^0)$. Let f denote maps of the type

$$U_{i_0} \times \cdots \times U_{i_a} \times \cdots U_{i_p} \to U_{i_0} \times \cdots \times \hat{U}_{i_a} \times \cdots U_{i_p}.$$

In higher degrees, the differential is determined by the formula

$$\mathrm{pr}_{i_0 \ldots i_{p+1}} \circ d = \sum_{a=0}^{p+1} (-1)^a \, {}^p j_{i_0 \ldots \hat{i}_a \ldots i_p *}(\eta(A_{i_0 \ldots i_p})) \circ \mathrm{pr}_{i_0 \ldots \hat{i}_a \ldots i_{p+1}},$$

where η is the unit of the adjunction $\mathrm{id} \to {}^p f_* \, {}^p f^*$. Let $\mathscr{U}$ be the cover of X as above and $s : I \to J$ satisfying $s_i : U_i \to V_{s(i)}$ as above. The chain map induced by s has the explicit description

$$\mathrm{pr}_{i_A} \circ s^* = {}^p j_{s(i_A)*}(\eta(A_{i_A})) \circ \mathrm{pr}_{s(i_A)},$$

where i_A is a multi-index $i_0 i_1 \ldots i_p$, $s(i_A) = s(i_0)\ldots s(i_p)$, and η is the unit of the adjunction $\mathrm{id} \to {}^p s_{i_A *}\, {}^p s_{i_A}^*$ on $V_{s(i_A)}$, where $s_{i_A} : U_{i_0} \times \cdots \times U_{i_p} \to V_{s(i_0)} \times \cdots \times V_{s(i_p)}$ is deduced by taking the product of the maps s_i.

Now given another map $t : I \to J$ satisfying $t_i : U_i \to V_{t(i)}$, let $0 \leq a \leq p$ and let f now denote maps of the sort

$$U_{i_0} \times \cdots \times U_{i_p} \to V_{s(i_0)} \times \cdots \times V_{s(i_a)} \times V_{t(i_a)} \times \cdots \times V_{t(i_p)},$$

where here we use the map $U_{i_a} \to V_{s(i_a)} \times V_{t(i_a)}$. We set up a homotopy h by the formula

$$\mathrm{pr}_{i_0 \ldots i_p} \circ h = \sum_{a=0}^{p} (-1)^a\, {}^p j_{s(i_0)\ldots s(i_a)t(i_a)\ldots t(i_p) *}(\eta(A_{s(i_0)\ldots s(i_a)t(i_a)\ldots t(i_p)})\circ \mathrm{pr}_{s(i_0)\ldots s(i_a)t(i_a)\ldots t(i_p)}$$

where here η is the unit of the adjunction $\mathrm{id} \to {}^p f_*\, {}^p f^*$.

Now we can follow the argument of Stacks `01FP`.

3.1.2. (3.1.2.7) The morphism $K_{n+1} \to K_n$ in $D\mathscr{A}$ can be represented by $K_{n+1} \xleftarrow{\alpha} K'_{n+1} \xrightarrow{\beta} K_n$ with α a quasi-isomorphism and β a homotopy class of morphisms. Replacing K_n by $K_n \oplus \mathrm{cone}(K'_{n+1})$ allows us to write a commutative square

$$\begin{tikzcd} K'_{n+1} \arrow[r, "(\beta,\iota)"] \arrow[d, "\alpha"] & K_n \oplus \mathrm{cone}(K'_{n+1}) \\ K_{n+1} \arrow[r] & K_n \arrow[u, "\gamma"'] \end{tikzcd}$$

where ι denotes the canonical map $K'_{n+1} \to \mathrm{cone}\, K'_{n+1}$. Now (β, ι) is injective and γ is a homotopy equivalence so that it can be inverted. Doing this finitely many times allows us to produce a filtered complex of objects (K, F) of $\mathscr{A}$ with the sequence of $F^i K$ isomorphic in $D\mathscr{A}$ to the given sequence.

(3.1.2.8) The intersection of two subobjects A, B of an object C in an abelian category $\mathscr{A}$ is defined as the pullback

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

or equivalently by the exact sequence

$$0 \to \mathrm{A} \cap \mathrm{B} \to \mathrm{A} \oplus \mathrm{B} \to \mathrm{C}.$$

The sum $\mathrm{A}+\mathrm{B}$ is defined as the image of $\mathrm{A} \oplus \mathrm{B}$ under the same map $\mathrm{A} \oplus \mathrm{B} \to \mathrm{C}$ defined by the monomorphisms $\mathrm{A} \to \mathrm{C}$ and $\mathrm{B} \to \mathrm{C}$. The inclusion

$$\sum_{i+j=p} \mathrm{F}^i \cap \mathrm{G}^j \subset \bigcap_{i+j=p+1} \mathrm{F}^i + \mathrm{G}^j$$

is evident from the fact that if $a+b=p$, every pair (i,j) satisfying $i+j=p+1$ also satisfies $i \leq a$ or $j \leq b$. For the reverse inclusion, find some a in the intersection; then we have expressions $a = f_i + g_j$ for $f_i \in \mathrm{F}^i, g_j \in \mathrm{G}^j, i+j = p+1$; as the filtration is finite & decreasing there exists an n such that $a = f_n$ and $g_n = 0$. Then $a - f_{n+1} = g_{p+1-(n+1)}$ is in $\mathrm{F}^n \cap \mathrm{G}^{p-n}$. Likewise $f_{n+i} - f_{n+i+1} = g_{p+1-(n+i+1)} - g_{p+1-(n+i)}$ is in $\mathrm{F}^{n+i} \cap \mathrm{G}^{p-(n+i)}$ for $i \geq 0$. By finiteness, $f_{n+i} = 0$ for $i \gg 0$, so that

$$a = \sum_{i=0}^{\infty} f_{n+i} - f_{n+i+1} \qquad (f_{n+i} - f_{n+i+1} \in \mathrm{F}^{n+i} \cap \mathrm{G}^{p-(n+i)})$$

has only finitely many nonzero terms.

3.1.3. This paragraph should be read along with Illusie *Complexe cotangent et déformations I* Ch. V §1, cited [**I2**]. Equations (3.1.3.4) and (3.1.3.5) are written with the notation Hom^{p+q} and $\mathrm{Hom}^{p+q}_{\mathrm{D}\mathscr{A}}$ and $\mathrm{Hom}^{p+q}_{\mathrm{DF}\mathscr{A}}$ instead of Ext^{p+q} and $\mathrm{Ext}^{p+q}_{\mathrm{D}\mathscr{A}}$ and $\mathrm{Ext}^{p+q}_{\mathrm{DF}\mathscr{A}}$. This notation, which continues into (3.1.4) and beyond, is standard but creates something of a conflict, as it should not be conflated with the Hom^n defined in the first part of (3.1.3), as the Hom^n there specifies a component of a familiar chain complex. For objects K, L of $\mathrm{D}\mathscr{A}$, $\mathrm{Ext}^n(\mathrm{K},\mathrm{L}) = \mathrm{Hom}_{\mathrm{D}\mathscr{A}}(\mathrm{K}, \mathrm{L}[n])$; in these notes we will tend to write Ext^n instead of Hom^n even when Deligne writes Hom^n.

3.1.4. The statement of the proposition of course has grammatical grouping 'lorsque $\{n = 0 \text{ ou } -1\}$ et $i > j$' and 'lorsque $\{n = 0 \text{ ou } -1\}$ et $n+i-j < 0$'. The proof of (i) is obtained from the distinguished triangle given by taking the sequence of cohomology and using the vanishing of H^0 and H^{-1} of $\mathrm{F}^{-\infty}/\mathrm{F}^0$. In the proof of (ii), the differential at $(0,0)$ on the E_r page for goes $(0,0) \xrightarrow{d} (r, 1-r)$; as $\mathrm{E}_1^{pq} = 0$ for $p < 0$, $\mathrm{E}_\infty^{00} = \cap_r \ker d_r$.

The assumptions give that $E_r^{pq} = 0$ whenever $q < 0$ and either $p + q = 0$ or $p + q = 1$, so that $\ker d_r = E_2^{00}$ for $r \geq 2$ and therefore $E_2^{00} = E_\infty^{00}$. The assumption that $E_1^{pq} = 0$ if $n = 0$ and $q < 0$ implies that the only nonzero graded piece in the filtration on $\mathrm{Hom}_{\mathrm{DF}\mathscr{A}}(K, L) = \mathrm{Ext}^0_{\mathrm{DF}\mathscr{A}}(K, L)$ is the 0^{th} and it coincides with E_2^{00}.

Note: $\mathrm{Ext}^n(\mathrm{Gr}_F^i K, \mathrm{Gr}_F^j L) = \mathrm{Ext}^{n+i-j}(\mathrm{Gr}_F^i K[i], \mathrm{Gr}_F^j L[j])$ so for example the condition $\mathrm{Ext}^n(\mathrm{Gr}_F^i K[-i], \mathrm{Gr}_F^j L[-j]) = 0$ for $n < 0$ equates to $\mathrm{Ext}^{n-j+i}(\mathrm{Gr}_F^i K, \mathrm{Gr}_F^j L) = 0$ for $n < 0$, which implies $\mathrm{Ext}^n(\mathrm{Gr}_F^i K, \mathrm{Gr}_F^j L) = 0$ whenever $i > j$ and $n < 1$.

Remark (vague). As discussed in the note to (3.1.7) below, considering the Gr_F^i as filtered objects in $\mathscr{D}F$, $\mathrm{Hom}^n_{\mathscr{D}F}(\mathrm{Gr}_F^i K, \mathrm{Gr}_F^j K) = 0$ when $i > j$ (any n). Of course, forgetting the filtration, $\mathrm{Hom}^n_{\mathscr{D}}(\omega\, \mathrm{Gr}_F^i K, \omega\, \mathrm{Gr}_F^j K)$ need not vanish. The result of (3.1.4) (i) can be seen as saying that this is precisely the difference between computing Hom in $\mathscr{D}F$ versus in $\mathscr{D}$, and that moreover it only need be checked for $n = 0$ and -1.

3.1.6. The filtration T_i is now taken to be an *increasing* filtration, corresponding to the *decreasing* filtration T^{-i}. Under this correspondence $(\mathrm{Gr}_i^T K)[i]$ becomes $(\mathrm{Gr}_T^{-i} K)[i]$. $\mathrm{Hom}^n(A, B)$ should be written $\mathrm{Ext}^n(A, B) = \mathrm{Hom}(A, B[n])$, null for $n < 0$ when A, B are in $\mathscr{C}$ by (1.3.1) (i).

3.1.7. *A priori* there is the matter of the uniqueness of the arrow of degree 1 $\mathrm{Gr}_F^i K \to \mathrm{Gr}_F^{i+1} K$ 'defined' by the distinguished triangle $(\mathrm{Gr}_F^{i+1}, F^i/F^{i+2}, \mathrm{Gr}_F^i)$. Recall [**BBD**, 1.1.10] which says that this arrow is unique (and therefore actually defined by the property that $(\mathrm{Gr}_F^{i+1}, F^i/F^{i+2}, \mathrm{Gr}_F^i)$ is distinguished) if $\mathrm{Hom}^{-1}_{\mathscr{D}F}(\mathrm{Gr}_F^{i+1} K, \mathrm{Gr}_F^i K) = 0$.

$$\begin{aligned}\mathrm{Hom}^{-1}(\mathrm{Gr}_F^{i+1} K, \mathrm{Gr}_F^i K) &= \mathrm{Hom}(\mathrm{Gr}_F^{i+1} K, (\mathrm{Gr}_F^i K)[-1]) \\ &= \mathrm{Hom}(\mathrm{Gr}_F^{i+1} K, \mathrm{Gr}_F^i(K[-1])) = 0\end{aligned}$$

(all Hom are in $\mathscr{D}F$) as of course $F^{i+1}\,\mathrm{Gr}_F^i = 0$.

For $d^{i+1} \circ d^i = 0$ the point is that in the commutative diagram of distinguished triangles below, the two morphisms $F^{i+1}/F^{i+3} \to \mathrm{Gr}_F^{i+1}$ coincide.

$$\begin{tikzcd}
\mathrm{Gr}_F^i[-1] \arrow[r] & F^{i+1}/F^{i+3} \arrow[r] \arrow[d] & F^i/F^{i+3} \arrow[d] & \\
& \mathrm{Gr}_F^{i+1} \arrow[r] \arrow[d, equal] & F^i/F^{i+2} \arrow[r] \arrow[d] & \mathrm{Gr}_F^i \\
F^{i+1}/F^{i+3} \arrow[r] & \mathrm{Gr}_F^{i+1} \arrow[r] & \mathrm{Gr}_F^{i+2}[1] &
\end{tikzcd}$$

3.1.8. On the subject of the differential $d_1(f) = df - fd$, we have to compute it. Illusie discusses the differential in [**I2**, V 1.4.10]. The spectral sequence (3.1.3.4) is isomorphic to the spectral sequence

$$E_1^{pq} = H^{p+q}\,\mathrm{Gr}_F^p\,\mathrm{R\,Hom}(L, M) \Rightarrow \mathrm{Ext}^{p+q}(L, M).$$

We replace M by a filtered injective resolution of M, which is therefore (non-canonically) termwise-split as a direct sum of its graded pieces, which are also injective (`05TP`), and we compute in the abelian category. General considerations (`012N`) dictate that the differential d_1 is equal to the coboundary map associated to the short exact sequence of complexes

$$0 \to \mathrm{Gr}^{p+1}(\mathrm{Hom}(L, M)) \to F^p/F^{p+2}\,\mathrm{Hom}(L, M) \to \mathrm{Gr}^p(\mathrm{Hom}(L, M))) \to 0,$$

where the differential on $\mathrm{Hom}(L, M)$ is defined by

$$\mathrm{Hom}^n(L, M) = \prod_i \mathrm{Hom}(L^i, M^{i+n}), \quad d^p = d_L + (-1)^{n+1} d_M.$$

We need only consider $n = 0$ and for simplicity we look at a single i at a time. Basically we need to translate this differential via the isomorphism

$$E_1^{pq} \simeq \oplus_k \mathrm{Ext}^{p+q}(\mathrm{Gr}^k L, \mathrm{Gr}^{k+p} M), \quad \text{or, more properly, through}$$
$$\mathrm{Gr}^p(\mathrm{Hom}(L^i, M^i)) \simeq \oplus_k \mathrm{Hom}(\mathrm{Gr}^k L^i, \mathrm{Gr}^{k+p} M^i).$$

The diagrams on the next page, combined with the fact that $d^0 = d_{\mathrm{L}} - d_{\mathrm{M}}$, show that d_1, the coboundary map induced by d^0 and the relevant exact sequence, coincides with

$$\prod_i \mathrm{Hom}(\mathrm{Gr}^k \mathrm{L}^i, \mathrm{Gr}^{k+p} \mathrm{M}^i) \ni (f_i) \mapsto (f_i d''_{\mathrm{L}} - d'_{\mathrm{M}} f_{i-1}),$$

for each k, where d'_{M} and d''_{M} are induced on Hom by coboundaries associated to the exact sequences

$$0 \to \mathrm{Gr}^{k+p+1} \mathrm{M} \to \mathrm{F}^{k+p}\mathrm{M}/\mathrm{F}^{k+p+2}\mathrm{M} \to \mathrm{Gr}^{k+p} \mathrm{M} \to 0 \quad \text{and}$$
$$0 \to \mathrm{Gr}^k \mathrm{L} \to \mathrm{F}^{k-1}/\mathrm{F}^{k+1}\mathrm{L} \to \mathrm{Gr}^{k-1} \mathrm{L} \to 0, \quad \text{respectively.}$$

So (f_i) belongs to $\ker d_1$ if $f_i d''_{\mathrm{L}} - d'_{\mathrm{M}} f_{i-1} = 0$ for all i, and all that is left to observe is that d'_{M} and d''_{L} correspond with Deligne's differential d.

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & \mathrm{Gr}^{p+1}(\mathrm{Hom}(\mathrm{L}^i,\mathrm{M}^i)) \arrow[r] \arrow[d] & \mathrm{F}^p/\mathrm{F}^{p+2}(\mathrm{Hom}(\mathrm{L}^i,\mathrm{M}^i)) \arrow[r] \arrow[d, "d^0=d_\mathrm{L}-d_\mathrm{M}"] & \mathrm{Gr}^p(\mathrm{Hom}(\mathrm{L}^i,\mathrm{M}^i)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Gr}^{p+1}(\mathrm{Hom}(\mathrm{L}^{i-1},\mathrm{M}^i)\oplus\mathrm{Hom}(\mathrm{L}^i,\mathrm{M}^{i+1})) \arrow[r, "\alpha"] \arrow[d] & \mathrm{F}^p/\mathrm{F}^{p+2}(\mathrm{Hom}(\mathrm{L}^{i-1},\mathrm{M}^i)\oplus\mathrm{Hom}(\mathrm{L}^i,\mathrm{M}^{i+1})) \arrow[r] \arrow[d] & \mathrm{Gr}^p(\mathrm{Hom}(\mathrm{L}^{i-1},\mathrm{M}^i)\oplus\mathrm{Hom}(\mathrm{L}^i,\mathrm{M}^{i+1})) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
$$

Let $a,b\in\mathbf{Z}$ (we care about $(a,b)=(i-1,i),(i,i)$, or $(i,i+1)$) and fix retractions to the monomorphisms $\mathrm{F}^p/\mathrm{F}^{p+k}\mathrm{M}^b := \mathrm{F}^p\mathrm{M}^b/\mathrm{F}^{p+k}\mathrm{M}^b \hookrightarrow \mathrm{F}^{p+1}/\mathrm{F}^{p+k}\mathrm{M}^b$ so that we can identify M^b with the direct sum of its graded pieces, which are almost all zero.

$$\mathrm{F}^p/\mathrm{F}^{p+2}\,\mathrm{Hom}(\mathrm{L}^a,\mathrm{M}^b)\simeq\oplus_k\mathrm{F}^p/\mathrm{F}^{p+2}\,\mathrm{Hom}(\mathrm{L}^a,\mathrm{Gr}^k\,\mathrm{M}^b)\simeq\oplus_k\,\mathrm{Hom}(\mathrm{F}^{k-p-1}/\mathrm{F}^{k-p+1}\mathrm{L}^a,\mathrm{Gr}^k\,\mathrm{M}^b)$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Gr}^{p+1}\,\mathrm{Hom}(\mathrm{L}^a,\mathrm{M}^b) \arrow[r] \arrow[dd, "\wr"] & \mathrm{F}^p/\mathrm{F}^{p+2}\,\mathrm{Hom}(\mathrm{L}^a,\mathrm{M}^b) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Gr}^p\,\mathrm{Hom}(\mathrm{L}^a,\mathrm{M}^b) \arrow[r] \arrow[ddd, "\wr"] & 0 \\
 & & \oplus_k\,\mathrm{Hom}(\mathrm{F}^{k-p-1}/\mathrm{F}^{k-p+1}\mathrm{L}^a,\mathrm{Gr}^k\,\mathrm{M}^b) \arrow[ur] \arrow[d, "\mathrm{Gr}"] & & \\
 & \oplus_k\,\mathrm{Hom}(\mathrm{Gr}^{k-p-1}\,\mathrm{L}^a,\mathrm{Gr}^k\,\mathrm{M}^b) \arrow[r] \arrow[ur] \arrow[d, phantom, "\wr"] & \oplus_k\,\mathrm{Hom}(\mathrm{Gr}^{k-p-1}\,\mathrm{L}^a\oplus\mathrm{Gr}^{k-p}\,\mathrm{L}^a,\mathrm{Gr}^k\,\mathrm{M}^b) \arrow[d, phantom, "\wr"] & & \\
0 \arrow[r] & \oplus_k\,\mathrm{Hom}(\mathrm{Gr}^{k-p}\,\mathrm{L}^a,\mathrm{Gr}^{k+1}\,\mathrm{M}^b) \arrow[r] & \oplus_k\,\mathrm{Hom}(\mathrm{Gr}^{k-p}\,\mathrm{L}^a,\mathrm{F}^k/\mathrm{F}^{k+2}\mathrm{M}^b) \arrow[r] & \oplus_k\,\mathrm{Hom}(\mathrm{Gr}^{k-p}\,\mathrm{L}^a,\mathrm{Gr}^k\,\mathrm{M}^b) \arrow[r] & 0
\end{tikzcd}
$$

The above diagram commutes. The bottom row and diagonal row are induced by the exact sequences

$$0\to\mathrm{Gr}^{k+p+1}\,\mathrm{M}^b\to\mathrm{F}^{k+p}/\mathrm{F}^{k+p+2}\mathrm{M}^b\to\mathrm{Gr}^{k+p}\,\mathrm{M}^b\to 0\quad\text{and}$$

$$0\to\mathrm{Gr}^k\,\mathrm{L}^a\to\mathrm{F}^{k-1}/\mathrm{F}^{k+1}\mathrm{L}^a\to\mathrm{Gr}^{k-1}\,\mathrm{L}^a\to 0,\quad\text{respectively.}$$

Returning to the proof of (3.1.8), let's examine why the filtered complex X in the first solution is taken by G to a bounded complex of objects of $\mathscr{C}$ isomorphic to the given one. In the (unconventionally indexed) double complex

$$\begin{tikzcd}
 & & & & \\
{} \arrow[r] & K^{i+1,j-1} \arrow[r] \arrow[u] & K^{i+1,j} \arrow[r] \arrow[u] & K^{i+1,j+1} \arrow[r] \arrow[u] & {} \\
{} \arrow[r] & K^{i,j-1} \arrow[r] \arrow[u] & K^{i,j} \arrow[r, "d''"] \arrow[u, "d'"] & K^{i,j+1} \arrow[r] \arrow[u] & {} \\
{} \arrow[r] & K^{i-1,j-1} \arrow[r] \arrow[u] & K^{i-1,j} \arrow[r] \arrow[u] & K^{i-1,j+1} \arrow[r] \arrow[u] & {} \\
 & \arrow[u] & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

the terms X^n correspond to the diagonals with slope -1 capturing all terms with biindex summing to n.

$$\mathrm{Gr}^i X \simeq (K^i[-i], (-1)^i d'') = K^i[-i].$$

It remains only to show that the morphism of degree one in the distinguished triangle associated to the exact sequence

$$0 \to \mathrm{Gr}^{i+1} X \xrightarrow{f} F^i/F^{i+2}X \xrightarrow{g} \mathrm{Gr}^i X \to 0$$

coincides with d''^i. Actually we will show that it coincides with $-d''^i$. This is OK, as any complex $L = (L, d)$ is isomorphic to the complex $L' = (L, -d)$ by the morphisms $(-1)^i \mathrm{id} : L^i \to L^i$ which induce an isomorphism of chain complexes $L \xrightarrow{\sim} L'$. Therefore we will still have shown that $0 \to K^a \to \cdots \to K^b \to 0$ is in the essential image of G. We have an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{Gr}^{i+1} X \arrow[r, "f"] & F^i/F^{i+2}X \arrow[r, "g"] & \mathrm{Gr}^i X \arrow[r] & \mathrm{Gr}^{i+1} X[1] \\
\mathrm{Gr}^{i+1} X \arrow[r] \arrow[u, equal] & \mathrm{Cyl}(f) \arrow[r] \arrow[u] & C(f) \arrow[r, "\delta"] \arrow[u, "\gamma"] & \mathrm{Gr}^{i+1} X[1] \arrow[u, equal]
\end{tikzcd}$$

where $\delta : C(f) = \mathrm{Gr}^{i+1} X[1] \oplus F^i/F^{i+2}X \to \mathrm{Gr}^{i+1} X[1]$ corresponds componentwise to projection to the first factor and $\gamma : C(f) \to \mathrm{Gr}^i X$ is given by the composition $C(f) \xrightarrow{\mathrm{pr}_2} F^i/F^{i+2}X \xrightarrow{g} \mathrm{Gr}^i X$, where here pr_2 denotes abusively the morphism of complexes induced by componentwise projection to the second factor.

$$\begin{aligned}
&C(f) = K^{i+1}[i] \oplus K^i[-i] \oplus K^{i+1}[-i-1] \quad \text{with differential}\\
&d^n(k^{i+1,n-i}, k^{i,n-i}, k^{i+1,n-i-1})\\
&\quad = ((-1)^i d'', (-1)^i d'', f(k^{i+1,n-i}) + d' k^{i,n-i} + (-1)^{i+1} d'' k^{i+1,n-i-1}).
\end{aligned}$$

It suffices to show that $-d' \circ \gamma$ is homotopic to δ. Consider the homotopy given by

$$\begin{aligned}
&h^n : C(f)^n = K^{i+1,n-i} \oplus K^{i,n-i} \oplus K^{i+1,n-i-1} \xrightarrow{\mathrm{pr}_3} K^{i+1,n-i-1} = (K^{i+1}[-i])^{n-1}\\
&dh(k^{i+1,n-i}, k^{i,n-i}, k^{i+1,n-i-1}) = (-1)^i d'' k^{i+1,n-i-1}\\
&hd(k^{i+1,n-i}, k^{i,n-i}, k^{i+1,n-i-1}) = d'(k^{i,n-i}) + (-1)^{i+1} d'' k^{i+1,n-i-1} + k^{i+1,n-i}\\
&\text{so that} \quad dh + hd = k^{i+1,n-i} + d'(k^{i,n-i}).
\end{aligned}$$

Therefore h defines a homotopy between the two morphisms

$$C(f) \underset{-d'\circ\gamma}{\overset{\delta}{\rightrightarrows}} K^{i+1}[-i].$$

As for the construction of the differential of the complex X, let's check that

$$d_2 = H_0 + H_1 + H_2 = (-1)^i d'' + d' + \sum_i (-1)^{i+1} H_2^i$$

satisfies (*); i.e. that $d_2 \circ d_2$ is of filtration ≥ 3. To do this, it suffices to look at the terms of $d_2 \circ d_2$ which are of filtration < 3. We denote by $(\geq n)$ terms of filtration $\geq n$. It suffices to compute what $d_2 \circ d_2$ does to $K^{i,j}$.

$$\begin{aligned}
d_2 \circ d_2 &= d_1 \circ d_1 + d_1 \circ H_2 + H_2 \circ D_1 + H_2 \circ H_2\\
&= d''^{i+1} \circ d''^i + (-1)^{i+2} d'' \circ H_2 + H_2 \circ (-1)^i d''\\
&= d'' \circ H_2^i + H_2^i \circ d'' + (-1)^{2i+3} d'' \circ H_2^i + (-1)^{2i+1} H_2^i \circ d'' + (\geq 3)\\
&= 0 + (\geq 3)
\end{aligned}$$

as $H_2 \circ H_2$ is of filtration 4 and $d' \circ H_2$ and $H_2 \circ d'$ are of filtration 3.

In the case $p \geq 2$, $\varphi^i : \mathrm{K}^i \to \mathrm{K}^{i+p+1}[1-p]$ is a morphism of complexes. This is because the component of degree $p+1$ of $d_p \circ d_p$ is bihomogenous of degree $(p+1, 1-p)$: indeed, it is a sum of morphisms obtained by composing two morphisms bihomogenous of degree $(a, -a+1)$ and $(b, -b+1)$, respectively, so that $a+b = p+1$. Let's verify that with

$$\mathrm{H}_{p+1} = \sum_i (-1)^{i+p} \mathrm{H}^i_{p+1}, \quad d_{p+1} \circ d_{p+1} \text{ is of filtration} \geq p+2.$$

We check what $d_{p+1} \circ d_{p+1}$ does to $\mathrm{K}^{i,j}$.

$$\begin{aligned} d_{p+1} \circ d_{p+1} &= \Big(d_p + \sum_i (-1)^{i+p} \mathrm{H}^i_{p+1}\Big) \circ \Big(d_p + \sum_i (-1)^{i+p} \mathrm{H}^i_{p+1}\Big) \\ &= \varphi + \mathrm{H}_0 \circ \mathrm{H}_{p+1} + \mathrm{H}_{p+1} \circ \mathrm{H}_0 + (\geq p+2) \\ &= \varphi^i + (-1)^{i+p+1} d'' \circ (-1)^{i+p} \mathrm{H}^i_{p+1} + (-1)^{i+p} \mathrm{H}^i_{p+1} \circ (-1)^i d'' + (\geq p+2) \\ &= \varphi^i - d'' \circ \mathrm{H}^i_{p+1} + (-1)^p \mathrm{H}^i_{p+1} \circ d'' + (\geq p+2) \\ &= 0 + (\geq p+2), \end{aligned}$$

in light of the identity that shows φ^i homotopic to 0 (as a morphism of complexes $\mathrm{K}^i \to \mathrm{K}^{i+p+1}[1-p]$) with homotopy H^i_{p+1}.

2ème solution. Let's check that the object (C, F) of $\mathscr{D}^b\mathrm{F}_{\text{bête}}$ defined up to isomorphism by the distinguished triangle

$$\begin{tikzcd} & (\mathrm{A}, \mathrm{F}) \arrow[dr, "\tilde{f}"] & & \\ (\mathrm{A}, \mathrm{F}[1]) \arrow[ur, "\mathrm{id}_\mathrm{A}"] \arrow[rr] & & (\mathrm{A}, \mathrm{F}) \arrow[r] & (\mathrm{C}, \mathrm{F}) \arrow[r] & {} \end{tikzcd}$$

is sent by G to a complex in $\mathrm{C}^b(\mathscr{C})$ isomorphic to $0 \to \mathrm{K}^a \to \cdots \to \mathrm{K}^b \to 0$. There is some ambiguity in the definition of A, B and f. As we will see, we want $f^{p+1} : \mathrm{K}^p \to \mathrm{K}^{p+1}$ to coincide with $-d^p$, G(B) to coincide with the bête truncation $\sigma_{\geq p+1}(\cdots \to \mathrm{K}^i \xrightarrow{d^i} \cdots)$ of the desired complex, and G(A) to coincide with the translation of the bête truncation $\sigma_{\leq p}(\cdots \to \mathrm{K}^i \xrightarrow{d^i} \cdots)[-1]$. We have a commutative diagram with distinguished rows and columns.

$$
\begin{tikzcd}
\mathrm{Gr}_{\mathrm{F}}^{i+2}\,\mathrm{A} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{B} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{C} \arrow[r] \arrow[d] & {} \\
\mathrm{F}^{i+1}/\mathrm{F}^{i+3}\mathrm{A} \arrow[r] \arrow[d] & \mathrm{F}^{i}/\mathrm{F}^{i+2}\mathrm{B} \arrow[r] \arrow[d] & \mathrm{F}^{i}/\mathrm{F}^{i+2}\mathrm{C} \arrow[r] \arrow[d] & {} \\
\mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{A} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{B} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{C} \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd}
$$

When $i > p$, $\mathrm{Gr}_{\mathrm{F}[1]}^{i}\,\mathrm{A} = \mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{A} = 0$, the left column vanishes, and we find an isomorphism of bête-truncated complexes $\sigma_{>p}\mathrm{G}(\mathrm{B}) \xrightarrow{\sim} \sigma_{>p}\mathrm{G}(\mathrm{C})$:

$$
\begin{tikzcd}
(\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{C})[p+1] \arrow[r, "d_{\mathrm{C}}^{p+1}"] & \cdots \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{b}\,\mathrm{C})[b] \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{b+1}\,\mathrm{C})[b+1] \arrow[r] & \cdots \\
(\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{B})[p+1] \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \cdots \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{b}\,\mathrm{B})[b] \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & (\mathrm{Gr}_{\mathrm{F}}^{b+1}\,\mathrm{B})[b+1] \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \cdots \\
\mathrm{K}^{p+1} \arrow[r, "d^{p+1}"] & \cdots \arrow[r] & \mathrm{K}^{b} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

When $i \leq p$, $\mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{B} = 0$ and $\mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{C} \xrightarrow{\sim} (\mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{A})[1]$. When $i < p$, moreover the entire middle column vanishes, finding an isomorphism of bête truncated complexes $\sigma_{<p}\mathrm{G}(\mathrm{C}) \xrightarrow{\sim} \sigma_{\leq p}(\mathrm{G}(\mathrm{A})[1])$:

$$
\begin{tikzcd}
\cdots \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{a-1}\,\mathrm{C})[a-1] \arrow[r] \arrow[d, "\sim"] & (\mathrm{Gr}_{\mathrm{F}}^{a}\,\mathrm{C})[a] \arrow[r, "d_{\mathrm{C}}^{a}"] \arrow[d, "\sim"] & \cdots \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{p}\,\mathrm{C})[p] \arrow[d, "\sim"] \\
\cdots \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{a-1}\,\mathrm{A})[a] \arrow[r] \arrow[d, "\sim"] & (\mathrm{Gr}_{\mathrm{F}}^{a}\,\mathrm{A})[a+1] \arrow[r] \arrow[d, "\sim"] & \cdots \arrow[r] & (\mathrm{Gr}_{\mathrm{F}}^{p}\,\mathrm{A})[p+1] \arrow[d, "\sim"] \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{K}^{a} \arrow[r, "d^{a}"] & \cdots \arrow[r] & \mathrm{K}^{p}
\end{tikzcd}
$$

We will be done if we can join these diagrams with a commutative diagram

$$\begin{tikzcd}
(\mathrm{Gr}_{\mathrm{F}}^{p}\,\mathrm{C})[p] \arrow[r, "d_{\mathrm{C}}^{p}"] \arrow[d, "\sim"] & (\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{C})[p+1] \\
(\mathrm{Gr}_{\mathrm{F}}^{p}\,\mathrm{A})[p+1] \arrow[r, dashed] \arrow[d, "\sim"] & (\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{B})[p+1] \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\mathrm{K}^{p} \arrow[r, "d^{p}"] & \mathrm{K}^{p+1}
\end{tikzcd}$$

When $i = p$, part of our earlier commutative diagram looks like

$$\begin{tikzcd}
& \mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{B} \arrow[r, "\sim"] \arrow[d, "\beta"', "\sim"] & \mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{C} \arrow[d] & \\
\mathrm{F}^{p+1}/\mathrm{F}^{p+3}\mathrm{A} \arrow[r, "\alpha"] \arrow[d, "\gamma"', "\sim"] & \mathrm{F}^{p}/\mathrm{F}^{p+2}\mathrm{B} \arrow[r] & \mathrm{F}^{p}/\mathrm{F}^{p+2}\mathrm{C} \arrow[r] \arrow[d] & {} \\
\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{A} & & \mathrm{Gr}_{\mathrm{F}}^{p}\,\mathrm{C} \arrow[d, "d_{\mathrm{C}}^{p}[-p]"] & \\
& & {} &
\end{tikzcd}$$

with distinguished row and column, inducing an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{B} \arrow[r] \arrow[d, "\sim"] & \mathrm{F}^{p}/\mathrm{F}^{p+2}\mathrm{C} \arrow[r] \arrow[d, equal] & (\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{A})[1] \arrow[r, "-(\beta^{-1}\circ\alpha\circ\gamma^{-1})[1]"] & {} \\
\mathrm{Gr}_{\mathrm{F}}^{p+1}\,\mathrm{C} \arrow[r] & \mathrm{F}^{p}/\mathrm{F}^{p+2}\mathrm{C} \arrow[r] & \mathrm{Gr}_{\mathrm{F}}^{p}\,\mathrm{C} \arrow[r, "d_{\mathrm{C}}^{p}[-p]"] \arrow[u, "\sim"'] & {}
\end{tikzcd} .$$

The commutative diagram

$$
\begin{tikzcd}
 & & \mathrm{K}^p[-p-1] \arrow[r, "f^{p+1}[-p-1]"] & \mathrm{K}^{p+1}[-p-1] \\
\mathrm{Gr}^p_{\mathrm{F}[1]}\mathrm{A} \arrow[rr, equal] & & \mathrm{Gr}^{p+1}_{\mathrm{F}}\mathrm{A} \arrow[u, "\wr"] \arrow[r, "\delta"] \arrow[d] & \mathrm{Gr}^{p+1}_{\mathrm{F}}\mathrm{B} \arrow[u, "\wr"] \arrow[d, "\beta"', "\sim"] \\
\mathrm{F}^{p+1}/\mathrm{F}^{p+3}\mathrm{A} = (\mathrm{F}[1])^p/(\mathrm{F}[1])^{p+2}\mathrm{A} \arrow[u, "\sim", "\gamma"'] \arrow[rr, "\mathrm{id}_{\mathrm{A}}"] \arrow[rrr, bend right, "\alpha"] & & \mathrm{F}^p/\mathrm{F}^{p+2}\mathrm{A} \arrow[r, "\tilde{f}"] & \mathrm{F}^p/\mathrm{F}^{p+2}\mathrm{B}
\end{tikzcd}
$$

shows that $\delta = \beta^{-1} \circ \alpha \circ \gamma^{-1} : \mathrm{Gr}^{p+1}_{\mathrm{F}}\mathrm{A} \to \mathrm{Gr}^{p+1}_{\mathrm{F}}\mathrm{B}$ has the property that $-\delta[p+1]$ makes a commutative diagram

$$
\begin{tikzcd}
(\mathrm{Gr}^p_{\mathrm{F}}\mathrm{C})[p] \arrow[r, "d^p_{\mathrm{C}}"] \arrow[d, "\sim"] & (\mathrm{Gr}^{p+1}_{\mathrm{F}}\mathrm{C})[p+1] \\
(\mathrm{Gr}^p_{\mathrm{F}}\mathrm{A})[p+1] \arrow[r, "-\delta[p+1]"] \arrow[d, "\sim"] & (\mathrm{Gr}^{p+1}_{\mathrm{F}}\mathrm{B})[p+1] \arrow[u, "\sim"] \arrow[d, "\sim"] \\
\mathrm{K}^p \arrow[r, "-f^{p+1}"] & \mathrm{K}^{p+1}
\end{tikzcd}
$$

This is why we should take $f^{p+1} = -d^p$.

Remark. There is a related construction involving décalage as defined by Deligne in *Théorie de Hodge II* (1.3.3). (The choice of name, décalage, refers not to a shift of the filtration, but rather to a shift in the page numbering of the spectral sequences associated to the filtration.) The décalage functor $(\mathrm{X}, \mathrm{F}) \mapsto (\mathrm{X}, \mathrm{Dec}(\mathrm{F}))$ is defined componentwise by

$$(\mathrm{Dec}\,\mathrm{F})^a(\mathrm{X}^i) := \ker(d : \mathrm{F}^{i+a}(\mathrm{X}^i) \to \mathrm{F}^{i+a}(\mathrm{X}^{i+1})/\mathrm{F}^{i+a+1}(\mathrm{X}^{i+1})).$$

By construction, the differential on $(\mathrm{Dec}\,\mathrm{F})^a$ coincides with the coboundary map on cohomology so that if we define (for each a) the complex $\mathrm{H}^a(\mathrm{X}, \mathrm{F})^\bullet$ with $\mathrm{H}^a(\mathrm{X}, \mathrm{F})^i := \mathrm{H}^i\,\mathrm{Gr}^{i+a}_{\mathrm{F}}\mathrm{X}$ with differential coming from the coboundary, the obvious map

$$\mathrm{Gr}^a_{\mathrm{Dec}\,\mathrm{F}}\mathrm{X} \longrightarrow \mathrm{H}^a(\mathrm{X}, \mathrm{F})$$

is a quasi-isomorphism. There is a t-structure on the filtered derived category DF with

$$(\mathrm{DF})^{\leq 0} = \{(\mathrm{X},\mathrm{F}) \text{ s.t. } \mathrm{Gr}^i\,\mathrm{X} \text{ is acyclic in degrees} > i\}$$
$$(\mathrm{DF})^{\geq 0} = \{(\mathrm{X},\mathrm{F}) \text{ s.t. } \mathrm{Gr}^i\,\mathrm{X} \text{ is acyclic in degrees} < i\}$$

whose heart coincides with the category of bête-filtered complexes. The truncation $\tau_{\leq 0}$ for this t-structure is the functor

$$(\mathrm{X},\mathrm{F}) \mapsto ((\mathrm{Dec}\,\mathrm{F})^0\mathrm{X}, \mathrm{F}\cap(\mathrm{Dec}\,\mathrm{F})^0)$$

which lands in $(\mathrm{DF})^{\leq 0}$ since $(\mathrm{Dec}\,\mathrm{F})^0$ lies in F^{i+1} in degrees $> i$. Under the equivalence of categories G, the corresponding cohomological functor is the functor $\mathrm{H}^{\bullet}$. This is (3.11) in Beilinson's *Notes on Absolute Hodge Cohomology*.

3.1.9. The isomorphisms (3.1.9.2) render anticommutative the given square because the isomorphisms $\mathrm{Gr}^i_{\mathrm{F}}(\mathrm{K}[1]) = (\mathrm{Gr}^i_{\mathrm{F}}\,\mathrm{K})[1]$ together with their translations render anticommutative the square

$$\begin{array}{ccc}
\mathrm{Gr}^i_{\mathrm{F}}(\mathrm{K}[1]) & \xrightarrow{d_1[-i]} & \mathrm{Gr}^{i+1}_{\mathrm{F}}(\mathrm{K}[1])[1] \\
\| & & \| \\
(\mathrm{Gr}^i_{\mathrm{F}}\,\mathrm{K})[1] & \xrightarrow{d_0[1-i]} & (\mathrm{Gr}^{i+1}_{\mathrm{F}}\,\mathrm{K})[2]
\end{array}$$

where we have added subscripts for clarity. As discussed in the note to (3.1.7) above, $d_0[1-i] : (\mathrm{Gr}^i_{\mathrm{F}}\,\mathrm{K})[1] \to (\mathrm{Gr}^{i+1}_{\mathrm{F}}\,\mathrm{K})[2]$ is the translation of the morphism $d_0[-i]$ defined by the distinguished triangle

$$\mathrm{Gr}^{i+1}_{\mathrm{F}}\,\mathrm{K} \xrightarrow{u} \mathrm{F}^i/\mathrm{F}^{i+2}\mathrm{K} \xrightarrow{v} \mathrm{Gr}^i_{\mathrm{F}}\,\mathrm{K} \xrightarrow{d_0[-i]}$$

On the other hand, $d_1[-i] : \mathrm{Gr}^i_{\mathrm{F}}(\mathrm{K}[1]) \to \mathrm{Gr}^{i+1}_{\mathrm{F}}(\mathrm{K}[1])$ is the morphism defined by the triangle in the second row

$$\begin{array}{ccccccc}
(\mathrm{Gr}^{i+1}_{\mathrm{F}}\,\mathrm{K})[1] & \xrightarrow{u[1]} & (\mathrm{F}^i/\mathrm{F}^{i+2}\mathrm{K})[1] & \xrightarrow{v[1]} & (\mathrm{Gr}^i_{\mathrm{F}}\,\mathrm{K})[1] & \longrightarrow & \\
\| & & \| & & \| & & \\
\mathrm{Gr}^{i+1}_{\mathrm{F}}(\mathrm{K}[1]) & \longrightarrow & \mathrm{F}^i/\mathrm{F}^{i+2}(\mathrm{K}[1]) & \longrightarrow & \mathrm{Gr}^i_{\mathrm{F}}(\mathrm{K}[1]) & \xrightarrow{d_1[-i]} &
\end{array}$$

once we show that this triangle is distinguished. If we can show there is a morphism of degree 1 $d_1[-i]$ making this triangle distinguished, this morphism of degree 1 will be unique by the considerations of the note to (3.1.7). The diagram

$$\begin{tikzcd}
(\mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{K})[1] \arrow[r, "-u[1]"] \arrow[d, "\mathrm{id}"] & (\mathrm{F}^i/\mathrm{F}^{i+2}\mathrm{K})[1] \arrow[r, "-v[1]"] \arrow[d, "-\mathrm{id}"] & (\mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{K})[1] \arrow[r, "-d_0[1-i]"] \arrow[d, "\mathrm{id}"] & {} \\
(\mathrm{Gr}_{\mathrm{F}}^{i+1}\,\mathrm{K})[1] \arrow[r, "u[1]"] \arrow[d, equal] & (\mathrm{F}^i/\mathrm{F}^{i+2}\mathrm{K})[1] \arrow[r, "v[1]"] \arrow[d, equal] & (\mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{K})[1] \arrow[r, "-d_0[1-i]"] \arrow[d, equal] & {} \\
\mathrm{Gr}_{\mathrm{F}}^{i+1}(\mathrm{K}[1]) \arrow[r] & \mathrm{F}^i/\mathrm{F}^{i+2}(\mathrm{K}[1]) \arrow[r] & \mathrm{Gr}_{\mathrm{F}}^{i}(\mathrm{K}[1]) \arrow[r, "d_1[-i]"] & {}
\end{tikzcd}$$

is an isomorphism of triangles with the top row distinguished. We conclude $d_1[-i] = -d_0[1-i]$ after the canonical isomorphisms $\mathrm{Gr}_{\mathrm{F}}^{i}(\mathrm{K}[1])[j] = (\mathrm{Gr}_{\mathrm{F}}^{i}\,\mathrm{K})[1+j]$.

3.1.11. The point is that we already computed the differential on the E_1 page to be $d_1(f) = df - fd$ in the note to 3.1.8, and the image of $d_1 : \mathrm{E}_1^{-1,0} \to \mathrm{E}_1^{0,0}$ consists of precisely the homotopies. As computed in 3.1.8, the kernel of $d_1 : \mathrm{E}_1^{0,0} \to \mathrm{E}_1^{1,0}$ consists of precisely the morphisms of complexes in $\mathrm{C}^b(\mathscr{C})$.

3.1.12. We can choose a representative f for a homotopy class of morphisms of complexes in $\mathrm{K}^b(\mathscr{C})$. Typos & clarifications: '$\mathrm{G}(\mathrm{C}(f), \tilde{\mathrm{F}}) = \mathrm{C}(\mathrm{G}(f))$. En particulier, si (K, F) et (L, F) sont bêtes, $(\mathrm{C}(f), \tilde{\mathrm{F}})$ l'est aussi, et $\mathrm{C}(\omega f) = \omega(\mathrm{C}(f), \tilde{\mathrm{F}})$ s'identifie à $\mathrm{real}(\mathrm{C}(f))$.' The isomorphism $\mathrm{G}(\mathrm{C}(f), \tilde{\mathrm{F}}) = \mathrm{C}(\mathrm{G}(f))$ relies on the earlier calculation $\mathrm{G}(\mathrm{K}[1], \mathrm{F}[1]) = (\mathrm{GK})[1]$. Interestingly,

$$\mathrm{G}(\mathrm{K}, \mathrm{L}, \mathrm{C}(f)) = \omega((\mathrm{K}, \mathrm{F}), (\mathrm{L}, \mathrm{F}), (\mathrm{C}(f), \tilde{\mathrm{F}})) = (\mathrm{K}, \mathrm{L}, \mathrm{C}(f))$$

where the middle triangle is not distinguished, but after forgetting the filtration, it is.

3.1.13. We will compute the differentials of $\mathrm{G}(\Sigma)$ to verify that $\mathbf{s}\Sigma^* = \mathrm{G}(\Sigma)$, but before we do, a few words on the matter of sign that was also an issue in the discussion of both solutions to (3.1.8) in the note to 3.1.8. Given a map $f : \mathrm{A}^* \to \mathrm{B}^*$ of complexes (considered as a double complex with first degree concentrated in 0 and 1) and the filtration by first degree on the associated simple complex K, it is easy to compute that the coboundary ∂ on cohomology associated to the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Gr}^1 \mathrm{K} \arrow[r, "v"] \arrow[d, equal] & \mathrm{F}^0/\mathrm{F}^2\mathrm{K} \arrow[r] \arrow[d, equal] & \mathrm{Gr}^0 \mathrm{K} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{B}[-1] \arrow[r] & \mathrm{B}[-1] \oplus \mathrm{A} \arrow[r] & \mathrm{A} \arrow[r] & 0 \end{tikzcd}$$

coincides with f, as the differential on $\mathrm{F}^0/\mathrm{F}^2\mathrm{K} \simeq \mathrm{B}[-1] \oplus \mathrm{A}$ is given by

$$\begin{pmatrix} d_{\mathrm{B}[-1]} & f \\ 0 & d_{\mathrm{A}} \end{pmatrix}.$$

The issue appears to be that when one defines the differential on the cone of a morphism of complexes $u : \mathrm{L} \to \mathrm{M}$ in the usual way as

$$\begin{pmatrix} d_{\mathrm{L}[1]} & 0 \\ u & d_{\mathrm{M}} \end{pmatrix},$$

then given the exact sequence of complexes with coboundary ∂

$$0 \to \mathrm{L} \xrightarrow{u} \mathrm{M} \to \operatorname{coker} u \to 0,$$

the morphism of degree one δ in the distinguished triangle $(\mathrm{L}, \mathrm{Cyl}(u), \mathrm{C}(u))$ coincides on cohomology with $-\partial$ after the isomorphism with $(\mathrm{L}, \mathrm{M}, \operatorname{coker} u)$. (This is consistent with Weibel's book but Gelfand-Manin say, I believe incorrectly, ∂.)

Applying this in the case of $u : \mathrm{Gr}^1 \mathrm{K} \to \mathrm{F}^0/\mathrm{F}^2\mathrm{K}$, we expect that the map of degree one $\mathrm{Gr}^0 \mathrm{K} \to (\mathrm{Gr}^1 \mathrm{K})[1]$ will coincide with $-f$. As before, this isn't really a problem in view of the isomorphism

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{A} \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & \mathrm{B} \arrow[r, "g"] \arrow[d, "-\mathrm{id}"] & \mathrm{C} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathrm{A} \arrow[r, "-f"] & \mathrm{B} \arrow[r, "-g"] & \mathrm{C} \arrow[r] & 0. \end{tikzcd}$$

It is also possible to define the differential of the cone of the morphism u to be

$$\begin{pmatrix} d_{\mathrm{L}[1]} & 0 \\ -u & d_{\mathrm{M}} \end{pmatrix},$$

and although less standard, this would correct our sign, but we don't proceed this way.

Returning to the notation in the beginning of this paragraph, we will show that the differentials on $\mathrm{G}(\mathbf{s}\Sigma^*)$ coincide with the differentials of Σ with the sign reversed. In fact, it suffices to do so for the single morphism of complexes f. In the double complex

$$\begin{tikzcd} \uparrow & \uparrow \\ \mathrm{A}^p \arrow[r, "f^p"] \arrow[u] & \mathrm{B}^p \arrow[u] \\ \mathrm{A}^{p-1} \arrow[r, "f^{p-1}"] \arrow[u] & \mathrm{B}^{p-1} \arrow[u] \\ \uparrow \arrow[u] & \uparrow \arrow[u] \end{tikzcd}$$

we consider first degree concentrated in 0, 1, and we let K denote the associated simple complex. With v as above,

$$\begin{tikzcd} \mathrm{C}(v) \arrow[r, "\delta"] \arrow[d] & (\mathrm{Gr}^1 \mathrm{K})[1] \arrow[d, equal] \\ \mathrm{Gr}^0 \mathrm{K} \arrow[r, "f"] & (\mathrm{Gr}^1 \mathrm{K})[1] \end{tikzcd} \quad \simeq \quad \begin{tikzcd} \mathrm{B} \oplus \mathrm{B}[-1] \oplus \mathrm{A} \arrow[r, "\delta"] \arrow[d, "\gamma"] & \mathrm{B} \arrow[d, equal] \\ \mathrm{A} \arrow[r, "f"] & \mathrm{B} \end{tikzcd}$$

where δ is projection to the first factor, γ to the last, and we wish to show these squares commute up to homotopy. The differential on $\mathrm{C}(v) \simeq \mathrm{B} \oplus \mathrm{B}[-1] \oplus \mathrm{A}$ is given by

$$\begin{pmatrix} d_{\mathrm{B}} & & \\ \mathrm{id} & d_{\mathrm{B}[-1]} & f \\ & & d_{\mathrm{A}} \end{pmatrix}$$

and we compute $dh - hd$ for the homotopy $h^p : \mathrm{B}^p \oplus \mathrm{B}^{p-1} \oplus \mathrm{A}^p \to \mathrm{B}^{p-1}$ which is projection to the middle factor:

$$dh(b^p, b^{p-1}, a^p) = db^{p-1}$$
$$hd(b^p, b^{p-1}, a^p) = b^p + f(a^p) - db^{p-1}.$$

This shows that $\delta \sim -f$ so that if $\Sigma^* = \mathrm{A}^* \xrightarrow{f} \mathrm{B}^* \xrightarrow{g} \mathrm{C}^*$, the differentials of $\mathrm{G}(\mathbf{s}\Sigma^*)$ coincide with $-f$ and $-g$ which, as discussed above, is ok.

3.1.14. We check that the cone C of the inclusion $\iota : \tau_{\leq i}\mathrm{K}^* \to \tau'_{\leq i}\mathrm{K}^*$ is homotopic to the complex L given by $\ker d \to \mathrm{K}^i \to \operatorname{im} d$ in degrees $i-1, i, i+1$. The morphisms of complexes f and g are given componentwise by

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "f^{i-2}"] & \ker d \arrow[r] \arrow[d, "f^{i-1}"] & \mathrm{K}^i \arrow[r] \arrow[d, "f^i"] & \operatorname{im} d \arrow[r] \arrow[d, "f^{i+1}"] & 0 \arrow[r] \arrow[d, "f^{i+2}"] & \cdots \\
\cdots \arrow[r] & \mathrm{K}^{i-1} \arrow[r, "{d[1]}"] \arrow[ddr, "\iota^{i-1}"] & \ker d \arrow[r] \arrow[ddr, "\iota^{i}"] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
 & \oplus & \oplus & \oplus & \oplus & \oplus & \\
\cdots \arrow[r] & \mathrm{K}^{i-2} \arrow[r, "d"] \arrow[d, "g^{i-2}"] & \mathrm{K}^{i-1} \arrow[r, "d"] \arrow[d, "g^{i-1}"] & \mathrm{K}^i \arrow[r, "d"] \arrow[d, "g^i"] & \operatorname{im} d \arrow[r] \arrow[d, "g^{i+1}"] & 0 \arrow[r] \arrow[d, "g^{i+2}"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \ker d \arrow[r] & \mathrm{K}^i \arrow[r] & \operatorname{im} d \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

where f^{i-1} is inclusion into the first factor and f^i, f^{i+1} are inclusion into the second factor, while g^i, g^{i+1} are projection to the second factor, $g^j = 0$ for $j < i-1$, and $g^{i-1} = \mathrm{pr}_1 - d \circ \mathrm{pr}_2$. Evidently $g \circ f = \mathrm{id}_\mathrm{L}$ and we need to construct a homotopy h so that $f \circ g = \mathrm{id}_\mathrm{C} + dh + hd$. Set each $h^p : \mathrm{C}^p \to \mathrm{C}^{p-1}$ for $p > i-1$ to zero, while for $p \leq i-1$, let h^p be the projection to the second factor of C^p followed by inclusion into the first factor of C^{p-1} so that if we only draw arrows for the components of h which are nonzero, h is given by

$$
\begin{tikzcd}
\cdots & \mathrm{K}^{i-2} & \mathrm{K}^{i-1} & \ker d & 0 & 0 & 0 & \cdots \\
 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \\
 & \mathrm{K}^{i-3} & \mathrm{K}^{i-2} \arrow[uul, "\mathrm{id}"'] & \mathrm{K}^{i-1} \arrow[uul, "\mathrm{id}"'] & \mathrm{K}^i & \operatorname{im} d & 0 &
\end{tikzcd}
$$

Now let's examine why an isomorphism between $\mathrm{H}^i\mathrm{K}^*[-i]$ and a cone on $\mathrm{real}(\tau_{\leq i-1}\mathrm{K}^*) \to \mathrm{real}(\tau_{\leq i}\mathrm{K}^*)$ is enough to conclude that $\mathrm{real}(\tau_{\leq i}\mathrm{K}^*) \simeq \tau_{\leq i}(\mathrm{real}\,\mathrm{K}^*)$ and ${}^p\mathrm{H}^i\,\mathrm{real}\,\mathrm{K}^* \simeq \mathrm{H}^i\mathrm{K}^*$. As K^* is an object of $\mathrm{C}^b(\mathscr{C})$, there are integers a, b such that $\mathrm{K}^i = 0$ for $i < a$ and $i > b$. Then $\tau_{\leq a-1}\mathrm{K}^* = 0$ and we find $\mathrm{real}(\tau_{\leq a}\mathrm{K}^*) \simeq (\mathrm{H}^a\mathrm{K}^*)[-a]$. As $\mathrm{H}^i\mathrm{K}^*$ is in $\mathscr{C}$, proceeding inductively we find that ${}^p\mathrm{H}^j(\mathrm{real}(\tau_{\leq i}\mathrm{K}^*))$ equals $\mathrm{H}^j\mathrm{K}^*$ for $j \leq i$ and is null for $j > i$. As $\mathrm{H}^{b+1}\mathrm{K}^* = 0$, $\mathrm{real}(\tau_{\leq b}\mathrm{K}^*) \simeq \mathrm{real}(\tau_{\leq b+1}\mathrm{K}^*) = \mathrm{real}\,\mathrm{K}^*$ so that $\mathrm{real}(\tau_{\leq b}\mathrm{K}*) \simeq \mathrm{real}\,\mathrm{K}^* \simeq \tau_{\leq b}\,\mathrm{real}\,\mathrm{K}^*$ since as we have seen, ${}^p\mathrm{H}^j(\mathrm{real}\,\mathrm{K}^*) = 0$ for $j > b$. Now supposing $\mathrm{real}(\tau_{\leq i}\mathrm{K}^*) \simeq \tau_{\leq i}(\mathrm{real}\,\mathrm{K}^*)$ has been established, the isomorphism of distinguished triangles

$$
\begin{tikzcd}
\mathrm{real}(\tau_{\leq i-1}\mathrm{K}^*) \arrow[r] \arrow[d, dashed, "\sim"] & \mathrm{real}(\tau_{\leq i}\mathrm{K}^*) \arrow[r] \arrow[d, "\sim"] & (\mathrm{H}^i\mathrm{K}^*)[-i] \arrow[r] \arrow[d, "\sim"] & {} \\
\tau_{\leq i-1}(\mathrm{real}\,\mathrm{K}^*) \arrow[r] & \tau_{\leq i}(\mathrm{real}\,\mathrm{K}^*) \arrow[r] & {}^p\mathrm{H}^i(\mathrm{real}\,\mathrm{K}^*)[-i] \arrow[r] & {}
\end{tikzcd}
$$

achieves the same for $i-1$.

Now we show that the cone C^* on the morphism $\iota : \tau'_{\leq i-1}\mathrm{K}^* \to \tau_{\leq i}\mathrm{K}^*$ is homotopic to the complex L^* reduced to $\mathrm{im}\, d^{i-1} \to \ker d^i$ in degrees $i-1$ and i. The story is the same as what we just did. We define morphisms $f : \mathrm{L}^* \to \mathrm{C}^*$ and $g : \mathrm{C}^* \to \mathrm{L}^*$ which compose to morphisms homotopic to the identity.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{im}\, d^{i-1} \arrow[r] \arrow[d, "f^{i-1}"] & \mathrm{im}\, d \arrow[r] \arrow[d, "f^i"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{K}^{i-2} \arrow[r, "{d[1]}"] \arrow[ddr, "\iota^{i-2}"] & \mathrm{K}^{i-1} \arrow[r] \arrow[ddr, "\iota^{i-1}"] & \mathrm{im}\, d^{i-1} \arrow[r] \arrow[ddr, "\iota^{i}"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
& \oplus & \oplus & \oplus & \oplus & \oplus & \\
\cdots \arrow[r] & \mathrm{K}^{i-3} \arrow[r, "d"] \arrow[d] & \mathrm{K}^{i-2} \arrow[r] \arrow[d] & \mathrm{K}^{i-1} \arrow[r] \arrow[d, "g^{i-1}"] & \ker d^i \arrow[r] \arrow[d, "g^i"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathrm{im}\, d^{i-1} \arrow[r] & \mathrm{im}\, d \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Here the morphisms f^i, g^i is the obvious inclusion and projection, while f^{i-1} is the inclusion to the first factor and $g^{i-1} = \mathrm{pr}_1 - d^{i-1} \circ \mathrm{pr}_2$. Then $g \circ f = \mathrm{id}_{\mathrm{L}^*}$ and $f \circ g = \mathrm{id}_{\mathrm{C}^*} + dh + hd$ where h is the homotopy with all components nonzero except for the arrows in the below diagram.

$$
\begin{tikzcd}
& \mathrm{K}^{i-2} & \mathrm{K}^{i-1} & \mathrm{im}\, d^{i-1} & 0 & 0 & 0 & \\
\cdots & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
& \mathrm{K}^{i-3} & \mathrm{K}^{i-2} \arrow[uul, "\mathrm{id}"'] & \mathrm{K}^{i-1} \arrow[uul, "\mathrm{id}"'] & \ker d^i & 0 & 0 &
\end{tikzcd}
$$

The complex L^* is the cone of the morphism

$$\mathrm{im}(d^{i-1})[-i] \to \ker(d^i)[-i]$$

(shifted by $[-i]$ not $[i]$) and the point is that the realization functor sends an object D of $\mathscr{C}$ shifted by i to the same object shifted by i: the complex $\mathrm{D}[i]$ with one-step filtration $F^j(D[i]) = D[i]$ for $j \le -i$, $F^j(D[i]) = 0$ for $j > -i$ is bête-filtered with $Gr^{-i}(D[i])[-i] = D$ and all other $Gr^j = 0$, so that $\mathrm{real}(D[i]) = D[i]$. Of course, the exact sequence

$$0 \to \mathrm{im}\, d^{i-1} \to \ker d^i \to H^i K^* \to 0$$

gives rise to a distinguished triangle (where the cone is unique up to unique isomorphism)

$$(\mathrm{im}\, d^{i-1})[-i] \to (\ker d^i)[-i] \to (H^i K^*)[-i] \to .$$

Remark. The functor real is exact and t-exact.

3.1.16. The claim about the characterization of the $\mathrm{Hom}_{D\mathscr{C}}(A, B[n])$ is effectively the classical Tôhoku statement, with the twist that it suffices to efface morphisms one at a time, rather than the whole Hom group at once. Swan calls this weaker effaceability property described in 3.1.16 'weak effaceability.' Grothendieck's statement about a δ-functor F^n being universal if F^n is effaceable for $n > 0$ holds with 'effaceable' replaced by 'weakly effaceable' if F takes values in a module category. The δ-functor $\mathrm{Hom}_{D\mathscr{C}}(A, -[n])$ takes values in abelian groups and is weakly effaceable. See Buchsbaum, *Satellites and Universal Functors* (4.2) & (4.3) and Swan, *Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions* (1.1) & (6.1) for the full story.

The bit about every object of $D^b\mathscr{C}$ se dévissant en objets de $\mathscr{C}$ is combined with the exact sequence of Hom^i of proposition 1-2 of 'état 0' of Verdier's thesis in SGA $4\frac{1}{2}$.

3.1.17. As remark (ii) is a corollary of remark (i), let it suffice to prove (i). There is some trivial ambiguity: where remark (i) writes 'pour A, B in $\mathscr{C}$' it means 'for any A, B in $\mathscr{C}$.' On the one hand, if $f \in \mathrm{Hom}_{\mathscr{D}}(A, B[N])$ is in the image of real, say $\mathrm{real}(\tilde{f}) = f$, then $\tilde{f}$ can be represented in $\mathrm{Hom}_{D\mathscr{C}}(A, B[N])$ by a Yoneda Ext and either the monomorphism at the left end or the epimorphism at the right will efface $\tilde{f}$, and, after realization, f (see below). On the other hand, let's suppose f is effaceable in the sense of (3.1.16) and we have an epimorphism $u : A' \twoheadrightarrow A$ and monomorphism $v'[-N] : B \hookrightarrow B'$ such that $v' f u = 0$. In light of (1.2.3), these morphisms give rise to distinguished triangles (A'', A', A) and (B, B', B'') in $\mathscr{D}$ which are unique up to unique isomorphism, and, in view of (1.1.9), a morphism of triangles

$$\begin{tikzcd} \mathrm{A}' \arrow[r, "u"] \arrow[d, dashed, "g"] & \mathrm{A} \arrow[r] \arrow[d, "f"] & \mathrm{A}''[1] \arrow[r] \arrow[d, dashed, "h"] & {} \\ \mathrm{B}''[\mathrm{N}-1] \arrow[r] & \mathrm{B}[\mathrm{N}] \arrow[r, "v'"] & \mathrm{B}'[\mathrm{N}] \arrow[r] & {} \end{tikzcd}$$

where now the two dashed arrows lie in a $\mathrm{Hom}_{\mathscr{C}}^{\mathrm{N}-1}$. All the data of this morphism of triangles (except f) is lifted uniquely to $\mathrm{D}\mathscr{C}$, where we can complete it to a morphism of triangles (in possibly several different ways) and then apply real. We would like to show that real establishes a bijection between the morphisms $\tilde{f}$ which complete the lift of the above to a morphism of triangles in $\mathrm{D}\mathscr{C}$ and those f which complete the morphism of triangles in $\mathscr{D}$. This will simultaneously show that our original f is in the image of real and prove that real : $\mathrm{Hom}_{\mathrm{D}\mathscr{C}}^{\mathrm{N}}(\mathrm{A}, \mathrm{B}[\mathrm{N}]) \to \mathrm{Hom}_{\mathscr{D}}^{\mathrm{N}}(\mathrm{A}, \mathrm{B}[\mathrm{N}])$ is injective.

We have a commutative diagram with exact rows and columns on the next page. From this diagram we see that if $g \in \mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}''[\mathrm{N}])$ and $h \in \mathrm{Hom}(\mathrm{A}''[1], \mathrm{B}'[\mathrm{N}])$ such that there is at least one $f \in \mathrm{Hom}(\mathrm{A}, \mathrm{B}[\mathrm{N}])$ such that fu is in the image of g and $v'f$ is in the image of h, the set of all $f \in \mathrm{Hom}(\mathrm{A}, \mathrm{B}[\mathrm{N}])$ satisfying this same property is in bijection with

$$\begin{aligned} &\mathrm{X} \cap \mathrm{Y}, \quad \text{where} \\ &\mathrm{X} := \operatorname{coker}(\mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}[\mathrm{N}]) \to \mathrm{Hom}(\mathrm{A}''[1], \mathrm{B}[\mathrm{N}])) \quad \text{and} \\ &\mathrm{Y} := \operatorname{coker}(\mathrm{Hom}^{-1}(\mathrm{A}, \mathrm{B}'[\mathrm{N}]) \to \mathrm{Hom}^{-1}(\mathrm{A}, \mathrm{B}''[\mathrm{N}])). \end{aligned}$$

Both X and Y are considered as subgroups of $\mathrm{Hom}(\mathrm{A}, \mathrm{B}[\mathrm{N}])$. The point is that real sends X and Y (Hom taken in $\mathrm{D}\mathscr{C}$) isomorphically to X and Y, respectively (Hom taken in $\mathscr{D}$) by the hypothesis that real is an isomorphism on Hom^n for $n < \mathrm{N}$. Therefore real does the same for $\mathrm{X} \cap \mathrm{Y}$. This proves our claim.

$$
\begin{tikzcd}
& \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}', \mathrm{B}[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}', \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}', \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}^{-1}(\mathrm{A}, \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-1}(\mathrm{A}, \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}, \mathrm{B}[\mathrm{N}]) \arrow[r, "v'"] \arrow[u, "u"] & \mathrm{Hom}(\mathrm{A}, \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}, \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}^{-1}(\mathrm{A}''[1], \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-1}(\mathrm{A}''[1], \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}''[1], \mathrm{B}[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}''[1], \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}(\mathrm{A}''[1], \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}^{-2}(\mathrm{A}', \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-2}(\mathrm{A}', \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}'[\mathrm{N}]) \arrow[r] \arrow[u] & \mathrm{Hom}^{-1}(\mathrm{A}', \mathrm{B}''[\mathrm{N}]) \arrow[r] \arrow[u] & \cdots \\
& \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Annihilating Yoneda Ext*s*. Now we verify that given $f \in \mathrm{Hom}_{\mathrm{D}\mathscr{C}}(\mathrm{A}, \mathrm{B}[\mathrm{N}])$ represented by the Yoneda Ext

$$\mathrm{Q} := \cdots \to 0 \to \mathrm{B} = \mathrm{K}^{-\mathrm{N}} \xrightarrow{\alpha} \mathrm{K}^{-\mathrm{N}+1} \to \cdots \to \mathrm{K}^0 \xrightarrow{\beta} \mathrm{A} \to 0 \to \cdots$$

either the monomorphism α or the epimorphism β will efface f. The postcomposition with $\alpha[\mathrm{N}]$ is represented by the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \mathrm{A} \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & \mathrm{B} = \mathrm{K}^{-\mathrm{N}} \arrow[r, "\alpha"] \arrow[u] \arrow[d, "\mathrm{id}_{\mathrm{B}}"] & \mathrm{K}^{-\mathrm{N}+1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & \mathrm{K}^0 \arrow[r] \arrow[u, "\beta"'] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{B} \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{K}^{-\mathrm{N}+1} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

with β inducing a quasi-isomorphism, and the point is that the morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{B} = \mathrm{K}^{-\mathrm{N}} \arrow[r, "\alpha"] \arrow[d, "\alpha"] & \mathrm{K}^{-\mathrm{N}+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{K}^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{K}^{-\mathrm{N}+1} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is homotopic to 0 via the obvious homotopy. On the other hand, the precomposition with β is represented by the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{K}_0 \arrow[r] \arrow[d, "\beta"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \mathrm{A} \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & \mathrm{B} = \mathrm{K}^{-\mathrm{N}} \arrow[r, "\alpha"] \arrow[u] \arrow[d, "\mathrm{id}_{\mathrm{B}}"] & \mathrm{K}^{-\mathrm{N}+1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & \mathrm{K}^0 \arrow[r] \arrow[u, "\beta"'] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{B} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and here the point is that this morphism is in the same equivalence class as the morphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{K}_0 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{B} = \mathrm{K}^{-\mathrm{N}} \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & \mathrm{K}^{-\mathrm{N}+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{K}^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{B} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

as the diagram

$$\begin{tikzcd}
 & \mathrm{K}^0 \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
\mathrm{K}^0 \arrow[dr, "\beta"] & & \mathrm{Q} \arrow[dl, "\beta"] \\
 & \mathrm{A} &
\end{tikzcd}$$

commutes, where here the arrow decorations refer only to the degree 0 component of the morphism, the other components being obvious. Now it is obvious this composition is null.

4.1.1. The following proof of Artin's theorem was given to me by Sasha Beilinson. The η notation, along with the construction of a retraction, is discussed after the proof.

Below $\mathrm{D}(\mathrm{X}) := \mathrm{D}^b_c(\mathrm{X}, \mathbf{Z}/\ell^n), \mathrm{D}(\mathrm{X})^{\leq 0} := \{\mathscr{F} \in \mathrm{D}(\mathrm{X}) : \dim \operatorname{supp} \mathrm{H}^i\mathscr{F} \leq -i\}$, i.e., this is ${}^p\mathrm{D}(\mathrm{X})^{\leq 0}$ where p is the middle perverse t-structure.

Theorem (*Artin*). — *For an affine map* $f : \mathrm{X} \to \mathrm{Y}$ *of schemes of finite type over a field* k *(with* char k *prime to* ℓ*) the functor* $f_* : \mathrm{D}(\mathrm{X}) \to \mathrm{D}(\mathrm{Y})$ *is right* t*-exact.*

Proof. — Pick $\mathscr{F} \in \mathrm{D}(\mathrm{X})^{\leq 0}$; we want to show that $f_*\mathscr{F} \in \mathrm{D}(\mathrm{Y})^{\leq 0}$. Let $d(\mathscr{F})$ be the dimension of support of $\mathscr{F}$. We use induction by $d(\mathscr{F})$, so we assume that for every f, k as in the theorem and $\mathscr{G} \in \mathrm{D}(\mathrm{X})^{\leq 0}$ with $d(\mathscr{G}) < d(\mathscr{F})$ one has $f_*\mathscr{G} \in \mathrm{D}(\mathrm{Y})^{\leq 0}$.

(o) *We can assume that* k *is algebraically closed* (since f_* commutes with the base change to an algebraic closure of k).

(i) *It is enough to show that for every closed point $y \in \mathrm{Y}$ the complex $(f_*\mathcal{F})_y$ is connective* (i.e., acyclic in degrees > 0): We need to check that for a point η of Y of dimension $\delta > 0$ the complex $(f_*\mathcal{F})_\eta[-\delta]$ is connective. Let $\mathrm{Z} \subset \mathrm{Y}$ be the closure of η. Replacing Y by an étale neighborhood of η, choose a retraction $\mathrm{Y} \to \mathrm{Z}$ (it exists since k is perfect). Consider Y as a Z-scheme and f as a map of Z-schemes. Let $f^o : \mathrm{X}^o \to \mathrm{Y}^o$ be the map of the generic fibers (over $\eta = \operatorname{Spec} k_\eta \in \mathrm{Z}$); this is an affine morphism of k_η-schemes. Since $\mathcal{F}|_{\mathrm{X}^o} \in \mathrm{D}(\mathrm{X}^o)^{\leq -\delta}$ and $d(\mathcal{F}|_{\mathrm{X}^o}) \leq d(\mathcal{F}) - \delta$, one has $f^o_*(\mathcal{F}|_{\mathrm{X}^o})[-\delta] \in \mathrm{D}(\mathrm{Y}^o)^{\leq 0}$ by the induction assumption applied to f^o, k_η, and $\mathcal{F}|_{\mathrm{X}^o}[-\delta]$. Now $(f_*\mathcal{F})|_{\mathrm{Y}^o} = f^o_*(\mathcal{F}|_{\mathrm{X}^o})$, hence $(f_*\mathcal{F})_\eta[-\delta] = f^o_*(\mathcal{F}|_{\mathrm{X}^o})_\eta[-\delta]$ is connective, q.e.d.

(ii) *The case when f is an open embedding $j : \mathrm{X} \hookrightarrow \overline{\mathrm{X}}$ with $\mathrm{Q} := \overline{\mathrm{X}} - \mathrm{X}$ a Cartier divisor:* We can assume that y as in (i) lies in Q which is a principal divisor $h = 0$. Let K be the field of fractions of the henselian local ring at $0 \in \mathbf{A}^1_k$, $\tilde{\mathrm{K}}$ its separable closure, $\mathrm{G} := \mathrm{Gal}(\tilde{\mathrm{K}}/\mathrm{K})$, and $\Psi = \Psi_h : \mathrm{D}(\mathrm{X}) \to \mathrm{D}(\mathrm{Q})$ be the nearby cycles functor. One has $(j_*\mathcal{F})_y = \mathrm{R}\Gamma(\mathrm{G}, \Psi(\mathcal{F})_y)$, so, since G has cohomological dimension 1, it is enough to check that $\Psi(\mathcal{F})_y[-1]$ is connective. By definition, $\Psi(\mathcal{F})_y$ is inductive limit of complexes $\mathrm{R}\Gamma(\mathrm{U}_{\tilde{\mathrm{K}}}, \mathcal{F}_{\mathrm{U}_{\tilde{\mathrm{K}}}})$ where $\mathrm{U}/\overline{\mathrm{X}}$ runs the category of affine étale neighborhoods of y, $\mathrm{U}_{\tilde{\mathrm{K}}} := \mathrm{U} \times_{\mathbf{A}^1} \operatorname{Spec} \tilde{\mathrm{K}}$, $\mathcal{F}_{\mathrm{U}_{\tilde{\mathrm{K}}}}$ is the pullback of $\mathcal{F}$ by the map $\mathrm{U}_{\tilde{\mathrm{K}}} \to \mathrm{X}$. Since $\mathcal{F}_{\mathrm{U}_{\tilde{\mathrm{K}}}} \in \mathrm{D}(\mathrm{U}_{\tilde{\mathrm{K}}})^{\leq -1}$ and $\mathrm{D}(\mathcal{F}_{\mathrm{U}_{\tilde{\mathrm{K}}}}) < d(\mathcal{F})$, each complex $\mathrm{R}\Gamma(\mathrm{U}_{\tilde{\mathrm{K}}}, \mathcal{F}_{\mathrm{U}_{\tilde{\mathrm{K}}}})[-1]$ is connective by the induction assumption applied to the affine map $\mathrm{U}_{\tilde{\mathrm{K}}} \to \operatorname{Spec} \tilde{\mathrm{K}}$, and so $\Psi(\mathcal{F})_y[-1]$ is connective.

(iii) *The case when f is the projection $p : \mathrm{X} = \mathbf{A}^1 \times \mathrm{Y} \to \mathrm{Y}$:* For y as in (i) consider the complementary embeddings $i_y : \mathrm{X}_y = \mathbf{A}^1_y \hookrightarrow \mathrm{X}$, $j_y : \mathrm{X} - \mathrm{X}_y \hookrightarrow \mathrm{X}$. Applying $p_*(-)_y$ to the exact triangle $j_{y!}j_y^*\mathcal{F} \to \mathcal{F} \to i_{y*}i_y^*\mathcal{F}$ we see that it is enough to show that $(p_*i_{y*}i_y^*\mathcal{F})_y$ and $(p_*j_{y!}j_y^*\mathcal{F})_y$ are connective.

$(p_*i_{y*}i_y^*\mathcal{F})_y$ is connective: One has $(p_*i_{y*}i_y^*\mathcal{F})_y = \mathrm{R}\Gamma(\mathbf{A}^1_k, i_y^*\mathcal{F})$, so it is enough to check that for every successive quotient of the (usual) canonical filtration on $i_y^*\mathcal{F}$ the complex $\mathrm{R}\Gamma(\mathbf{A}^1_k, \mathrm{Gr}_n\, i_y^*\mathcal{F})$ is connective. Since $\mathrm{Gr}_{>0}\,\mathcal{F} = 0$ and $\mathrm{Gr}_0\,\mathcal{F}$ is supported at finitely many points, we are reduced to the claim that for a (usual) sheaf $\mathcal{G}$ on $\mathbf{A}^1_k$ one has $\mathrm{H}^{>1}(\mathbf{A}^1_k, \mathcal{G}) = 0$ which is SGA 4 IX 5.7.

$(p_* j_{y!} j_y^* \mathscr{F})_y$ *is connective:* One has $\mathscr{G} := j_{y!} j_y^* \mathscr{F} \in \mathrm{D}(\mathrm{X})^{\geq 0}$. Consider the open embedding $j : \mathrm{X} = (\mathbf{P}^1 - \{\infty\}) \times \mathrm{Y} \hookrightarrow \overline{\mathrm{X}} := \mathbf{P}^1 \times \mathrm{Y}$. Let $\overline{p} : \overline{\mathrm{X}} \to \mathrm{Y}$ be the projection, and $\bar{i}_y : \mathbf{P}^1_k \to \overline{\mathrm{X}}$ be the embedding $\bar{i}_y(a) = (a, y)$. Then $p = \overline{p} j$, $(p_*\mathscr{G})_y = (\overline{p}_* j_* \mathscr{G})_y = \mathrm{R}\Gamma(\mathbf{P}^1_k, \bar{i}_y^* j_* \mathscr{G})$ be proper base change, and so $(p_*\mathscr{G})_y = (j_*\mathscr{G})_{\bar{i}_y(\infty)}$ since $i_y^* \mathscr{G} = 0$. We are done by (ii) applied to j and $\mathscr{G}$.

(iv) *The general case:* It is enough to write f as a composition $f = f_n f_{n-1} \dots f_0$ of affine maps f_i such that our claim is true for each f_i (indeed, the sheaves $\mathscr{F}_i := (f_i f_{i-1} \dots f_0)_* \mathscr{F}$ satisfy $d(\mathscr{F}_i) \leq d(\mathscr{F})$, and so $\mathscr{F}_i = f_{i*} \mathscr{F}_{i-1} \in \mathrm{D}^{\leq 0}$ by induction by i). Now locally on Y we can factor f as composition $\mathrm{X} \hookrightarrow \mathbf{A}^n \times \mathrm{Y} \to \mathrm{Y}$ where $\hookrightarrow$ is a closed embedding and $\to$ is the projection. Thus $f = f_n f_{n-1} \dots f_0$ where f_0 is $\hookrightarrow$ and f_i is the projection $\mathbf{A}^{n-i+1} \times \mathrm{Y} \to \mathbf{A}^{n-i} \times \mathrm{Y}$ for $i > 0$. Our claim is true for f_0 since f_{0*} is t-exact and for f_i, $i > 0$, by (iii), and we are done. $\square$

Constructing retractions. In the above, η is used simultaneously for a point of the topological space of the scheme Y and for a geometric point centered on this (scheme-theoretic) point. Let $\overline{\eta}$ be a geometric point of Y centered on a point η of Y and $\mathrm{Z} = \overline{\{\eta\}}$. As k is perfect, the smooth locus of Z is nonempty, so we may assume that Z factors as $\mathrm{Z} \xrightarrow{h} \mathbf{A}^\delta \to \operatorname{Spec} k$ for h étale. Suppose $\mathrm{Y} = \operatorname{Spec} \mathrm{A}$, $\mathrm{Z} = \operatorname{Spec} \mathrm{B}$ and form the pullback

$$\begin{tikzcd}
\mathrm{A}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & k[x_1, \dots, x_\delta] \arrow[d] \\
\mathrm{A} \arrow[r] & \mathrm{B}
\end{tikzcd}$$

of rings; it is easy to find a retract $k[x_1, \dots, x_\delta] \to \mathrm{A}'$. Let $\mathrm{Y}' = \operatorname{Spec} \mathrm{A}'$.

$$\begin{tikzcd}
\mathrm{Z} \arrow[r, hook] \arrow[d, "h"] & \mathrm{Y} \arrow[d] \\
\mathbf{A}^\delta \arrow[r, hook] & \mathrm{Y}' \arrow[l, dashed, bend right]
\end{tikzcd}$$

The map $\mathrm{Z} \times_{\mathbf{A}^\delta} \mathrm{Y} \to \mathrm{Y}$ is étale and the base change by $\mathrm{Z} \to \mathrm{Y}$ is given by $\mathrm{Z} \times_{\mathbf{A}^\delta} \mathrm{Z} \to \mathrm{Z}$, which admits the diagonal as section. As h is étale, this diagonal is is an isomorphism onto a connected component of $\mathrm{Z} \times_{\mathbf{A}^\delta} \mathrm{Z}$ (SGA 1 I 9.3) and we identify Z with this component. Let U denote $\mathrm{Z} \times_{\mathbf{A}^\delta} \mathrm{Y}$ minus the closed subscheme $\mathrm{Z} \times_{\mathbf{A}^\delta} \mathrm{Z} - \mathrm{Z}$; U is an étale neighborhood of $\overline{\eta}$ and is equipped with a retract $U \to Z$, namely the one

$$Z \hookrightarrow U = Z \times_{\mathbf{A}^\delta} Y - (Z \times_{\mathbf{A}_\delta} Z - Z) \to Z$$

induced by the first projection $Z \times_{\mathbf{A}^\delta} Y \to Z$.

4.1.7. The Čech spectral sequence is also called the Cartan-Leray spectral sequence and its existence in an arbitrary topos is established in SGAA, Exp. V 3.3.

4.1.8. The only thing worth mentioning is that the entire second paragraph is implicitly local to U_i. After all, on U_i we have that $\tau_{\leq -i}K$ is in ${}^pD^{\leq 0}$, and to show that $K|U_i$ is in ${}^pD^{\leq 0}$, it suffices to show that $H^q(\tau_{>-i}K)|U_i$ is 0 for all q (i.e. for all $q > -i$). Proceeding by descending induction on $q > -i$, the induction step consists of showing that $H^i(V \cap W_i, \mathscr{L}) = 0$ for all affine open V implies that $\mathscr{L} = 0$.

4.2.5. As f^* and $R\mathscr{H}\!om$ commute with reduction modulo ℓ^n, it is enough to prove the statement in the category $D^b_{ctf}(X, \mathbf{Z}/\ell^n)$, and the equality $\Gamma H^0 R\mathscr{H}\!om = H^0 R\Gamma R\mathscr{H}\!om$ holds in $D(X, \mathbf{Z}/\ell^n)$ because, as $R\mathscr{H}\!om$ is in $D^{\geq 0}_c$, $H^0 = \tau_{\leq 0} \ker d_0$, and Γ commutes with the formation of kernels. For (4.2.5.3), the retraction $\mathscr{H} \to {}^\circ f_* f^*\mathscr{H}$ is ${}^\circ f_*(\eta(f^*\mathscr{H})$ where η is the unit $\mathrm{id} \to {}^\circ e_* e^*$:

$$\mathscr{H} \to {}^\circ f_* f^*\mathscr{H} \to {}^\circ f_* {}^\circ e_* e^* f^* \mathscr{H} = \mathscr{H},$$

and it remains to check that the first arrow is surjective. The references in the rest of this paragraph are to SGAA Exposé XV. By, (1.1), it suffices to show that $H^0(Y', \mathscr{H}) \to H^0(X', f'^*\mathscr{H})$ is bijective for each $Y' \to Y$ étale. Replacing Y by Y′, we know by (1.5) that f is (-1)-acyclic; i.e. that $\alpha : H^0(Y, \mathscr{H}) \to H^0(X, f^*\mathscr{H})$ is injective, as f is surjective. Moreover, f is locally acyclic as it is smooth. Then, (1.16) shows that α is surjective iff for every geometric point $\overline{y}$ of Y algebraic over a point y of Y, $\overline{\alpha} : H^0(\overline{y}, \mathscr{H}_{\overline{y}}) \to H^0(X_{\overline{y}}, f^*\mathscr{H}|X_{\overline{y}})$ is, so we may assume Y is the spectrum of an algebraically closed field, in which case X is connected, as the fibers of f were assumed geometrically connected. Then α is seen to be bijective by another application of (1.1), which reduces the matter to the corresponding question for a constant sheaf. Note that the existence of a retraction discussed above is irrelevant to this argument.

4.2.6. This paragraph, as written, is nonsense. The correct (equivalent) statements are

(a) u^* identifies $\mathscr{A}$ with a subcategory of $\mathscr{B}$ closed under subquotients.
(b) the unit of adjunction $\eta_! : \mathrm{id}_{\mathscr{B}} \to u^* u_!$ is a natural epimorphism;
(b′) the counit of adjunction $\varepsilon_* : u^* u_* \to \mathrm{id}_{\mathscr{B}}$ is a natural monomorphism.

Note that statement (a) is different from $\mathscr{A}$ being épaisse, as an épaisse subcategory is also closed under extensions. A general remark: an adjunction $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ with unit η and counit ε is called idempotent if it can be factored as a reflection and a coreflection, in which case many things are true; see the nLab page 'idempotent adjunction.' In particular, εF and ηG are natural isomorphisms. In our situation, both $u_! \dashv u^*$ and $u^* \dashv u_*$ are idempotent adjunctions as they are reflective and coreflective, respectively, so we get that $\varepsilon_* u^*$ and $\eta_! u^*$ are natural isomorphisms. Of course, $u_!$ is right exact and u_* is left exact. We prove (a)⇔(b′); dual arguments give (a)⇔(b).

To prove (a)⇒(b′), let B be an object of $\mathscr{B}$. In the commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \ker \varepsilon_*(B) \arrow[r] & u^* u_* B \arrow[r] & B \\
0 \arrow[r] & u^* u_* \ker \varepsilon_*(B) \arrow[r] \arrow[u, "\alpha"] & u^* u_* u^* u_* B \arrow[r, "\gamma"] \arrow[u, "\beta"] & u^* u_* B \arrow[u]
\end{tikzcd}$$

$\ker \varepsilon_*(B)$ is in the essential image of $\mathscr{A}$ as $\mathscr{A}$ is closed under subobjects. Therefore α, β, and γ are isomorphisms, which shows $u^* u_* \ker \varepsilon_*(B) = 0$ and therefore $\ker \varepsilon_*(B) = 0$.

Now let's prove (b′)⇒(a). In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & u^* B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & u^* u_* A \arrow[r] \arrow[u] & u^* u_* u^* B \arrow[r] \arrow[u] & u^* u_* C \arrow[u] &
\end{tikzcd}$$

with exact rows, the middle arrow is an isomorphism and the outer arrows are monomorphisms. By the four-lemma, the first arrow is also an epimorphism, hence an isomorphism, identifying A in the essential image of $\mathscr{A}$. Therefore this essential image is closed under subobjects and hence also under subquotients, as u^* is exact.

Identifying $\mathscr{A}$ with its essential image, a full subcategory of $\mathscr{B}$, every object B of $\mathscr{B}$ has a largest subobject in $\mathscr{A}$, viz. $u^*u_*\mathrm{B}$, and a largest quotient in $\mathscr{A}$, viz. $u^*u_!\mathrm{B}$. To see this, simply observe that both candidates are indeed in $\mathscr{A}$, and if A is in $\mathscr{A}$ and a subobject of $\mathscr{B}$, then $\mathrm{A} \simeq u^*u_*\mathrm{A} \hookrightarrow u^*u_*\mathrm{B} \hookrightarrow \mathrm{B}$, and dually.

The example adjunction is backwards: $u_!(\mathrm{X} \to \mathrm{Y}) = (\mathrm{Y} \xrightarrow{\sim} \mathrm{Y})$ is the left adjoint, $u_*(\mathrm{X} \to \mathrm{Y}) = (\mathrm{X} \xrightarrow{\sim} \mathrm{X})$ is the right adjoint, since the diagrams

$$\begin{tikzcd} \mathrm{A} \arrow[r] & \mathrm{B} \arrow[d] \\ \mathrm{C} \arrow[r, "\sim"] & \mathrm{D} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathrm{A} \arrow[r, "\sim"] \arrow[d] & \mathrm{B} \\ \mathrm{C} \arrow[r] & \mathrm{D} \end{tikzcd}$$

have unique completions to commutative squares. Then $\varepsilon_* : u^*u_* \to \mathrm{id}_{\mathscr{B}}$ needn't be a natural monomorphism, as a commutative diagram of the sort

$$\begin{tikzcd} \mathrm{A} \arrow[r] \arrow[d] & \mathrm{B} \arrow[d] \\ \mathrm{C} \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"] & \mathrm{C} \arrow[d] \\ \mathrm{C} \arrow[r] & \mathrm{D} \end{tikzcd}$$

determines the morphism $\mathrm{A} \to \mathrm{C}$ but isn't enough to determine the morphism $\mathrm{B} \to \mathrm{D}$.

4.2.6.1. The Jordan-Hölder theorem holds in any abelian category, and since u^* is exact and preserves simple objects, identifying $\mathscr{A}$ with its essential image, it follows that the components of any A in $\mathscr{A}$ also belong to $\mathscr{A}$. To show that $\mathscr{A}$ is closed under subobjects, and therefore subquotients, it will suffice to show that if $0 \to \mathrm{B} \to \mathrm{A} \to \mathrm{S} \to 0$ with A in $\mathscr{A}$, B in $\mathscr{B}$, and S simple, then B is in $\mathscr{A}$. Since S is simple, it is a component of A and therefore in $\mathscr{A}$. The five-lemma, applied to the diagram with exact rows obtained by applying ε_*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{B} \arrow[r] & \mathrm{A} \arrow[r] & \mathrm{S} \\ 0 \arrow[r] & u^*u_*\mathrm{B} \arrow[r] \arrow[u] & u^*u_*\mathrm{A} \arrow[r] \arrow[u, "\sim"] & u^*u_*\mathrm{S}, \arrow[u, "\sim"] \end{tikzcd}$$

shows that indeed $u^*u_*\mathrm{B} \xrightarrow{\sim} \mathrm{B}$, and B is in $\mathscr{A}$.

4.2.6.2. Notes on the proof are below; the proof depends on the middle perversity insofar as the commutativity of intermediate extension with $f^*[d]$ relies on the relative dimension coinciding with the change in perversity between an irreducible component and its inverse image by f. The proof doesn't work for $\mathbf{Z}_\ell$-sheaves as it relies on 4.3.1 which fails for $\mathbf{Z}_\ell$-cohomology.

Commutation of $f^*[d]$ with intermediate extension. We wish to show that $f^*[d]j_{*!} = j_{*!}f^*[d]$. The transitivity of $j_{*!}$ for j the inclusion of a locally closed subset allows us to factorize the inclusion of V as the open immersion $\mathrm{V} \hookrightarrow \overline{\mathrm{V}}$ followed by the closed immersion $\overline{\mathrm{V}} \hookrightarrow \mathrm{Y}$; the latter posing no problem as in this case lower shriek, lower star, and intermediate extension all coincide and commute with $f^*[d]$ by smooth base change. Reduced to the case of open immersion $\mathrm{Y} = \overline{\mathrm{V}}$, let m be minimal such that $\mathrm{F} = \mathrm{Y} - \mathrm{V}$ is of dimension $\leq m$, and put $t := p(2m)$. We can find a stratification of F into strata satisfying 2.2.10 (a) so that for A in $\mathrm{D}^b_c(\mathrm{V}, \mathrm{R}/m^n)$, the $\mathrm{H}^i j_*\mathrm{A}$ are locally constant on each stratum for $i \geq t$. Let U_n (resp. F_n) denote the union of strata S in this stratification of F satisfying $p(\mathrm{S}) \leq n$ (resp. $p(\mathrm{S}) \geq n$). For such an S, $\overline{\mathrm{S}} - \mathrm{S}$ is a union of strata of dimension strictly less than dim S. Therefore each S in the stratification of $\mathrm{U}_n - \mathrm{U}_{n-1}$ is closed in U_n. By transitivity of the intermediate extension, it will suffice to extend A from V to $\mathrm{V} \cup \mathrm{U}_t$, as the complement of the latter in Y is F_{t+1}, closed; $\mathrm{V} \cup \mathrm{U}_t - \mathrm{V} = \mathrm{U}_t$ is also closed in $\mathrm{V} \cup \mathrm{U}_t$, and nonempty. U_t is a disjoint union of equidimensional strata smooth over $\overline{k}$ (hence by the note to 2.2.10, the étale topology sees them as smooth), so that the $\mathrm{H}^i j_*\mathrm{A}$ are locally constant on U_t for $i \geq t$. We have reduced to the setting of 2.2.4: $\mathrm{Y} = \overline{\mathrm{V}}$, $\mathrm{F} = \mathrm{Y} - \mathrm{V}$ smooth of dimension m, $\mathrm{H}^i j_*\mathrm{A}$ are locally constant on F for $i \geq t := p(2m)$, and $j_{!*}\mathrm{A} = \tau^{\mathrm{F}}_{\leq t-1} j_*\mathrm{A}$. As f is smooth of relative dimension d, $f^{-1}\mathrm{F}$ is also equidimensional, now of dimension $m + d$. As pullbacks of locally constant sheaves, the $\mathrm{H}^i j_* f^*\mathrm{A}$ are still locally constant on $f^{-1}\mathrm{F}$. If $t' = p(2(d+m)) = -d - m$, then shifting by d and applying $\tau^{\mathrm{F}}_{<t'}$ is the same as applying $\tau^{\mathrm{F}}_{<t}$ and then shifting by d. It only remains to verify that f^* commutes with $\tau^{\mathrm{F}}_{\leq 0}$, but this is easy: letting $i : \mathrm{F} \hookrightarrow \mathrm{Y}$ be the closed immersion, $(f^*\tau^{\mathrm{F}}_{\leq 0}\mathrm{A}, f^*\mathrm{A}, f^* i_* \tau_{>0} i^*\mathrm{A})$ defines $f^*\tau^{\mathrm{F}}_{\leq 0}\mathrm{A}$, but $f^* i_* \tau_{>0} i^* \mathrm{A} = i_* \tau_{>0} i^* f^* \mathrm{A}$ (notating i as usual for the base extension as well), hence also defines $\tau^{\mathrm{F}}_{\leq 0} f^*\mathrm{A}$.

Irreducibility of the inverse image of an irreducible local system on an irreducible scheme by a smooth morphism with geometrically connected fibers. SGA 1, Exp. IX, 5.6 shows that if $f : \mathrm{S}' \to \mathrm{S}$ is universally submersive (e.g. faithfully flat and quasi-compact) with geometrically connected fibers, and S is connected, then S′ is connected, and, choosing a geometric point s' of S′ and letting s be the image of s' in S, the homomorphism $\pi_1(\mathrm{S}', s') \to \pi_1(\mathrm{S}, s)$ is surjective. This immediately implies that if $\mathscr{L}$ is an irreducible local system on S, $f^*\mathscr{L}$ is an irreducible local system on S′. Of course, smooth with geometrically connected fibers means smooth with geometrically irreducible fibers as these fibers are themselves smooth (SGA 1, Exp II, 2.1) and therefore regular. It is a topological fact (Stacks `004Z`) that if Y is irreducible and $f : \mathrm{X} \to \mathrm{Y}$ is open with irreducible fibers, then X is irreducible. This verifies that the if V is irreducible and f is smooth with connected fibers, the inverse image on $f^{-1}\mathrm{V}$ of an irreducible local system on V is again irreducible.

4.2.7. Let $\mathrm{X}', \mathrm{X}, \mathrm{Y}$ be of finite type over a field k and $f : \mathrm{X}' \to \mathrm{X}$. We wish to show that $\boxtimes$ commutes with direct image; i.e. that the below arrow is an isomorphism.

$$f_*\mathrm{K} \boxtimes \mathrm{L} \to (f \times \mathrm{id})_*(\mathrm{K} \boxtimes \mathrm{L})$$

We will compute locally about a geometric point $\xi := (x, y) \to \mathrm{X} \times \mathrm{Y}$ (x, y geometric points of X, Y respectively), so that all the objects in $\mathrm{X}_x \leftarrow (\mathrm{X} \times \mathrm{Y})_\xi \to \mathrm{Y}_y$ are spectra of strictly henselian local rings. Let $t \to \mathrm{X}_x$ be a geometric point centered on the generic point of an irreducible component of X_x. *Th. finitude* 2.16 gives that $\mathrm{Y} \to \operatorname{Spec} k$ is universally locally acyclic, so for any M in $\mathrm{D}^+(\mathrm{X} \times \mathrm{Y}, \mathbf{Z}/\ell)$,

$$\Gamma((\mathrm{X} \times \mathrm{Y})_\xi, \mathrm{M}) = \Gamma((\mathrm{X} \times \mathrm{Y})_{\xi,t}, \mathrm{M}),$$

where $(\mathrm{X} \times \mathrm{Y})_{\xi,t}$ denotes the geometric fiber in t of $(\mathrm{X} \times \mathrm{Y})_\xi \to \mathrm{Y}_y$. To pass from X_x to $(\mathrm{X}_x)_t = t$ we can first pass to the limit of Zariski neighborhoods of t, which is the spectrum of an artinian local ring, then kill nilpotents and extend scalars. As lower star commutes with smooth base change and the étale topology doesn't see nilpotents, we may therefore assume $\mathrm{X} = t$, the spectrum of a separably closed extension $k(t)$ of k, and $\mathrm{Y} = \mathrm{Y}_y$. As $\mathrm{X}_x \to \operatorname{Spec} \overline{k} \leftarrow \mathrm{Y}_y$, $\operatorname{Spec} \overline{k} \times_k \mathrm{Y}_y$ is the disjoint union of copies of Y_y, and

ξ picks one of them; i.e. $(\operatorname{Spec}\overline{k}\times_k \mathrm{Y}_y)_\xi = \mathrm{Y}_y$. So we may assume $\mathrm{Y}_y \to \overline{k}$, and write

$$\Gamma((\mathrm{X}\times\mathrm{Y})_{\xi,t},\mathrm{M}) = \Gamma(\mathrm{Y}_y \times_{\overline{k}} k',\mathrm{M}) = \Gamma(\mathrm{Y}_y,\mathrm{M}),$$

where the second equality is Arcata V 3.3. We are reduced to $k=\overline{k}$, $\mathrm{X}=\overline{k}$, $\mathrm{Y}=\mathrm{Y}_y$, in which case the formula is

$$\Gamma(\mathrm{X}',\mathrm{K})\otimes\Gamma(\mathrm{Y},\mathrm{L}) \to \Gamma(\mathrm{X}'\times\mathrm{Y},\mathrm{K}\boxtimes\mathrm{L}),$$

which is obtained from the Künneth formula of *Th. finitude* 1.11 by smooth base change and passage to the limit along $\mathrm{Y}_y \to \mathrm{Y}$.

4.2.8. It would appear that $\boxtimes$ is only right t-exact in $\mathbf{Z}_\ell$-cohomology, due to the possible appearance of Tor.

4.3.1. This is a theorem in $\mathbf{Q}_\ell$-cohomology and not in $\mathbf{Z}_\ell$-cohomology because the latter has few irreducible objects; in particular, the category of lisse $\mathbf{Z}_\ell$-sheaves is not artinian; the irreducible lisse $\mathbf{Z}_\ell$ sheaves are torsion. . . .

4.3.3. 'la monodromie de $\mathscr{L}$ ne change pas parréstriction à U' $\rightsquigarrow$ [**SGA1**, I 10.3].

4.3.4. The dévissage of the perverse sheaf $\mathscr{F}$ should occur over an irreducible affine smooth open so that we can apply the results of 4.1.10–4.1.12. The sequence 4.1.10.1 has outer terms supported on $\mathrm{X}-\mathrm{U}$ and so reduces the problem for $\mathscr{F}$ to that for $j_!j^*\mathscr{F} = j_!\mathscr{L}[\dim \mathrm{U}]$; 4.1.12.3 then reduces the problem for $j_!\mathscr{L}[\dim\mathrm{U}]$ to that for $j_{!*}\mathscr{L}[\dim\mathrm{U}]$. This allows us to proceed by induction on the length of $\mathscr{L}$. Let $\mathscr{L}'\subset\mathscr{L}$ be a simple lisse subsheaf. It suffices to remark that the kernel of $j_{*!}\mathscr{L}'[\dim\mathrm{U}] \to j_{*!}\mathscr{L}[\dim\mathrm{U}]$ is supported on $\mathrm{X}-\mathrm{U}$, and the restriction of the cokernel to U has strictly lesser length than that of $\mathscr{L}'$.

4.4. $\rightsquigarrow$ note to Appendix A.

4.5.1. The formula for E_2^{pq} is direct from

$$\mathrm{H}^0_c(\tilde{\mathrm{U}}_1,\mathscr{L}) = \mathrm{H}^2(\tilde{\mathrm{U}}_1,\mathscr{L}^\vee(1))^\vee = ((\mathscr{L}^\vee(1))^{\pi_1(\tilde{\mathrm{U}}_1,\tilde{t})})^\vee = (\mathscr{L}(-1))_{\pi_1(\tilde{\mathrm{U}}_1,\tilde{t})},$$

where the first equality is Poincaré duality.

The multiplicativity of χ in d_1 follows from the Euler-Poincaré formula by the corresponding statement for the constant sheaf (which follows from Hurwitz's formula) and the fact that the number of points of the fiber above each $x \in \mathrm{X}_1 - \mathrm{U}_1$ is equal to d_1.

The equality $\mathrm{H}^q(\tilde{\mathrm{Y}}, \mathrm{K}_1) = \mathrm{H}^q(\tilde{\mathrm{Y}}, \Psi_{\overline{\eta}}\mathrm{K}_1)$ follows from the fact that [**SGA 7**, XIII, 2.1.8.3] is an isomorphism as $\tilde{\mathrm{pr}}$ is proper.

The second-to-last displayed equation is the definition of the Swan conductor [**SGA5**, X, 6.2.1]. One can understand the last displayed equation as an equality of constructible sheaves of modules over the torsion ring $\mathbf{Z}/\ell[\mathrm{Q}]$. In particular, this is a statement in a ringed topos where the ring is noncommutative. This poses no problem: $\mathscr{H}\!om$ still sends injective sheaves to (flasque) sheaves acyclic for $\mathrm{R}\Gamma$ [**SGAA**, V, 6.1 (3)], so that in particular $\mathrm{R}\Gamma\,\mathrm{R}\mathscr{H}\!om = \mathrm{R}\,\mathrm{Hom}$, and (as Sw is projective) we have:

$$\begin{aligned}\mathrm{H}^{q+1}(\tilde{\mathrm{Y}}, \mathscr{H}\!om_{\mathrm{Q}}(\mathrm{Sw}, \Psi_{\overline{\eta}}\mathrm{K}_1[-1])) &= \mathrm{H}^{q+1}\,\mathrm{Hom}_{\mathrm{Q}}(\mathrm{Sw}, \mathrm{R}\Gamma(\tilde{\mathrm{Y}}, \Psi_{\overline{\eta}}\mathrm{K}_1[-1]))\\ &= \mathrm{Hom}_{\mathrm{Q}}(\mathrm{Sw}, \mathrm{H}^{q+1}(\tilde{\mathrm{Y}}, \Psi_{\overline{\eta}}\mathrm{K}_1[-1])).\end{aligned}$$

5.1.1. Recall the commutative diagram defining relative Frobenius (1.3.1) and that

$$\begin{tikzcd}
{[\mathscr{F}]} \arrow[r, "\mathrm{Fr}_q"] \arrow[d] & {[\mathscr{F}]} \arrow[d] \\
\mathrm{X} \arrow[r, "\mathrm{Fr}_q"] & \mathrm{X}
\end{tikzcd}$$

is in fact cartesian, which defines the Frobenius correspondence $\mathrm{F}_q^* : \mathrm{Fr}_q^*\,\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ as the inverse of the universal arrow; i.e. if $g : [\mathscr{F}] \to \mathrm{X}$ is the structure morphism,

$$\begin{tikzcd}
{[\mathscr{F}]} \arrow[drr, dashed, bend left, "\mathrm{Fr}_q"] \arrow[ddr, bend right, "g"'] & & \\
& \mathrm{Fr}_q^*\,[\mathscr{F}] \arrow[ul, "\mathrm{F}_q^*"', "\sim"] \arrow[r, "\pi_{\mathrm{X}_0/\mathrm{S}}"'] \arrow[d, "g^{(q)}"] & {[\mathscr{F}]} \arrow[d, "g"] \\
& \mathrm{X} \arrow[r, "\mathrm{Fr}_q"] & \mathrm{X}
\end{tikzcd}$$

commutes.

As Fr_q is integral, radicial, and surjective, the natural morphisms $\mathrm{Fr}_q^* \mathrm{Fr}_{q*} \to \mathrm{id} \to \mathrm{Fr}_{q*} \mathrm{Fr}_q^*$ are isomorphisms; this implies that for each $x \in \mathrm{X}$, the base change morphism [**SGAA**, XVIII 3.1.14.2]

$$\mathrm{Fr}_q^* \, i^!_{\mathrm{Fr}_q(x)} \to i^!_x \, \mathrm{Fr}_q^*$$

is an isomorphism. As Fr_q is a universal homeomorphism, this implies that Fr_q^* is left t-exact (2.2.12); as it is right t-exact (2.2.5), it is t-exact. As X is a variety, Fr_q is actually finite and $\mathrm{Fr}_{q*} = \mathrm{Fr}_{q!}$ is t-exact (2.2.5).

5.1.2. I find the proof of this proposition remarkable. The morphisms in the target category are those $f : \mathcal{F} \to \mathcal{G}$ making commutative squares

$$\begin{tikzcd}
\mathrm{Fr}_q^* \mathcal{F} \arrow[r, "\mathrm{Fr}_q^*(f)"] \arrow[d, "\mathrm{F}_q^*"', "\wr"] & \mathrm{Fr}_q^* \mathcal{G} \arrow[d, "\mathrm{F}_q^*"', "\wr"] \\
\mathcal{F} \arrow[r, "f"] & \mathcal{G}.
\end{tikzcd}$$

As F (i.e. F_q^*, recall *Rapport* 1.8) acts on $\mathrm{Hom}(\mathrm{K}, \mathrm{L})$ by conjugation $f \mapsto \mathrm{F} f \mathrm{F}^{-1}$, this is the same as the morphism f being fixed by F.

In the paragraph discussing Ext^1 in $\mathrm{M}(\mathrm{X})$, clearly an extension of type $\left(\mathcal{G} \oplus \mathcal{F}, \left(\begin{smallmatrix} \phi & \mathrm{U}\phi \\ 0 & \phi \end{smallmatrix}\right)\right)$ in the kernel of the 'forget ϕ' map on Ext^1. Such an extension is already trivial in the ϕ-category if it splits; i.e. if I can write down a map $\mathcal{G} \oplus \mathcal{F} \to \mathcal{G}$ of the form (id, f) compatible with $\left(\begin{smallmatrix} \phi & \mathrm{U}\phi \\ 0 & \phi \end{smallmatrix}\right)$; i.e. if I can find a f to make the diagram

$$\begin{tikzcd}
\mathrm{Fr}_q^* \mathcal{G} \oplus \mathrm{Fr}_q^* \mathcal{F} \arrow[r, dashed, "(\mathrm{id}, \mathrm{Fr}_q^*(f))"] \arrow[d, "\left(\begin{smallmatrix} \phi & \mathrm{U}\phi \\ 0 & \phi \end{smallmatrix}\right)"', "\wr"] & \mathrm{Fr}_q^* \mathcal{G} \arrow[d, "\phi"', "\wr"] \\
\mathcal{G} \oplus \mathcal{F} \arrow[r, "(\mathrm{id}, f)"] & \mathcal{G}
\end{tikzcd}$$

commute; i.e. if there is a $f : \mathcal{F} \to \mathcal{G}$ such that $\mathrm{U}\phi + f\phi = \phi \, \mathrm{Fr}_q^*(f)$; i.e. if $\mathrm{U} = \phi \, \mathrm{Fr}_q^*(f) \phi^{-1} - f$ for some $f : \mathcal{F} \to \mathcal{G}$.

5.1.3. The point is that in our case $\mathcal{P}(\mathrm{Y}_0)$ (a priori a subcategory of the category of pairs) is equivalent to the category of perverse sheaves on Y_0, so that conditions (a) and (b) are obvious for this subcategory.

In the following paragraph, 'sheaf' means 'perverse sheaf for perversity 0'; i.e. a complex concentrated in degree zero. Transposed into the language of [**WeilII**], condition (c) reads: 'every lisse subquotient in the category of Weil sheaves of a lisse étale sheaf is again a lisse étale sheaf.' The point is that in the language of [**SGA 7**, XIII 1.1], the category of pairs $(\mathcal{F}, \phi)$ where $\mathcal{F}$ is a sheaf (concentrated in degree zero) and $\phi : \mathrm{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is the same as the category of sheaves on X with action of $\mathrm{Gal}(\mathbf{F}, \mathbf{F}_q)$ compatible with the action of Galois on X.[3] In any event, we can ask that the action of Galois be not only compatible, but also *continuous,* and this is the distinction between Weil sheaves and étale sheaves [**WeilII**, 1.1.11], as [**SGA 7**, XIII 1.1.3] shows that the category of étale sheaves on X_0 is the same as the category of étale sheaves on X with a compatible, continuous action of Galois. This would mean that $\phi : \mathrm{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ would extend to an action of $\hat{\mathbf{Z}}$. Suppose $\mathcal{F} = \mathbf{Q}_\ell$ (so that $\mathrm{Fr}_q^* \mathcal{F} = \mathbf{Q}_\ell$). Then a choice of ϕ amounts to a choice of element in $\mathbf{Q}_\ell^\times$, but this ϕ extends to an action of $\mathrm{Gal}(\mathbf{F}, \mathbf{F}_q)$ iff it gives a continuous action of $\hat{\mathbf{Z}}$, which is true if the powers of ϕ preserve a $\mathbf{Z}_\ell$ lattice inside $\mathbf{Q}_\ell$. As there is only one, this amounts to asking that the choice of element of $\mathbf{Q}_\ell^\times$ corresponding to ϕ have valuation zero for the ℓ-adic valuation; i.e. be an ℓ-adic unit (element of $\mathbf{Z}_\ell^\times$). This is why (given a geometric point $\bar{x}$ of X) the category of lisse Weil sheaves is equivalent to the category of representations of $W(X_0, \bar{x})$, while the category of lisse étale sheaves is equivalent to the category of representations of $\pi_1(X_0, \bar{x})$ [**WeilII**, 1.1.12]. To see that condition (c) is satisfied for our category of lisse étale sheaves, therefore, one must simply remark that given a representation V_0 of $\pi_1(X_0, \bar{x})$ and a $W(X_0, \bar{x})$-subrepresentation $V_1 \subset V_0$, V_1 is in fact a $\pi_1(X_0, \bar{x})$-subrepresentation. Let $r_?$ equal $\dim V_?$, $? = 0, 1$; let E be a finite extension of $\mathbf{Q}_\ell$ such that the action of π_1 on V_0 factors through $\mathrm{GL}(r_0, E)$ and let R be the ring of integers of E. The action of π_1 on V_0 stabilizes a lattice $L_0 \simeq R^{r_0} \subset V_0$ so that we may identify the image of π_1 in $\mathrm{GL}(V_0)$ with a compact subgroup $G_\pi \subset \mathrm{GL}(r_0, R)$ and W_0 with a dense subgroup $G_W \subset G_\pi$. Finding a basis for V_1 lying in L_0 identifies a free submodule $L_1 \simeq R^{r_1} \subset L_0$ stable

[3]To digress further in that reference for a moment, given a sheaf $\mathcal{F}_0$, its base extension $\mathcal{F}$ carries a compatible action of Galois since the morphism $u(g)_* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is given on $U \to X$ étale as follows: $U \xrightarrow{u(g)} U$ is an isomorphism and the functor $\mathcal{F}$ applied to this morphism gives the isomorphism $\mathcal{F}(U) \xrightarrow{\sim} u(g)_* \mathcal{F}(U)$. That it is compatible over a field k which is not topologically cyclic follows from the fact that $\mathcal{F}$ is a functor.

by G_W such that $L_1 \otimes E = V_1$. The image of $G_\pi \times L_1 \to L_0 \subset V_0$ is compact and the image of $G_W \times L_1$ is dense in it; as the latter is contained in L_1, the former is too (L_1 is compact, hence a closed subgroup of L_0, hence contains the closure of the image of $G_W \times L_1$, which is the image of $G_\pi \times L_1$).

5.1.8. $(D^b_{\leq w})[1] = D^b_{\leq w+1}$, and, as D exchanges $D^b_{\leq -w}$ and $D^b_{\geq w}$, $(D^b_{\geq w})[1] = D^b_{\geq w+1}$.
'$D^b_{\leq w} \cap D^b_{\geq w+1}$ est réduit aux objets nuls' $\rightsquigarrow$ while the morphism

$$\mathrm{Hom}^1(K_0, K_0[-1]) \to \mathrm{Hom}^1(K, K[-1])$$

has kernel $\mathrm{Hom}(K_0, K_0[-1])_F$ by (5.1.2.5), this map takes $\mathrm{id} : K_0 \to K_0$ to $\mathrm{id} : K \to K$. Since, by 5.1.15 (iii), the map is null, $\mathrm{id} : K \to K$ is null, so K is null, so K_0 is null. . . .

5.1.14. (ii) Passing to stalks, one uses the stability of $D^b_{\leq w}$ under $\tau_{\leq i}$ and $\tau_{\geq i}$ to assume that one's complexes are concentrated in one degree, and then the statement is the same as for ordinary sheaves [**Weil II**, 1.2.5 (ii)].

(ii*) If K and L are in $D^b_{\leq w}$ and $D^b_{\geq w'}$, respectively, then DL is of weight $\leq -w'$ and so by (ii) $K \otimes DL$ is of weight $\leq w - w'$. Therefore

$$\begin{aligned} R\mathcal{H}om(K, L) &= R\mathcal{H}om(K, R\mathcal{H}om(DL, K_{X_0})) \\ &= R\mathcal{H}om(K \otimes DL, K_{X_0}) = D(K \otimes DL) \end{aligned}$$

is of weight $\geq w' - w$.

5.2.2. 'les $\mathcal{L}_0$ sont purs' $\rightsquigarrow$ as every sub-quotient of a mixed perverse sheaf is mixed, the sheaves $({}^\circ j_* \mathcal{L}_0)[1]$ are mixed, so the sheaves $\mathcal{L}_0$ are mixed, so the $\mathcal{L}_0$ are pure, since every irreducible lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{L}_0$ on a normal connected $\mathbf{F}_q$-variety is pure. (Following the argument of [**Weil II**, 1.8.11], $\mathcal{L}_0$ admits a filtration with successive quotients pure sheaves, and shrinking the open locus $\mathcal{L}_0$ remains irreducible. Therefore after shrinking enough, the filtration has only one nonzero graded piece: the restriction of $\mathcal{L}_0$. Now apply [**Weil II**, 1.8.10] to conclude that $\mathcal{L}_0$ was pure before restriction.)

$\sum_p \dim \mathrm{H}^{-1}(\mathrm{U}, \mathrm{Gr}_\mathrm{F}^p \mathcal{F}) \leq r \rightsquigarrow$ a graded piece with punctual support has $\dim \mathrm{H}^{-1} = 0$; a graded piece of the form $({}^\circ j_* \mathcal{L})[1]$ has

$$\mathrm{H}^{-1}(\mathrm{U}, ({}^\circ j_* \mathcal{L})[1]) = \mathrm{H}^0(\mathrm{U}, {}^\circ j_* \mathcal{L}) = \mathrm{H}^0(u^{-1}(\mathrm{Y}), \mathcal{L}).$$

The latter group is the global sections of $\mathcal{L}$, which is lisse, on $u^{-1}(\mathrm{Y})$, which is computed by taking invariants of monodromy acting on the stalk of $\mathcal{L}$, and which therefore has dimension less than or equal to the rank of $\mathcal{L}$.

We would like to find a g such that $\dim \mathrm{H}^1(\overline{\mathrm{U}}, {}^\circ k_* u^* \mathcal{L}) > r$. By the Euler-Poincaré formula (Laumon, *Transformation de Fourier* 2.2.1.2)

$$\sum_{i=0}^{2} (-1)^i \dim \mathrm{H}^i(\overline{\mathrm{U}}, {}^\circ k_* u^* \mathcal{L}) = (2-2g) r(\mathcal{L}) - \sum_{x \in |\overline{\mathrm{U}}|} a_x(\mathcal{L}),$$

where the numbers $a_x(\mathcal{L})$ are natural numbers and $r(\mathcal{L})$ is the generic rank of $\mathcal{L}$ (the number r is the sum of the $r(\mathcal{L})$ over all graded pieces). Therefore $\dim \mathrm{H}^1(\overline{\mathrm{U}}, {}^\circ k_* u^* \mathcal{L}) \geq (2g-2) r(\mathcal{L})$ and when $g > 0$, this number is $\geq 2g-2$. If g is chosen so that $2g-2 > r$, we have the desired lower bound.

In the end we have shown that each constituent in a Jordan-Hölder series for $\mathcal{F}$ is pure of weight $\geq w$. Therefore every constituent in such a series for $\mathrm{D}\mathcal{F}$ is pure of weight $\leq -w$, so $\mathrm{D}\mathcal{F}$ is mixed of weight $\leq -w$ and $\mathcal{F}$ is mixed of weight $\geq w$.

5.2.3. This remark is in effect a translation of the spectral sequence

$$\mathrm{E}_1^{pq} = \mathrm{H}^{p+q}(\mathrm{U}, \mathrm{Gr}_\mathrm{F}^p \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{U}, \mathcal{F}).$$

There are only two nonzero diagonals: the diagonals where $p+q = -1$ or 0. Considered as a presheaf of spectral sequences, the E_1-terms on the $p+q=-1$ diagonal are in the subcategory $\mathcal{N}$. The spectral sequence degenerates on E_2 so that the cokernel of $\mathrm{H}^{-1}(\mathrm{U}, \mathrm{Gr}_\mathrm{F}^{p-1} \mathcal{F}) \xrightarrow{d_1} \mathrm{H}^0(\mathrm{U}, \mathrm{Gr}_\mathrm{F}^p \mathcal{F})$ coincides with $\mathrm{E}_2^{p,-p} = \mathrm{E}_\infty^{p,-p} = \mathrm{Gr}_\mathrm{F}^p \mathrm{H}^0(\mathrm{U}, \mathcal{F})$ for the filtration induced on cohomology. The presheaf $\mathrm{U} \mapsto \mathrm{H}^{-1}(\mathrm{U}, \mathrm{Gr}_\mathrm{F}^{p-1} \mathcal{F})$ is in $\mathcal{N}$, so that $\int \mathrm{Gr}_\mathrm{F}^p \mathcal{F} \xrightarrow{\sim} \mathrm{Gr}_\mathrm{F}^p \int \mathcal{F}$ in $\mathrm{pf}(\mathcal{S})/\mathcal{N}$. We're done, since an exact sequence in the category of perverse sheaves on X is the same as a perverse sheaf with a one-step decreasing filtration F with perverse successive quotients as above, and as we have just seen, $\int$ applied to the exact sequence

$$0 \to \mathrm{Gr}^1_{\mathrm{F}} \mathscr{F} \to \mathscr{F} \to \mathrm{Gr}^0_{\mathrm{F}} \mathscr{F} \to 0$$

is isomorphic in $\mathrm{pf}(\mathcal{S})/\mathcal{N}$ to

$$0 \to \mathrm{Gr}^1_{\mathrm{F}} \int \mathscr{F} \to \int \mathscr{F} \to \mathrm{Gr}^0_{\mathrm{F}} \int \mathscr{F} \to 0.$$

5.2.4. The point is that regardless of whether $\mathrm{Gr}^p_{\mathrm{F}} \mathscr{F}_0$ is of type (a) or (b), $u : \mathrm{U} \to \mathrm{X}$ can be chosen so that $\mathrm{H}^0(\mathrm{U}, \mathrm{Gr}^p_{\mathrm{F}} \mathscr{F})$ grows linearly with $\deg u$. Modulo some bounded presheaf (i.e. belonging to the subcategory $\mathcal{N}$), $\mathrm{H}^0(\mathrm{U}, \mathrm{Gr}^p_{\mathrm{F}} \mathscr{F})$ is a subquotient of $\mathrm{H}^0(\mathrm{U}, \mathscr{F})$, so its image still grows linearly with $\deg u$. As $\mathrm{W}_{w-1}\mathrm{H}^0(\mathrm{U}, \mathscr{F})$ grows like $o(\deg u)$, there is a U so that $\mathrm{W}_{w-1} \mathrm{Gr}^p_{\mathrm{F}} \mathrm{H}^0(\mathrm{U}, \mathscr{F})$ does not contain the image of $\mathrm{H}^0(\mathrm{U}, \mathrm{Gr}^p_{\mathrm{F}} \mathscr{F})$ in $\mathrm{Gr}^p_{\mathrm{F}} \mathrm{H}^0(\mathrm{U}, \mathscr{F})$. Then the arguments provided show that either (a) $\mathrm{Gr}^p_{\mathrm{F}} \mathscr{F}$ injects into a quotient of $\mathrm{H}^0(\mathrm{U}, \mathrm{Gr}^p_{\mathrm{F}} \mathscr{F})$ of weight $\geq w$, or (b) $\mathrm{H}^1(\overline{\mathrm{U}}, {}^\circ k_* u^* \mathscr{L})$, which is pure, admits a nonzero quotient of weight $\geq w$. In any case, the same conclusion holds.

It remains to see that u can be chosen so that $\mathrm{H}^0(\mathrm{U}, \mathrm{Gr}^p_{\mathrm{F}} \mathscr{F})$ grows linearly with $\deg u$. In the punctual case (a) this is trivial. In case (b) as was analyzed using Euler-Poincaré, $\mathrm{H}^1(\overline{\mathrm{U}}, {}^\circ k_* u^* \mathscr{L})$ grows linearly with g. By the Hurwitz formula, provided that $\overline{\mathrm{U}}_1$ has genus > 1, the genus of a revêtement étale $u_1 : \overline{\mathrm{U}} \to \overline{\mathrm{U}}_1$ is linear with $\deg u_1$. It remains to be seen that one can find a nonconstant separable (i.e. generically étale) map $\overline{\mathrm{U}}_1 \to \overline{\mathrm{X}}$ with $\overline{\mathrm{U}}_1$ a curve of genus > 1. When $\overline{\mathrm{X}} = \mathbf{P}^1$, this is easy, as any curve admits a nonconstant separable map to $\mathbf{P}^1$ corresponding to any choice of separating transcendence base for its function field. When $\overline{\mathrm{X}}$ has genus 1, by the Hurwitz formula it will suffice to find a finite separable branched cover of $\overline{\mathrm{X}}$ which is not everywhere unramified. By [**SGA1**, XIII 2.12], finding such a covering that is tamely ramified at its branch points is a purely topological problem: if $\mathrm{V} \subset \overline{\mathrm{X}}$ is the open obtained by removing n distinct closed points $a_1, \ldots, a_n$, then the tame fundamental group $\pi_1^t(\mathrm{V})$ is generated by $2 + n$ elements x, y, σ_j, with $1 \leq j \leq n$, such that σ_j is a generator of the tame inertia at a_j, subject to the relation

$$xyx^{-1}y^{-1} \cdot \prod_{1 \leq j \leq n} \sigma_j = 1. \tag{$*$}$$

For every finite group G of order prime to the characteristic p generated by the elements $\overline{x}, \overline{y}, \overline{\sigma}_j$ and satisfying the relation $(*)$, there exists a revêtement étale of V, of group G, corresponding to a homomorphism $\pi_1^t(\mathrm{U}) \to \mathrm{G}$ sending x, y, σ_j to $\overline{x}, \overline{y}, \overline{\sigma}_j$. If m is any integer not divisible by p, then the above tells us there exists a galois cover $\mathrm{U}_1 \to \mathrm{V}$ with galois group $\mathbf{Z}/m\mathbf{Z}$ so that the tame inertia at a_1, a_2 acts nontrivially: if we write $\mathbf{Z}/m\mathbf{Z}$ multiplicatively as $\langle q | q^m = e \rangle$, then we can take, e.g., $x = q, y = q, \sigma_1 = q, \sigma_2 = q^{-1}$. If $\overline{\mathrm{U}}_1$ denotes the normalization of $\overline{\mathrm{X}}$ in U_1, then $\overline{\mathrm{U}}_1 \times_{\mathrm{X}} \mathrm{V} = \mathrm{U}_1$, $\overline{\mathrm{U}}_1$ is smooth and complete, and the map $u_1 : \overline{\mathrm{U}}_1 \to \overline{\mathrm{X}}$ is nonconstant, finite, and separable. As σ_1 and σ_2 act nontrivially on $u_1^{-1}(\mathrm{V})$, $\overline{\mathrm{U}}_1 \to \overline{\mathrm{X}}$ is tamely ramified over a_1 and a_2.[4] By the Hurwitz formula, the genus of U_1 is > 1.

5.2.5. 'On laisse au lecteur le soin' $\rightsquigarrow$ it suffices to check the conclusion over an affine Zariski covering of X_0 and the hypothesis is inherited by each affine Zariski open $\mathrm{V}_0 \subset \mathrm{X}_0$; if $\varphi : \mathrm{V}_0 \hookrightarrow \mathbf{A}_0^n$ denotes the embedding of V_0 into affine space, the hypothesis is inherited by $\varphi_* \mathscr{F}_0$ and the conclusion holds for $(\mathbf{A}_0^n, \varphi_* \mathscr{F}_0)$ iff it does for $(\mathrm{V}_0, \mathscr{F}_0)$.

5.2.6. To be clear, one needs (4.5.4) with $\mathrm{X}_a = \overline{\mathrm{U}}$, $\mathrm{X} = \overline{\mathrm{U}}^n$, $\mathrm{U} = \mathrm{U}^n$, f the open immersion $\mathrm{U}^n \hookrightarrow \overline{\mathrm{U}}^n$, $\widetilde{\mathrm{X}}_a = \overline{\mathrm{U}}_\alpha$, and $\widetilde{\mathrm{X}} = \overline{\mathrm{U}}_\alpha^n$. As U_α is affine and $\mathrm{U}_\alpha \to \mathbf{A}^n$ is étale, $\mathrm{H}^0(\mathrm{U}_\alpha^n, \mathscr{F})$ is of weight $\geq w$ (see the paragraph after (5.2.1)). (This argument is the one of (5.2.4).)

5.2.7. '${}^p u^! \mathscr{F}$ est donc nul' $\rightsquigarrow$ en effet, if ${}^p u^! \mathscr{F} \neq 0$, it would have a nontrivial simple perverse subsheaf $0 \neq \mathscr{T} \subset {}^p u^! \mathscr{F}$ and $u_* \mathscr{T}$ would be a simple (1.4.26) perverse subsheaf (4.1.10.2) of $\mathscr{F}$ with support contained in H.

5.2.8. The morphism $\mathrm{Fr} : \mathrm{X} \to \mathrm{X}$ is integral, radicial, surjective so that $\mathrm{Fr}^* \mathrm{Fr}_* \mathscr{F} \xrightarrow{\sim} \mathscr{F} \xrightarrow{\sim} \mathrm{Fr}_* \mathrm{Fr}^* \mathscr{F}$. The perverse sheaf $\mathrm{Fr}^* \mathscr{G}$ has punctual support (base change for $i : \mathrm{Supp}\, \mathscr{G} \hookrightarrow \mathrm{X}$) and is the maximal perverse subsheaf of $\mathrm{Fr}^* \mathscr{G}$ with punctual support, since if $\mathrm{Fr}^* \mathscr{G} \subset \mathscr{G}'$, $\mathscr{G} \xrightarrow{\sim} \mathrm{Fr}_* \mathrm{Fr}^* \mathscr{G} \subset \mathrm{Fr}_* \mathscr{G}'$ and as $\mathrm{Fr}_* \mathscr{G}'$ has punctual support, it must

[4] Indeed, replacing $\overline{\mathrm{X}}$ by $\overline{\mathrm{X}}_{(a_1)}$, its henselization at a_1, $\mathrm{U}_{(a_1)} := \overline{\mathrm{U}} \times_{\overline{\mathrm{X}}} \overline{\mathrm{X}}_{(a_1)}$ splits into the direct product of henselian traits, indexed by the geometric points in the fiber of a_1. If there as as many as the degree of $\mathrm{U}_1 \to \mathrm{V}$, $\mathrm{U}_{(a_1)} \to \mathrm{X}_{(a_1)}$ is étale and the tame inertia must act trivially.

equal $\mathscr{G}$, so that $\mathrm{Fr}^*\mathscr{G} \xrightarrow{\sim} \mathrm{Fr}^*\mathrm{Fr}_*\mathscr{G}' \xrightarrow{\sim} \mathscr{G}$. We have a diagram

$$\begin{tikzcd} \mathrm{Fr}^*\mathscr{G} \arrow[r, hook] & \mathrm{Fr}^*\mathscr{F} \arrow[d, "\wr"] \\ \mathscr{G} \arrow[r, hook] & \mathscr{F}. \end{tikzcd}$$

Tracing the diagram right down is a monomorphism with image having punctual support, hence factors through $\mathscr{G} \hookrightarrow \mathscr{F}$ yielding a map $\mathrm{Fr}^*\mathscr{G} \hookrightarrow \mathscr{G}$; tracing right up the same is true, giving a factorization through $\mathrm{Fr}^*\mathscr{G} \hookrightarrow \mathrm{Fr}^*\mathscr{F}$ and a map $\mathscr{G} \hookrightarrow \mathrm{Fr}^*\mathscr{G}$. Both compositions give the identity, hence an isomorphism $\mathrm{Fr}^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$ making the above diagram commute. Now cite (5.1.2).

5.3.2. *Deuxième preuve* (a) Let $\mathrm{F}_0 := \mathrm{X}_0 - \mathrm{U}_0$ and $i : \mathrm{F}_0 \hookrightarrow \mathrm{X}_0$; as F_0 is finite by assumption we can use the formula $j_{!*} = \tau^{\mathrm{F}_0}_{\leq -1} j_*$ of (2.2.4), which also shows that $v^* j_{!*} = j_{!*} v^*$ as v^* is t-exact and commutes with lower star. Then the correct formula would be $j_{!*} v^* \mathscr{F}_0 = \tau^{\mathrm{F}_0}_{\leq -1} j_* v^* \mathscr{F}_0$ fitting into the distinguished triangle $(\tau^{\mathrm{F}_0}_{\leq -1} j_* v^* \mathscr{F}_0, j_* v^* \mathscr{F}_0, i_* \tau_{\geq 0} i^* j_* v^* \mathscr{F}_0)$, where $i_* \tau_{\geq 0} i^* j_* v^* \mathscr{F}_0$ is acyclic off degree 0 and supported on F_0. In particular, its hypercohomology is acyclic off degree 0, giving the injection $\mathrm{H}^0(\mathrm{V}, v^* j_{!*}\mathscr{F}) \hookrightarrow \mathrm{H}^0(\mathrm{V}, j_* v^* \mathscr{F})$.

(b) Let's start by saying v_0 is in V_0 if $f^{-1}(v_0)$ does not contain the support of a simple perverse quotient of $\mathscr{F}_0$ (4.1.10.1). The support of the finitely many constituents of $\mathscr{F}_0$ is contained in the closure of finitely many points of X_0.

$i^* j_{!*} \mathscr{F}_0[-1]$ is perverse $\rightsquigarrow$ recall that i is the immersion of a closed fiber $f^{-1}(v_0)$ chosen with the property that it does not contain the support of any simple quotient of $\mathscr{F}_0$. The perverse sheaf $i_* {}^p\mathrm{H}^0 i^* j_{!*}\mathscr{F}_0$ is a quotient of $j_{!*}\mathscr{F}_0$ (1.4.10.1). Applying j^*, we get a quotient of $\mathscr{F}_0$ supported on $f^{-1}(v_0)$, which must be zero. Therefore $i_* {}^p\mathrm{H}^0 i^* j_{!*}\mathscr{F}_0$ is supported on $f^{-1}(v_0) \cap \mathrm{F}_0$, and must be zero by (1.4.25).

$i^*(j_{!*}\mathscr{F}_0)[-1] = j_{!*}(i^*\mathscr{F}_0[-1]) \rightsquigarrow$ over a nonempty open subset of $\mathbf{A}^1$, j_* commutes with arbitrary base change (*Th. finitude* 1.9), so we may assume $i^* j_* = j_* i^*$ after possibly shrinking V_0. Let F_0 be as above; then $j_{!*}\mathscr{F}_0 = {}^p\tau^{\mathrm{F}}_{\leq -1} j_* \mathscr{F}_0$. Let $\mathrm{K}_0 = j_*\mathscr{F}_0$. In order to conclude, it would suffice to show that $i^*({}^p\tau^{\mathrm{F}}_{\leq -1}\mathrm{K}_0)[-1] = {}^p\tau^{\mathrm{F}}_{\leq -1} i^*\mathrm{K}_0[-1]$; i.e. that the triangle $(i^*({}^p\tau^{\mathrm{F}}_{\leq -1}\mathrm{K}_0)[-1], i^*\mathrm{K}_0[-1], k_* {}^p\tau_{>-1} k^*(i^*\mathrm{K}_0[-1]))$ is distinguished, where k

is defined by the diagram

$$\begin{tikzcd} F_0 \cap f^{-1}(v_0) \arrow[r, hook, "k"] \arrow[d, hook, "i"] & f^{-1}(v_0) \arrow[d, hook, "i"] \\ F_0 \arrow[r, hook, "k"] & X_0. \end{tikzcd}$$

In other words, setting $L_0 := k^*K_0$, we must show ${}^p\tau_{>-1}i^*L_0[-1] \sim i^*({}^p\tau_{>-1}L_0)[-1]$. As $i^*[-1]$ has perverse amplitude in $[0,1]$, we need only find an open locus of v_0 so that if $\mathcal{L}_0 := {}^pH^{-1}L_0$, $i^*\mathcal{L}_0[-1]$ is perverse (then the first and third terms of the distinguished triangle $(i^*({}^p\tau_{\leq -1}L_0)[-1], i^*L_0[-1], i^*({}^p\tau_{>-1}L_0)[-1])$ will belong to $\mathcal{D}^{\leq -1}$ and $\mathcal{D}^{>-1}$, respectively). As remarked above, this will be true if $F_0 \cap f^{-1}(v_0)$ does not contain the support of any simple perverse quotient of $\mathcal{L}_0$, so, shrinking V_0, we can make it true for every $v_0 \in |V_0|$.

Let M_0 be any complex in $D^b_c(X_0)$. For each closed point of $\mathbf{A}^1_0$, we have distinguished triangles exchanged by duality involving the unipotent nearby and vanishing cycles

$$\Psi^{\mathrm{un}}_f(M_0) \xrightarrow{\mathrm{can}} \Phi^{\mathrm{un}}_f(M_0) \to i^*M_0 \to$$
$$i^!M_0 \to \Phi^{\mathrm{un}}_f(M_0) \xrightarrow{\mathrm{var}} \Psi^{\mathrm{un}}_f(M_0)(-1) \to .$$

There is an nonempty open locus of $\mathbf{A}^1_0$ over which f is locally acyclic rel. M_0 (*Th. finitude* 2.13). For each closed point of this open locus, therefore, we have $\Phi_f(M_0) = 0$ and an isomorphism $i^*M_0[-2](-1) \sim i^!M_0$. Specializing M_0 to $j_{!*}\mathcal{F}_0$, we have found that we can shrink our $V_0 \subset \mathbf{A}^1_0$ to make

$$i^*(j_{!*}\mathcal{F}_0)[-2](-1) \xrightarrow{\sim} i^!(j_{!*}\mathcal{F}_0) \tag{*}$$

hold for all closed points in V_0.

Let's see how the induction hypothesis combined with the above imply that $j_{!*}\mathcal{F}_0$ is of weight $\geq w$ on $f^{-1}(V_0)$. By (5.1.9), it will suffice to show that for each closed point x_0 in $f^{-1}(V_0)$, letting g denote its inclusion in $f^{-1}(V_0)$, $g^!(j_{!*}\mathcal{F}_0)$ is of weight $\geq w$, and as x_0 lies in $f^{-1}(f(x_0))$, it will suffice to show that $i^!(j_{!*}\mathcal{F}_0)$ is of weight $\geq w$. For this it will suffice by (*) to show that $i^*(j_{!*}\mathcal{F}_0)$ is of weight $\geq w$; i.e. that $i^*(j_{!*}\mathcal{F}_0)[-1] = j_{!*}(i^*\mathcal{F}_0[-1])$ is of weight $\geq w-1$. Restricting (*) to U_0 finds $i^*\mathcal{F}_0[-1] \xrightarrow{\sim} i^!\mathcal{F}_0[1](1)$ is of weight $\geq w-1$; now use the induction hypothesis.

To finish, one uses the projection trick. By assumption X_0 is a closed subscheme of some $\mathbf{A}^n_0$ and we have shown that for each of the n projections to $\mathbf{A}^1_0$, $j_{!*}\mathcal{F}_0$ is of weight $\geq w$ on $\mathrm{pr}_i^{-1}(V_{0,i})$ for some nonempty open $V_{0,i} \subset \mathbf{A}^1_0$. Then the same is true on $\cup_{i=1}^n \mathrm{pr}_i^{-1}(V_{0,i})$, and it suffices to show that its complement Q_0 (in $\mathbf{A}^n_0$, hence also in X_0) is finite. This is obvious as the residue field of any point of $\mathbf{A}^n_0$ is generated by $x_1, \ldots, x_n$ and the image of each of these coordinate functions in the residue field of any point of Q_0 is algebraic over $\mathbf{F}_q$, so every irreducible component of Q_0 has dimension 0.

5.3.5. We have as a consequence of (5.3.6) that $W_{i-1}W_i\mathcal{F}_0 = W_{i-1}\mathcal{F}_0$ since the left hand side is indeed a subobject of $\mathcal{F}_0$, has all simple subquotients of weight $\leq i-1$, and we have the extension

$$0 \to W_i\mathcal{F}_0/W_{i-1}W_i\mathcal{F}_0 \to \mathcal{F}_0/W_{i-1}W_i\mathcal{F}_0 \to \mathcal{F}_0/W_i\mathcal{F}_0 \to 0$$

in which the outer terms are of weight $\geq i-1$.

5.3.7. We can restrict the long exact sequence of cohomology by j^* and the cokernel of the map $H^{-d}(\mathcal{F}_0)|Y_0 \to H^{-d}(\mathcal{G}_0)|Y_0$ is a (constructible) subsheaf of $H^{-d+1}(\mathcal{K}_0)|Y_0$, where $\mathcal{K}_0 := \ker(\mathcal{F}_0 \to \mathcal{G}_0)$. Let i denote the inclusion of the generic point of Y_0: then i^* of this cokernel is zero, since $H^{-d+1}i^*\mathcal{K}_0 = 0$ (2.2.12).

5.3.8. C.f. note to Weil II (3.4.5).

5.3.9. The analysis makes repeated use of (5.1.2), as it deduces sheaves on X_0 or X_1 from F_q^* or $F_{q^n}^*$-stable subsheaves of the perverse sheaf $\mathcal{F}$, which comes from a perverse sheaf on X_0. Recall the basics of Fr_q from the note to (5.1.1); in particular Fr_q^* and Fr_{q*} are t-exact and the morphisms of adjunction $\mathrm{Fr}_q^*\,\mathrm{Fr}_{q*} \to \mathrm{id} \to \mathrm{Fr}_{q*}\,\mathrm{Fr}_q^*$ are isomorphisms.

If $\mathcal{F}_0$ is simple (but not *a priori* mixed – the distinction is irrelevant after L. Lafforgue's work), then $\mathcal{F}$ is readily seen to be semisimple by following the proof of (5.3.8) to find the existence of a nonzero $\mathcal{F}'_0 \subset \mathcal{F}_0$ so that $\mathcal{F}'$ is semisimple. As $\mathcal{F}_0$ is by assumption simple, $\mathcal{F}'_0 = \mathcal{F}_0$.

Suppose $\mathscr{G}$ is a simple perverse sheaf on X. Then Fr_q^* takes simple perverse sheaves to simple perverse sheaves: $\mathrm{Fr}_q^*\mathscr{G} \neq 0$ since $\mathscr{G} \xrightarrow{\sim} \mathrm{Fr}_{q*}\mathrm{Fr}_q^*\mathscr{G}$, and if $\mathscr{H} \subset \mathrm{Fr}_q^*\mathscr{G}$, then $\mathrm{Fr}_{q*}\mathscr{H} \subset \mathrm{Fr}_{q*}\mathrm{Fr}_q^*\mathscr{G} \xleftarrow{\sim} \mathscr{G}$ so $\mathrm{Fr}_{q*}\mathscr{H}$ must be zero, and $\mathrm{Fr}_q^*\mathrm{Fr}_{q*}\mathscr{H} \xrightarrow{\sim} \mathscr{H} = 0$. Therefore Fr_q^* acts on A, the set of isomorphism classes of simple constituents of $\mathscr{F}$, with inverse Fr_{q*}. So if $\mathrm{Fr}_q^*\, a = a'$, then $\mathrm{Fr}_q^*\mathscr{F}_a \subset \mathscr{F}_{a'}$; applying Fr_{q*} finds $\mathscr{F}_a \xrightarrow{\sim} \mathrm{Fr}_{q*}\mathrm{Fr}_q^*\mathscr{F}_a \subset \mathrm{Fr}_{q*}\mathscr{F}_{a'} \subset \mathscr{F}_a$ so that actually $\mathrm{Fr}_q^*\mathscr{F}_a = \mathscr{F}_{a'}$ and $\mathscr{F}_a = \mathrm{Fr}_{q*}\mathscr{F}_{a'}$.

$\mathscr{F}_{0\mathrm{C}} \simeq \pi_*\mathscr{F}_{1a} \rightsquigarrow (\pi^*, \pi_*)$ are adjoint functors on perverse sheaves. By (5.1.2), the category of perverse sheaves $\mathscr{F}_0$ on X_0 embeds fully faithfully into the category of perverse sheaves on X with an isomorphism $\mathrm{Fr}_q^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$, and the functor π^* corresponds to forgetting the isomorphism $\mathrm{Fr}_q^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ and remembering only the isomorphism $\mathrm{Fr}_{q^d}^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ induced by it. Continuing to use π^*, π_* for the corresponding adjoint functors on the essential image, $\pi_*\mathscr{F}_{1a} = \pi_*(\mathscr{F}_a, \mathrm{F}_{q^d}^*)$ corresponds to the perverse sheaf on X stable under Fr_q^* with the property that given any perverse sheaf $\mathscr{G}$ on X stable under Fr_q^* in the essential image, to give a Fr_q^*-equivariant morphism $\mathscr{G} \to \pi_*(\mathscr{F}_a, \mathrm{F}_{q^d}^*)$ is the same as giving a $\mathrm{Fr}_{q^d}^*$-equivariant morphism $\mathscr{G} \to \mathscr{F}_a$.

Claim: to give a Fr_q^*-equivariant morphism $\mathscr{G} \to \mathscr{F}_{\mathrm{C}}$ is the same as giving a $\mathrm{Fr}_{q^d}^*$-equivariant morphism $\mathscr{G} \to \mathscr{F}_a$ for any perverse sheaf $(\mathscr{G}, \mathrm{F}_q^*)$ on X.

To see the claim, as $\mathscr{F}_{\mathrm{C}} = \oplus\mathscr{F}_{a'}$ is a product of the $\mathscr{F}_{a'}$ ($a' \in \mathrm{A}$), to give a morphism $\alpha : \mathscr{G} \to \mathscr{F}_{\mathrm{C}}$ is the same as giving morphisms $\alpha_{a'} : \mathscr{G} \to \mathscr{F}_{a'}$ for all $a' \in \mathrm{A}$. The condition that α be F_q^*-equivariant is that $\mathrm{F}_{q^n}^*\alpha_{a'} = \alpha_b$ for all n, where $b = \mathrm{Fr}_{q^n}^*(a')$; i.e.

$$\begin{tikzcd}
\mathrm{Fr}_{q^n}^*\mathscr{G} \arrow[r, "\mathrm{Fr}_{q^n}^*\alpha_{a'}"] \arrow[d, "\mathrm{F}_{q^n}^*"', "\wr"] & \mathrm{Fr}_{q^n}^*\mathscr{F}_{a'} \arrow[d, "\mathrm{F}_{q^n}^*"', "\wr"] & \subset & \mathscr{F}_{\mathrm{C}} \arrow[d, "\mathrm{F}_{q^n}^*"', "\wr"] \\
\mathscr{G} \arrow[r, "\alpha_b"] & \mathscr{F}_b & \subset & \mathscr{F}_{\mathrm{C}}
\end{tikzcd}$$

must commute for all n. This shows that such a F_q^*-equivariant α is determined totally by α_a, and that moreover, giving an F_q^*-equivariant α is the same as giving an α_a with the property that $\mathrm{F}_{q^d}^*\alpha_a = \alpha_a$, i.e. that α_a is $\mathrm{F}_{q^d}^*$-equivariant. This proves the claim.

The claim combined with the adjunction give isomorphisms of functors

$$\mathrm{Hom}_{\mathrm{F}_q^*}(-, \mathscr{F}_{\mathrm{C}}) \sim \mathrm{Hom}_{\mathrm{F}_{q^d}^*}(-, \mathscr{F}_a) \sim \mathrm{Hom}_{\mathrm{F}_q^*}(-, \pi_*(\mathscr{F}_a, \mathrm{F}_{q^d}^*))$$

on the essential image of the functor given in (5.1.2). This shows $\mathcal{F}_{\mathrm{C}} \simeq \pi_*(\mathcal{F}_a, \mathrm{F}^*_{q^d})$ in this essential image and therefore $\mathcal{F}_{\mathrm{C}0} \simeq \pi_* \mathcal{F}_{1a}$.

Any endomorphism $\mathcal{G}_a$ is null or an isomorphism, so $\operatorname{End} \mathcal{G}_a$ is a finite-dimensional division algebra over $\overline{\mathbf{Q}}_\ell$. But any such coincides with $\overline{\mathbf{Q}}_\ell$, so every automorphism of $\mathcal{G}_a$ is of the form $\lambda \cdot \mathrm{id}$ with $\lambda \in \overline{\mathbf{Q}}_\ell^*$. That every finite-dimensional division algebra D over an algebraically closed field F is trivial follows from the following elementary argument: given $a \in \mathrm{D}$, the powers of a are linearly dependent over F and satisfy some monic polynomial $f(x) \in \mathrm{F}[x]$ of least degree. As F is algebraically closed, $f(x) = (x - \lambda) g(x)$ for some $\lambda \in \mathrm{F}$; by minimality of f, $g(a) \neq 0$, so $a = \lambda \in \mathrm{F}$ and $\mathrm{D} = \mathrm{F}$.

$\mathrm{ev} : \mathcal{G}_a \otimes \mathrm{V}_a \xrightarrow{\sim} \mathcal{F}_a$ est $\mathrm{F}^*_{q^d}$-equivariant $\rightsquigarrow$ we must check that $\mathrm{F}^*_{q^d}\,\mathrm{ev} = \mathrm{ev}$. In this paragraph we write F and Fr for $\mathrm{F}_{q^d}, \mathrm{Fr}_{q^d}$, respectively. $\mathrm{F}^*\,\mathrm{ev}$ is defined by the diagram

$$\begin{tikzcd} \mathrm{Fr}^* \mathcal{G}_a \otimes \mathrm{Fr}^* \mathrm{V}_a \arrow[r, "\mathrm{Fr}^*\,\mathrm{ev}"] \arrow[d, "\phi\otimes\mathrm{F}^*"', "\wr"] & \mathrm{Fr}^* \mathcal{F}_a \arrow[d, "\mathrm{F}^*"', "\wr"] \\ \mathcal{G}_a \otimes \mathrm{V}_a \arrow[r, dashed, "\mathrm{F}^*\,\mathrm{ev}"] & \mathcal{F}_a \end{tikzcd} \tag{$*$}$$

$\mathrm{F}^* : \mathrm{Fr}^*_{q^d} \mathrm{V}_a \xrightarrow{\sim} \mathrm{V}_a$ sends $\mathrm{Fr}^* f \in \operatorname{Hom}(\mathrm{Fr}^* \mathcal{G}_a, \mathrm{Fr}^* \mathcal{F}_a)$ to the morphism $\mathrm{F}^* f$ defined by

$$\begin{tikzcd} \mathrm{Fr}^* \mathcal{G}_a \arrow[r, "\mathrm{Fr}^* f"] \arrow[d, "\phi"', "\wr"] & \mathrm{Fr}^* \mathcal{F}_a \arrow[d, "\mathrm{F}^*"', "\wr"] \\ \mathcal{G}_a \arrow[r, dashed, "\mathrm{F}^* f"] & \mathcal{F}_a. \end{tikzcd} \tag{$\dagger$}$$

It is an isomorphism with inverse obtained by replacing upper stars by lower stars and reversing the direction of the vertical arrows. The assignment $f \mapsto \mathrm{F}^* f$ is an automorphism of $\operatorname{Hom}(\mathcal{G}_a, \mathcal{F}_a)$ also denoted by F^*. It is an automorphism since Fr^* is fully faithful and $\mathrm{F}^* : \operatorname{Hom}(\mathrm{Fr}^* \mathcal{G}_a, \mathrm{Fr}^* \mathcal{F}_a) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{G}_a, \mathcal{F}_a)$ is an isomorphism. (This defines the action $\mathrm{ev} \mapsto \mathrm{F}^*\,\mathrm{ev}$ coming from the analogous automorphism of $\operatorname{Hom}(\mathcal{G}_a \otimes \mathrm{V}_a, \mathcal{F}_a)$.)

The morphism ev : $\mathcal{G}_a \otimes \mathrm{Hom}(\mathcal{G}_a, \mathcal{F}_a) \to \mathcal{F}_a$ is evaluation. The commutativity of the diagram $(*)$ with ev in place of F^* ev results from the fact that the diagram

$$\begin{tikzcd}
& \mathrm{Fr}^* \mathcal{G}_a \arrow[r, "\mathrm{Fr}^* f"] \arrow[d, "\phi"', "\wr"] & \mathrm{Fr}^* \mathcal{F}_a \arrow[d, "\mathrm{F}^*"', "\wr"] \\
\mathrm{Fr}^* \mathcal{G}_a \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "\sim", "\phi"'] & \mathcal{G}_a \arrow[r, dashed, "\mathrm{F}^* f"] & \mathcal{F}_a
\end{tikzcd}$$

commutes with the dashed arrow the identity.

'$(\mathrm{V}_a, \mathrm{F}^*_{q^d})$ est indécomposable' $\rightsquigarrow$ a $\mathrm{F}^*_{q^d}$-equivariant decomposition of V_a, would, via $\mathcal{G}_a \otimes \mathrm{V}_a \xrightarrow{\sim} \mathcal{F}_a$, give rise to a $\mathrm{F}^*_{q^d}$-equivariant decomposition of $\mathcal{F}_a$, hence a decomposition of $\mathcal{F}_{1a}$, hence a decomposition of $\mathcal{F}_0 = \mathcal{F}_{0\mathrm{C}} \simeq \pi_* \mathcal{F}_{1a}$.

'on peut supposer $\mathrm{F}^*_{q^d}$ sur V_a unipotent' $\rightsquigarrow$ as $\mathrm{F}^*_{q^d}$ acts on V_a with only one Jordan block, there is an $f \in \mathrm{V}_a$ so that $\mathrm{F}^*_{q^d} f = \lambda f$ for some $\lambda \in \overline{\mathbf{Q}}^*_\ell$. As $\dim \mathrm{Hom}(\mathrm{Fr}^*_{q^d} \mathcal{G}_a, \mathcal{G}_a) = 1$, replacing ϕ by $\lambda\phi$, one sees via the diagram (†) that $\mathrm{F}^*_{q^d} f = f$, and therefore $\mathrm{F}^*_{q^d}$ acts unipotently on V_a. Moreover f identifies $(\mathcal{G}_a, \phi)$ as a subobject of $(\mathcal{F}_a, \mathrm{F}^*_{q^d})$ and if $(e_i)_{1 \le i \le n}$ is a base for V_a ($n = \dim \mathrm{V}_a$) putting V_a in Jordan normal form for the action of $\mathrm{F}^*_{q^n}$ (with the notation as in the top of the page) then e_n is the sole eigenvector and $e_n = f$. In this base (and with this ϕ), the action of $\mathrm{F}^*_{q^d}$ on V_a looks like

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \tag{‡}$$

and we have $\mathrm{F}^*_{q^n}$-equivariant isomorphisms $\mathcal{G}_a \otimes \mathrm{E}_n \simeq \mathcal{G}_a \otimes \mathrm{V}_a \xrightarrow{\sim} \mathcal{F}_a$ descending to an isomorphism $\mathcal{G}_1 \otimes \mathrm{E}_n \xrightarrow{\sim} \mathcal{F}_{1a}$.

'l'image inverse sur $\mathrm{Spec}(\mathbf{F}_{q^d})$ de E_n est isomorphe au faisceau analogue sur $\mathrm{Spec}(\mathbf{F}_{q^d})$' $\rightsquigarrow$ this observation is equivalent to the observation that if M denotes the unipotent Jordan block matrix of dimension n (written explicitly in (‡) above), then

M^d and M are similar matrices. This fact remains true over any field, provided the characteristic does not divide d. To see why, write $M = I + N$ with N nilpotent; then $M^d = I + dN + \ldots + dN^{d-1} + N^d$ is again unipotent, so all eigenvalues of M^d are 1 and M^d can be put in Jordan normal form. Provided $d \neq 0$, $\dim\ker(M^d - I) = 1$: e_n is in the kernel, and $e_2, \ldots, e_n$ are in the image of $M^d - I$, since $Ne_{n-1} = e_n$, $N^i e_{n-1} = 0$ for $i > 1$, and $(M^d - I)e_{n-j} = de_{n-j+1} \mod \operatorname{span}(e_{n-j+2}, \ldots, e_n)$ for $j > 0$. In conclusion, M^d can be put in Jordan normal form with only one Jordan block corresponding to the eigenvalue 1, and is therefore similar to M.

'$\pi_*(\mathcal{G}_1 \otimes E_n) \simeq \pi_*(\mathcal{G}_1) \otimes E_n$' $\rightsquigarrow$ [**SGAA**, XVII 5.2.9] la 'formule des projections.'

'$\pi_*(\mathcal{G}_1)$ est simple' $\rightsquigarrow$ clearly $\mathcal{G}_1$ is simple because $\mathcal{G}_a$ is simple, and (5.1.2) tells that subobjects of $\mathcal{G}_1$ correspond to ϕ-invariant subobjects of $\mathcal{G}_a$. Moreover, ϕ has been chosen so that $(\mathcal{G}_a, \phi)$ is isomorphic to a subobject of $(\mathcal{F}_a, F^*_{q^d})$, i.e. $\mathcal{G}_1 \hookrightarrow \mathcal{F}_{1a}$, so $\pi_*\mathcal{G}_1 \hookrightarrow \pi_*\mathcal{F}_{1a} \simeq \mathcal{F}_{0C}$ and the diagram

$$\begin{tikzcd}
\mathrm{Hom}_\phi(-, \mathcal{G}_a) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Hom}_{F^*_q}(-, \pi_*(\mathcal{G}_a, \phi)) \arrow[d, hook] \\
\mathrm{Hom}_{F^*_{q^d}}(-, \mathcal{F}_a) \arrow[r, "\sim"] & \mathrm{Hom}_{F^*_q}(-, \mathcal{F}_C)
\end{tikzcd}$$

commutes, since if $u : (\mathcal{G}_a, \phi) \hookrightarrow (\mathcal{F}_a, F^*_{q^d})$ and $f \in \mathrm{Hom}_\phi(-, \mathcal{G}_a)$, tracing the diagram down right gives $\pi_*(u \circ f) \circ \eta$ while tracing it right down gives $\pi_*(u) \circ (\pi_*(f) \circ \eta)$, where η is the counit of the adjunction. We have seen that the lower horizontal isomorphism is obtained by taking $F^*_{q^n}$-conjugates of the $F^*_{q^d}$-equivariant morphism to $\mathcal{F}_a$. The commutativity of the diagram implies that the resulting F^*_q-equivariant morphism to $\mathcal{F}_C$ lands in $\pi_*(\mathcal{G}_a, \phi)$ iff the original $F^*_{q^d}$-equivariant morphism to $\mathcal{F}_a$ lands in $\mathcal{G}_a$ ($F^*_{q^d}$ on $\mathcal{F}_a$ induces ϕ on $\mathcal{G}_a$). Moreover, the F^*_q-equivariant map to $\mathcal{F}_C$ is a monomorphism iff the $F^*_{q^d}$-equivariant map to $\mathcal{F}_a$ is, as $F^*_{q^n}$ is exact for all n so all the $F^*_{q^n}$-conjugates of a monomorphism are monomorphisms. We conclude that a F^*_q-equivariant monomorphism to $\pi_*(\mathcal{G}_a, \phi)$ is uniquely determined by a $F^*_{q^d}$-equivariant monomorphism to $\mathcal{G}_a$. As $\mathcal{G}_a$ is simple, any such must be null or an isomorphism, in which case the F^*_q-conjugates will be null or isomorphisms, respectively. Therefore $\pi_*(\mathcal{G}_a, \phi)$, i.e. $\pi_*\mathcal{G}_1$, is simple.

(ii) As $\dim \mathrm{V}_a = 1$ in this case, $\mathcal{G}_1 \simeq \mathcal{F}_{1a}$; we have seen that $\mathcal{G}_1$ is simple and corresponds to $(\mathcal{G}_a, \phi)$ on X, where $\mathcal{G}_a$ is simple. The isomorphism classes of the F_q^*-conjugates of $\mathcal{G}_a$ (which are in this case the conjugates of $\mathcal{F}_a$) are determined by the F_q^*-orbit A of a; as d is the least integer with the property that $\mathrm{F}_{q^d}^*$ stabilizes $\mathcal{F}_a = \mathcal{G}_a$, we have $\mathrm{F}_{q^i}^* \mathcal{G}_a \simeq \mathrm{F}_{q^j}^* \mathcal{G}_a$ iff $i \equiv j \mod d$.

5.3.10. The place we used that $\overline{\mathbf{Q}}_\ell$ is algebraically closed in the argument in (5.3.9) and its note is when we concluded that $\operatorname{End} \mathcal{G}_a$ is 1-dimensional since it is a finite-dimensional division algebra over $\overline{\mathbf{Q}}_\ell$.

5.3.11. These Homs are null by (1.4.25).

'Chacun d'eux est de l'une des formes $j_{!*}\mathcal{G}'$ ou $i_*\mathcal{G}''$' $\rightsquigarrow$ (1.4.26).

Indeed ${}^p i^! j_{!*} = 0 = {}^p i^* j_{!*}$ by (1.4.24) and of course $j^* i_* = 0$.

The list of compatibilities to verify is short (`0103`) and immediate on X from the above.

5.4.1. 'il existe un ouvert dense U_0' $\rightsquigarrow$ (2.2.12).

5.4.5. Rappelons que $\mathrm{Ext}^1 = \mathrm{Hom}^1_{\mathrm{D}}$ by (3.1.17) (ii). Since ${}^p\tau_{\leq i}\mathrm{K}_0$ is a successive extension of $({}^p\mathrm{H}^j\mathrm{K}_0)[-j]$ for $j \leq i$, and these are pure of the same weight as K_0 by (5.4.4), ${}^p\tau_{\leq i}\mathrm{K}_0$ is pure of the same weight for any i.

5.4.7. To see

$$\mathcal{H}^i(j_{!*}\mathcal{L}[d]) = \begin{cases} {}^\circ j_*\mathcal{L} & i = -d, \\ 0 & i < -d, \end{cases}$$

take the long exact sequence of ordinary ($p = 0$) cohomology associated to the distinguished triangle $(j_{!*}(\mathcal{L}[d]), j_*(\mathcal{L}[d]), i_*{}^p\tau_{>-1}i^*j_*(\mathcal{L}[d]))$ of (1.4.23), where $i : \mathrm{X} - \mathrm{U} \hookrightarrow \mathrm{X}$. Well actually one could replace X by $\overline{\mathrm{U}}$ and the point is that as $\dim \overline{\mathrm{U}} - \mathrm{U} < \dim \mathrm{U}$, $\mathcal{H}^i({}^p\tau_{>-1}i^*j_*(\mathcal{L}[d])) = 0$ for $i \leq -d$ (2.1.2.1).

To see $\mathrm{H}^{-d}(\mathrm{X}, j_{!*}\mathscr{L}[d]) = \mathrm{H}^0(\mathrm{X}, {}^\circ j_*\mathscr{L})$, use

$$\mathrm{E}_1^{pq} = \mathrm{H}^q(\mathrm{X}, \mathscr{H}^p(j_{!*}\mathscr{L}[d]) \Rightarrow \mathrm{H}^{p+q}(\mathrm{X}, j_{!*}\mathscr{L}[d]).$$

Since $j_{!*}\mathscr{L}[d]$ is a perverse sheaf on $\overline{\mathrm{U}}$ of dimension d, $\mathrm{E}_1^{pq} = 0$ for $p < -d$, and we've seen above that $\mathscr{H}^{-d}(j_{!*}\mathscr{L}[d]) = {}^\circ j_*\mathscr{L}$.

5.4.8. As $f : \mathrm{X}_x \to \mathrm{X}$ is flat, by going down the generic point η of each irreducible component of X_x is sent to the generic point ξ of an irreducible component of X. With $\mathrm{K} = j_{!*}(\mathscr{L}[d])[n]$, this shows that

$$\mathrm{H}^i(\mathrm{K})_{\overline{\eta}} = \begin{cases} 0 & \text{if no such } \xi \text{ is in U or if } i \neq -d-n, \\ \mathscr{L}_{\overline{\eta}} \ (= \mathscr{L}_{\overline{\xi}}) & \text{if such a } \xi \in \mathrm{U} \text{ and } i = -d-n. \end{cases}$$

At most one such ξ can be in U.

Claim: at most one η can belong to U'_x.

Let's assume the claim. By the note to (5.4.7),

$$\mathrm{H}^{-d-n}(\mathrm{X}_x, \mathrm{K}) = \mathrm{H}^0(\mathrm{X}_x, {}^\circ j_*\mathscr{L}) = \prod \mathrm{H}^0(\mathrm{U}'_x, \mathscr{L}),$$

the product taken over the various connected components U'_x of U_x. As the various connected components U'_x are disjoint, and at most one η belongs to each, one is indeed reduced to showing

$$\mathrm{H}^0(\mathrm{U}'_x, \mathscr{L}) \twoheadrightarrow (\mathscr{L}_{\overline{\eta}})^{\mathrm{Gal}(\overline{\eta}/\eta)}$$

if $\eta \in \mathrm{U}'_x$.

The claim follows from the fact that the irreducible components and connected components of U_x coincide, since if U'_x is irreducible and $\eta \in \mathrm{U}'_x$, η is automatically the generic point of U'_x. To see that the irreducible and connected components of U_x coincide, as $f : \mathrm{U}_x \to \mathrm{U}$ is flat, the generic points of irreducible components of the former map to the generic point ζ of the latter, so we see they are in bijection with the finitely many points of $f^{-1}(\zeta)$ (`07QQ`). Assume $\mathrm{X} = \operatorname{Spec}\mathrm{A}$, then with $\mathrm{X}_x = \operatorname{Spec}\mathrm{C}$, the ideals of C corresponding to the finitely many points of $f^{-1}(\zeta)$ are each finitely generated, so there exists some étale neighborhood $g : \operatorname{Spec}\mathrm{B} = \mathrm{X}' \to \mathrm{X}$ of x so that the image of B in C contains all the generators of the prime ideals of C corresponding to the points of $f^{-1}(\zeta)$, and therefore the map $X_x \to X'$ sends the fiber $f^{-1}(\zeta)$ injectively into $g^{-1}(\zeta)$. As $g^{-1}(U)$ is smooth and g is flat, the points of $g^{-1}(\zeta)$ are in bijection with the irreducible components of $g^{-1}(U)$, which are the same as the connected components of $g^{-1}(U)$. We're done, because this gives a partition of $U_x = f^{-1}(U)$ into a disjoint union of subschemes $\sqcup U'_x$ so that each U'_x contains precisely one point of $f^{-1}(\zeta)$, hence is irreducible.

5.4.9. (a) As the $H^i f_* L_0$ are lisse on U, the stalks $H^i(Y_x, L)$ are all the same for various geometric points x of U. Looking at (5.4.7), we indeed have $H^i(Y, L)$ on the left, and we are reduced to showing for K of the form $(j_{!*}\mathscr{L}[d])[n]$ with j the immersion of a smooth connected open locus V in an irreducible component of X of dimension d with generic point η, $\mathscr{L}$ lisse on V, and $U \subset X$ open with the property that the $H^i K$ are lisse on U, that $(H^i K)_x^{\pi_1(U,x)} = (H^i K)_{\overline{\eta}}^{\mathrm{Gal}(\overline{\eta}/\eta)}$, as $f_* L$ is a direct sum of such K. As the note to (5.4.7) shows that $H^i K = {}^\circ j_* \mathscr{L}$ when $i = -d-n$ and vanishes on V in other degrees, the condition that the $H^* K$ are lisse on U means that either they are all zero, or $K|U$ is acyclic off degree $-d-n$, $\eta \in U$ and ${}^\circ j_* \mathscr{L}$ is lisse on U. Then indeed

$$(H^{-d-n}K)_x^{\pi_1(U,x)} = H^0(U, {}^\circ j_* \mathscr{L}) = H^0(V, \mathscr{L}) = L_{\overline{\eta}}^{\mathrm{Gal}(\overline{\eta}/\eta)} = (H^{-d-n}K)_{\overline{\eta}}^{\mathrm{Gal}(\overline{\eta}/\eta)}.$$

(b) Suppose $\tilde{x}$ is centered on η, a generic point of an irreducible component of $X_{(x)}$. Then the local monodromy is $\mathrm{Gal}(\tilde{x}/\eta)$ acting on $(H^* f_* L)_{\tilde{x}} = H^*(Y_{\tilde{x}}, L)$ and the statement is a literal translation of (5.4.8).

Intermezzo: SGA 5 Exposé VII. In this intermezzo and in this intermezzo only, we write the R when writing a right-derived functor, so $?_*$ means $R^0 ?_*$, etc.

See also the notes to this exposé.

Cup-products. There are several notions of cup-product, each with its own merits, and we discuss three below.

First and most direct is the method of `0B68`. Let $f : X \to Y$ be a morphism of schemes and K, L in $D(X, \mathscr{A}_X)$. Using the results of Spaltenstein and Serpé, we can dispense once and for all with considerations of cohomological dimension or tor-dimension when discussing the definition of derived functors $\mathrm{R}f_*$ and $\otimes^{\mathbf{L}}$ in étale cohomology and define a map (called cup-product)

$$\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} \mathrm{R}f_*(\mathrm{L}) \to \mathrm{R}f_*(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{L}) \tag{$\dagger$}$$

by adjunction from the map $f^*(\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} \mathrm{R}f_*(\mathrm{L})) \to \mathrm{K} \otimes^{\mathbf{L}} \mathrm{L}$ obtained by compositing the isomorphism $f^*(\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} \mathrm{R}f_*(\mathrm{L})) = f^*\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} f^*\mathrm{R}f_*(\mathrm{L})$ with the counits of adjunction. The second method follows [**SGA5**, VII 2.1.1] and is more explicit: one proceeds in the style of Godement to define an explicit morphism of resolutions which, precomposed with a K-flat resolution, gives the above arrow.

To define the cup-product in ℓ-adic cohomology, consideration must be given to questions of finiteness, and so one should ask that X and Y be of finite type over a regular scheme of dimension ≤ 1 so that $\mathrm{R}f_*$ is of finite cohomological dimension and the six functors preserve the property of being of finite tor-dimension. Then one should ask that K, L be in $\mathrm{D}^b_{\mathrm{ctf}}(\mathrm{X}, \mathscr{A}_{\mathrm{X}})$ and proceed similarly.

The cup-product induces maps

$$\mathrm{R}^a f_*(\mathrm{K}) \otimes \mathrm{R}^b f_*(\mathrm{L}) \to \mathrm{H}^{a+b}(\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} \mathrm{R}f_*(\mathrm{L})) \to \mathrm{R}^{a+b} f_*(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{L}).$$

where the first map is obtained by replacing $\mathrm{R}f_*\mathrm{K}$ or $\mathrm{R}f_*\mathrm{L}$ by a K-flat resolution and then using the general recipe that bilinearly associates to elements $x \in \mathrm{H}^a\mathrm{K}$, $y \in \mathrm{H}^b\mathrm{L}$ an element of $\mathrm{H}^{a+b}(\mathrm{K} \otimes \mathrm{L})$. This recipe represents x and y by cycles so that $x \otimes y$ is a cycle in the total complex $\mathrm{K} \otimes \mathrm{L}$ of the double complex; the class of $x \otimes y$ is independent of the choice of representatives and the association is clearly bilinear (`068G`).

It is possible to interpret the cup product with the fundamental class of a divisor Y (or more general kinds of closed loci such as locally complete intersections) in a variety X as a composition of morphism of restriction followed by Gysin morphism (as alluded to in [**SGA5**, VII §4]). To do so it is useful to review the definition of cup-product as it appears in [**SGA4**$\frac{1}{2}$, Cycle 1.2] (the third and final definition of cup-product we will use). Let $i : \mathrm{Y} \hookrightarrow \mathrm{X}$ be the inclusion; we will suppose for simplicity that Y is a (Cartier) divisor since the case we are concerned with is the inclusion of a hyperplane in projective space.

As defined in [**SGA**$4\frac{1}{2}$, Cycle 2.1], Y determines a class in $H^2_Y(X, \mu)$. Here $R\Gamma_Y = R\Gamma \circ Ri^!$ is endowed with the natural 'forget supports' map given by the counit $i_*Ri^! \to \mathrm{id}$. Forgetting supports of the class in $H^2_Y(X, \mu)$ gives the Chern class in $H^2(X, \mu)$ as defined in [**SGA5**, VII 2.2].

The cup-product as defined in [**SGA**$4\frac{1}{2}$, Cycle 1.2] is the map, for $\mathscr{F}, \mathscr{G}$ abelian sheaves on X,

$$\Gamma_Y(X, \mathscr{F}) \otimes \Gamma(Y, \mathscr{G}) \to \Gamma_Y(X, \mathscr{F} \otimes \mathscr{G}) \tag{$*$}$$

which to a section s of $\mathscr{F}$ supported on Y and a section t of $\mathscr{G}$ over Y applies the following recipe: represent t by any section t' over X and tensor with s to produce $s \otimes t'$; as s is supported on Y, the result doesn't depend on the choice of t' and is also supported on Y. This prescription derives to give an arrow

$$R\Gamma_Y(X, K) \otimes^{\mathbf{L}} R\Gamma(Y, L) \to R\Gamma_Y(X, K \otimes^{\mathbf{L}} L) \tag{$\ddagger$}$$

for complexes K and L.

The advantage of this definition of cup-product is that it is more precise: the cup-product with the fundamental class of Y as defined previously factors through this map. Let ξ denote the fundamental class of Y in $H^2_Y(X, \mu)$ (or in $H^2(X, \mu)$) and L be in D(X). The diagram

$$\begin{tikzcd}
H^i(X, L) \arrow[r, "\xi\cup"] \arrow[d] & H^{i+2}(X, L)(1) \\
H^i(Y, L) \arrow[r, "\xi\cup"] & H^{i+2}_Y(X, L)(1) \arrow[u, "\text{forget}"']
\end{tikzcd}$$

commutes, where the upper cup product is the map (†) while the lower one is (‡).

The local version goes like this: ξ corresponds to a map $\mathscr{A}_Y \to Ri^!(\mu)[2]$ in D(Y), which via adjunction and the counit $i_*i^! \to \mathrm{id}$ gives a map $\mathscr{A}_X \to i_*Ri^!(\mu)[2] \to \mu[2]$ which is the map we encounter in [**SGA5**, VII 2.2] and its note. Given any K in D(X), the cup-product on cohomology above is obtained by applying $R\Gamma$ to the maps

$$\begin{tikzcd} K = K \otimes^{\mathbf{L}} \mathscr{A}_X \arrow[rr, "\xi"] \arrow[d] & & K[2](1) \\ i_*(i^*K \otimes^{\mathbf{L}} \mathscr{A}_Y) \arrow[r, "\xi"] & i_*(i^*K \otimes^{\mathbf{L}} Ri^!(\mu)[2]) \arrow[r] & i_*Ri^!(K(1)[2]), \arrow[u] \end{tikzcd} \tag{$\star$}$$

where the vertical arrows are units or counits. Here the map

$$i^*(\mathscr{F}) \otimes i^!(\mathscr{G}) \to i^!(\mathscr{F} \otimes \mathscr{G})$$

is the local version of $(*)$, which derives to give

$$i^*(K) \otimes^{\mathbf{L}} Ri^!(L) \to Ri^!(K \otimes^{\mathbf{L}} L),$$

whence (‡) by application of $R\Gamma$. This arrow is obtained by adjunction from the arrow

$$i_*(i^*(K) \otimes^{\mathbf{L}} Ri^!(L)) = K \otimes^{\mathbf{L}} i_*Ri^!(L) \to K \otimes^{\mathbf{L}} L.$$

1.1. (ii) $\rightsquigarrow$ [**SGAA**, XVIII 2.8].

1.3. (i) c) The short exact sequence of sheaves

$$0 \to \mathscr{A}_U \to \mathscr{A} \to \mathscr{A}_Y \to 0$$

gives a distinguished triangle of functors

$$R\mathscr{H}\!om(\mathscr{A}_Y, -) \to R\mathscr{H}\!om(\mathscr{A}, -) \to R\mathscr{H}\!om(\mathscr{A}_U, -) \to$$

and, as $\mathscr{H}\!om(\mathscr{A}_U, -) = \alpha_*\alpha^*$ [**SGAA**, V 6.5], applying Rp_*, a distinguished triangle

$$R_Y p_* \to Rp_* \to Rq_*\alpha^* \to$$

whence the long exact sequence

$$\to R^i_Y p_*(p^*\mathscr{L}) \to R^i p_*(p^*\mathscr{L}) \to R^i q_*(q^*\mathscr{L}) \to R^{i+1}_Y p_*(p^*\mathscr{L}) \to \dots.$$

where by [**SGAA**, V 6.1.2], the sheaf $R^i_Y p_* p^*\mathscr{L}$ is the sheaf associated to the presheaf

$$U \mapsto H^i_{Y \times_S U}(X \times_S U, p^*\mathscr{L}).$$

(One ends up with the same thing by sheafifying the associated long exact sequence of presheaves of cohomology with and without support.)

In light of the above, which shows $R_Y p_* = f_*\beta^!$, the correct reference for the isomorphism

$$R^{2r}_Y p_*(p^*\mathscr{L}) \xrightarrow{\sim} \mathscr{L} \otimes \mu_S^{\otimes -r}$$

is [**SGAA**, XVI 3.8].

(ii) a) The bizarre notation $i \neq (1, 2r)$ means 'i is not equal to 1 or $2r$.'

c) Evidently the trace morphism is $R^{2r}_! q(q^*\mathscr{L}) \to \mathscr{L} \otimes \mu_S^{\otimes -r}$. Let's see that it is an isomorphism. As q is lisse with geometrically connected fibers, by [**SGAA**, XV 1.16] it is 0-acyclic, so that the unit of adjunction $\mathrm{id} \to q_*q^*$ is a natural isomorphism, which implies that q^* is fully faithful, since for sheaves $\mathscr{F}, \mathscr{G}$ one has

$$\mathrm{Hom}(q^*\mathscr{F}, q^*\mathscr{G}) = \mathrm{Hom}(\mathscr{F}, q_*q^*\mathscr{G}) = \mathrm{Hom}(\mathscr{F}, \mathscr{G}).$$

Now to see that $Rq_!(q^*\mathscr{L}) = \mathscr{L}[-2r](-r)$, as $Rq^!K = q^*K[2r](r)$, just write

$$\begin{aligned}\mathrm{Hom}(L[-2r](-r), K) = \mathrm{Hom}(q^*L[-2r](-r), q^*K) &= \mathrm{Hom}(q^*L, q^*K[2r](r))\\ &= \mathrm{Hom}(q^*L, Rq^!K) = \mathrm{Hom}(Rq_!q^*L, K).\end{aligned}$$

2.1. The morphism of Godement resolutions

$$C^{\cdot}(X, \mathscr{A}_X) \otimes C^{\cdot}(X, \mathscr{A}_X) \to C^{\cdot}(X, \mathscr{A}_X \otimes \mathscr{A}_X)$$

is defined in Godement §6.6 using his (4.3.2), which describes sections of $C^p(X, \mathscr{A}_X)$ in terms of functions similar to Alexander-Spanier cochains. This gives a morphism

$$f_*C^{\cdot}(X, \mathscr{A}_X) \otimes f_*C^{\cdot}(X, \mathscr{A}_X) \to f_*C^{\cdot}(X, \mathscr{A}_X \otimes \mathscr{A}_X)$$

and as $f_*C^{\cdot}(X, \mathscr{A}_X)$ computes $Rf_*\mathscr{A}_X$, by taking a K-flat resolution $K \to f_*C^{\cdot}(X, \mathscr{A}_X)$ of Spaltenstein (`06Y7`), $K \otimes f_*C^{\cdot}(X, \mathscr{A}_X)$ computes $Rf_*\mathscr{A}_X \otimes^L Rf_*\mathscr{A}_X$ with a morphism to $f_*C^{\cdot}(X, \mathscr{A}_X) \otimes f_*C^{\cdot}(X, \mathscr{A}_X)$. When $Rf_*\mathscr{A}_X$ is in $D^-(Y)$, you don't need Spaltenstein and can use a usual bounded-above flat resolution. (I have no clue how Jouanolou intended to define this arrow without finiteness hypotheses on f in '77.)

2.2. ‘La classe ξ s’identifie à une flèche $\mathscr{A}_{\mathrm{P}} \to \mu_{\mathrm{P}}[2]$’ $\rightsquigarrow$ ξ is an element of

$$\begin{aligned}
\mathrm{H}^2(\mathrm{P}, \mu_{\mathrm{P}}) = \mathrm{Ext}^2(\mathscr{A}_{\mathrm{P}}, \mu_{\mathrm{P}}) &= \mathrm{Hom}_{\mathrm{D(P)}}(\mathscr{A}_{\mathrm{P}}, \mu_{\mathrm{P}}[2]) \\
&= \mathrm{Hom}_{\mathrm{D(P)}}((\mu_{\mathrm{P}}[2]) \otimes^{\mathrm{L}} (\mu_{\mathrm{P}}^{\otimes -1}[-2]), \mu_{\mathrm{P}}[2]) \\
&= \mathrm{Hom}_{\mathrm{D(P)}}(\mu_{\mathrm{P}}^{\otimes -1}[-2], \mathrm{R}\mathscr{H}\!om(\mu_{\mathrm{P}}[2], \mu_{\mathrm{P}}[2])) \\
&= \mathrm{Hom}_{\mathrm{D(P)}}(p^*\mu_{\mathrm{S}}^{\otimes -1}[-2], \mathscr{A}_{\mathrm{P}}) = \mathrm{Hom}_{\mathrm{D(S)}}(\mu_{\mathrm{S}}^{\otimes -1}[-2], \mathrm{R}p_*\mathscr{A}_{\mathrm{P}})
\end{aligned}$$

coinciding with η on the cohomology of dimension 0, not 2.

2.2.2. b) Of course the map has target $\oplus_i \mathrm{R}^{2i} p_*(\mu_{\mathrm{P}}^{\otimes i})$. The cup-product map is the one discussed at the beginning of this intermezzo.

In the proof, it suffices to show that ξ^r generates all of $\mathrm{H}^{2r}(\mathrm{P}, \mu^{\otimes i})$ to conclude that ξ^i does for $i < r$ since the cup-product is A-bilinear.

c) ‘en d’autres termes, on a $\mathrm{Tr}(\eta^r) = 1$’ $\rightsquigarrow$ $\xi : \mathscr{A}_{\mathrm{P}} \to \mu_{\mathrm{P}}[2]$ in D(P) gives $\mathscr{A}_{\mathrm{S}} \to \mathrm{R}p_*(\mu_{\mathrm{P}}[2])$ in D(S) hence by tensorization and cup product a map $\mathscr{A}_{\mathrm{S}} \to (\mathrm{R}p_*(\mu_{\mathrm{P}}[2]))^{\otimes r} \to \mathrm{R}p_*(\mu_{\mathrm{P}}^{\otimes r}[2r])$ inducing $\eta^r : \mathscr{A}_{\mathrm{S}} \to \mathrm{R}^{2r} p_*(\mu_{\mathrm{P}}^{\otimes r})$ on H^0.

The stability of Tr under base change is [**SGAA**, XVIII 2.9].

How to see (c)? Here is one way:

(1) Show that the cup product on $\mathbf{P}_k^n$ induces the cup product on $\mathbf{P}_k^{n-1}$.
(2) As Poincaré duality gives a perfect pairing $(x, y) \mapsto \mathrm{Tr}(x \cup y)$, induction on n reduces to the case $n = 1$.
(3) When $n = 1$, the Chern class map can be identified with $\deg : \mathrm{Pic}\, \mathbf{P}_k^1 \xrightarrow{\sim} \mathbf{Z}$.

(1) is formal and tautological since the cup-product is obtained from adjunction from the map $f^*\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} f^*\mathrm{R}f_*(\mathrm{L}) \to \mathrm{K} \otimes^{\mathbf{L}} \mathrm{L}$ obtained from the counits of adjunction, which are natural. Let $i : \mathbf{P}_k^{n-1} \hookrightarrow \mathbf{P}_k^n$ denote the imbedding at infinity; then the units of adjunction $\mathrm{K} \to i_*i^*\mathrm{K}$ and $\mathrm{L} \to i_*i^*\mathrm{L}$ give a commutative diagram

$$\begin{tikzcd}
f^*\mathrm{R}f_*(\mathrm{K}) \otimes^{\mathbf{L}} f^*\mathrm{R}f_*(\mathrm{L}) \arrow[r] \arrow[d] & \mathrm{K} \otimes^{\mathbf{L}} \mathrm{L} \arrow[d] & \\
f^*\mathrm{R}f_*(i_*i^*\mathrm{K}) \otimes^{\mathbf{L}} f^*\mathrm{R}f_*(i_*i^*\mathrm{L}) \arrow[r] & i_*i^*\mathrm{K} \otimes^{\mathbf{L}} i_*i^*\mathrm{L} \arrow[r, "\sim"] & i_*i^*(\mathrm{K} \otimes^{\mathbf{L}} \mathrm{L}).
\end{tikzcd}$$

(2) is obtained by assuming $n > 1$, $i > 0$, and that ξ and ξ^{n-i} are free elements of the A-modules $\mathrm{H}^2(\mathbf{P}_k^n, \mu)$ and $\mathrm{H}^{2(n-i)}(\mathbf{P}_k^n, \mu^{\otimes(n-i)})$, respectively. As $\mathrm{Tr}(- \cup -)$ induces a perfect pairing

$$\mathrm{H}^2(\mathbf{P}_k^{n-i+1}, \mu) \times \mathrm{H}^{2(n-i)}(\mathbf{P}_k^{n-i+1}, \mu^{\otimes(n-i)}) \to \mathrm{H}^{2(n-i+1)}(\mathbf{P}_k^{n-i+1}, \mu^{\otimes(n-i+1)}) \simeq \mathrm{A}$$

and the cup product on $\mathbf{P}_k^n$ induces the one on $\mathbf{P}_k^{n-i+1}$ by (1), $\xi \cup \xi^{n-i} = \xi^{n-i+1}$ is a free element of the A-module $\mathrm{H}^{2(n-i+1)}(\mathbf{P}_k^n, \mu^{\otimes(n-i+1)}) \xrightarrow{\sim} \mathrm{H}^{2(n-i+1)}(\mathbf{P}_k^{n-i+1}, \mu^{\otimes(n-i+1)})$.

Last, recall that on a smooth projective curve X over a separably closed field, one has

$$\mathrm{H}^2(\mathrm{X}, \mu) \simeq \mathrm{Pic}(\mathrm{X})/\nu\,\mathrm{Pic}(\mathrm{X}) \xrightarrow[\sim]{\deg} \mathrm{A}$$

coming from the Kummer exact sequence, and under this identification the the Chern class map is identified with the degree map. This shows (3).

5.4.10. The references for the definitions of the Chern class (it is the obvious one coming from the Kummer exact sequence), the cup product and Gysin morphism is [**SGA**$4\frac{1}{2}$, Cycle] & [**SGA5**, VII].

As mentioned above in the note to SGA 5 VII 2.2, $\ell \in \mathrm{H}^2(\mathrm{X}_0, \mathbf{Q}_\ell(1))$ corresponds to an element of $\mathrm{Hom}_{\mathrm{D}(\mathrm{X}_0)}(\mathbf{Q}_\ell, \mathbf{Q}_\ell[2](1))$ and by tensorization a morphism $\mathrm{K}_0 \to \mathrm{K}_0[2](1)$ for any K_0 in $\mathrm{D}^b(\mathrm{X}_0, \overline{\mathbf{Q}}_\ell)$.

The restriction of our relatively ample invertible sheaf on $\mathrm{X}_0/\mathrm{Y}_0$ over $f^{-1}(\mathrm{U}_0)$ for $\mathrm{U}_0 \subset \mathrm{Y}_0$ corresponds to the restriction of the cohomology class $\mathrm{H}^2(\mathrm{X}_0, \mathbf{Q}_\ell(1)) \to \mathrm{H}^2(\mathrm{U}_0, \mathbf{Q}_\ell(1))$: if $j : f^{-1}(\mathrm{U}_0) \hookrightarrow \mathrm{X}_0$, $j_!\mathbf{Q}_\ell \to \mathbf{Q}_\ell$ induces a map

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{X}_0)}(\mathbf{Q}_\ell, \mathbf{Q}_\ell[2](1)) \to \mathrm{Hom}_{\mathrm{D}(\mathrm{X}_0)}(j_!\mathbf{Q}_\ell, \mathbf{Q}_\ell[2](1)) = \mathrm{Hom}_{\mathrm{D}(\mathrm{U}_0)}(\mathbf{Q}_\ell, \mathbf{Q}_\ell[2](1)).$$

This shows that the map ℓ^n commutes with restriction and the problem is local. Since on each affine open of Y_0 some power of our relatively ample invertible sheaf on $\mathrm{X}_0/\mathrm{Y}_0$ is very ample (`02NP`), we may assume our invertible sheaf is the reciprocal image of $\mathcal{O}(1)$ along a closed immersion $\mathrm{X}_0 \hookrightarrow \mathbf{P}_0^d \times \mathrm{Y}_0$.

To define the universal family of hyperplane sections parametrized by Y_0', consider X_0 as a closed locus in $\mathbf{P}_0^d \times \mathrm{Y}_0$ and let $p : \mathbf{P}_0^d \times \mathrm{Y}_0 \to \mathbf{P}_0^d$ denote the projection. Then

H_0 is defined as a closed locus in X'_0 by the condition that (a, x) is in H_0 if $a(p(x)) = 0$; i.e. by the condition that $p(x)$ lie in the hyperplane defined by a.

5.4.11. The last part of (iii) is true if ${}^p\mathrm{H}^{d-1} f_* u^*\mathrm{K} \simeq u^*({}^p\mathrm{H}^{-d} u_* {}^p\mathrm{H}^{d-1} h_*(uv)^*\mathrm{K})[d]$ by (4.2.6.2). As $u^*[d]$ is fully faithful (4.2.5), the unit $\mathrm{id} \to {}^p\mathrm{H}^{-d} u_*(u^*[d])$ is a natural isomorphism. By (i) we can write

$$\begin{aligned} u^*({}^p\mathrm{H}^{-d} u_* {}^p\mathrm{H}^{d-1} f_* u^*\mathrm{K})[d] &\xleftarrow{\sim} u^*({}^p\mathrm{H}^{-d} u_* u^*({}^p\mathrm{H}^{-1} f_*\mathrm{K})[d])[d] \\ &\xleftarrow{\sim} u^*({}^p\mathrm{H}^{-1} f_*\mathrm{K})[d] \xrightarrow{\sim} {}^p\mathrm{H}^{d-1} f_* u^*\mathrm{K}. \end{aligned}$$

Therefore (5.4.11.1) does suffice to show the second part of (iii).

5.4.12. $\rightsquigarrow$ [**SGA5**, VII 2.2.1].

5.4.13. To see that H_0 is indeed a projective sub-bundle of X'_0 over X_0, we see easily that the fibers are indeed hyperplanes in $\check{\mathbf{P}}^d_0$: given $x \in |\mathrm{X}|$, the locus $(vu)^{-1}(x) \subset \mathrm{H}_0$ is described inside of $\check{\mathbf{P}}^d$ by the condition that a given hyperplane contain x. Moreover this condition trivializes over the open locus $\mathrm{U}_{j0} \subset \mathrm{X}_0$ given by points x with $p(x)$ missing the jth coordinate hyperplane by projection from the point $a \in \check{\mathbf{P}}^d$ with a 1 in the jth coordinate and 0 elsewhere. In more words, given $x \in |\mathrm{U}_j|$, $p(x)$ is not contained in the hyperplane a. The hyperplanes containing $p(x)$ form a hyperplane $\mathrm{L}_x \subset \check{\mathbf{P}}^d$ not containing a. Let $\mathrm{L} \subset \check{\mathbf{P}}^d$ denote the hyperplane given by the vanishing of the jth coordinate in $\check{\mathbf{P}}^d$. Then projection from a sends L_x isomorphically onto L and trivializes H_0 over U_{j0}.

5.4.14. It is easy to see that the morphism of restriction dualizes to give the Gysin morphism, but to see that the composition gives the $u^*[d]$ of the map ℓ is trickier. The $u^*[d]$, ${}^p\mathrm{H}^j$, f_* is all junk and (with $\mathcal{G} = u^*\mathcal{F}[d]$) what we are really looking at are maps

$$\mathcal{G} \to v_* v^* \mathcal{G} \quad \leftrightarrow \quad v_* v^! \mathcal{G} \to \mathcal{G}[2](1).$$

The key point is that with Y a hyperplane in $\mathrm{X} = \mathbf{P}^n_k$ and K constant on X, the isomorphism of purity $i^*\mathrm{K} \xrightarrow{\sim} i^!\mathrm{K}(1)[2]$ is obtained via the composition of the lower horizontal maps of ($\star$). One may replace K by $\mathcal{A}_{\mathrm{X}}$, in which case this composition corresponds to the cup-product with the fundamental class of [**SGA**$4\frac{1}{2}$, Cycle], and [**SGA**$4\frac{1}{2}$, Cycle 2.1.4] gives that indeed the class of Y (i.e. ξ) generates $\mathrm{R}^2 i^! \mu$. In other words, one could say that the isomorphism of purity in this situation is induced by cup-product with the fundamental class of the hyperplane.

In the situation of the universal family of hyperplane sections of X_0, the point is that the fiber over any geometric point $x \in \mathrm{X}$ of $\mathrm{H}_0 \subset \mathrm{X}'_0$ is just the inclusion of a hyperplane in $\check{\mathbf{P}}^d_x$, and the previous discussion applies.

Appendix A: t-exactness of nearby & vanishing cycles. To understand the argument in Appendix A, [**I**, §3.1] is very helpful. In Appendix A, the calculation of $\mathrm{R}\Gamma$ proceeds along identical lines to [**D**, 7.11.3 & 10.7].

There is a discrepancy between (4.4.2) the the proposition of Appendix A; namely (4.4.2) says that $\Psi_{\overline{\eta}} := \mathrm{R}\Psi_{\overline{\eta}}$ is right t-exact, while the appendix claims that the same functor has perverse amplitude -1; i.e. that $\Psi_{\overline{\eta}}[-1]$ is t-exact. This discrepancy is due to a difference of t-structures; both are relative to the middle perversity function $p_{1/2}$, but the dimension function in (4.4.2) is the naïve one, while the dimension function in the appendix is the rectified dimension function introduced by Artin in [**SGAA**, XIV 2.2], which is also the one described in Remark (i) to Appendix A. The point is that if S admits structure morphism to a field k and $x \in \mathrm{X}$ maps to the generic point of S,

$$\mathrm{tr.\,deg.}(k(x)/k) = \mathrm{tr.\,deg.}(k(x)/k(\eta)) + 1;$$

in general if x has image s in S, Artin rectifies the naïve dimension function by adding $\mathrm{tr.\,deg.}(k(x)/k(y))$ so that

$$\delta(x) = \dim \overline{\{y\}} + \mathrm{tr.\,deg.}(k(x)/k(y)).$$

This has the effect of shifting the naïve t-structure on $\mathrm{X}_{\overline{\eta}}$ by 1 to the left, so that if $\Psi_{\overline{\eta}}$ is t-exact with respect to the naïve t-structure (4.4.2), $\Psi_{\overline{\eta}}[-1]$ is t-exact with respect to the rectified t-structure. The point is that we would like to think of S as the henselization of a curve at a regular point; if S were instead the curve instead of its localization and $\mathrm{X} \to \mathrm{S}$ were still of finite type, any point x of X lying over the generic point of S would have strictly positive dimension ($\dim \overline{\{x\}} > 0$), and indeed if S were a curve of finite type over a field k, $\dim \overline{\{x\}}$ would be given by precisely $\mathrm{tr.\,deg.}(k(x)/k)$.

On the matter of invariants of a $\Lambda[\mathrm{G}]$-module when G is a finite group of order invertible in Λ: the functor 'invariants under G' $\mathrm{Hom}_{\Lambda[\mathrm{G}]}(\Lambda, -)$ is exact if Λ is projective as $\Lambda[\mathrm{G}]$-module. The canonical surjection $\Lambda[\mathrm{G}] \twoheadrightarrow \Lambda$ is split by

$$1 \mapsto \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} g =: \omega,$$

which recognizes Λ as projective $\Lambda[\mathrm{G}]$-module. Therefore when Q is a profinite group of order prime to ℓ acting on a finite Λ-module ($\Lambda \supset \mathbf{Z}/\ell^n$), the functor 'invariants under Q' is exact. This argument can be applied stalkwise, and gives more. Namely, if M is a $\Lambda[\mathrm{G}]$-module, the map $\mathrm{M} \twoheadrightarrow \mathrm{M}_\mathrm{G}$ factors as

$$\mathrm{M} \xrightarrow{\omega} \mathrm{M}^\mathrm{G} \hookrightarrow \mathrm{M}_\mathrm{G}.$$

This shows that the map $\mathrm{M}^\mathrm{G} \hookrightarrow \mathrm{M}_\mathrm{G}$ is an isomorphism; the inverse is given by the map which to a class $[m] \in \mathrm{M}_\mathrm{G}$ associates $\omega\, m$.

There is the matter of how to define the nearby cycles: in [**SGA 7**, XIII 2.1] the functor $\mathrm{R}\Psi$ is defined in a somewhat different way from the functor Ψ of [**SGA 7**, I 2.2]. Let (S, η, s) be a henselian trait and let $\tilde{\eta}, \overline{\eta}$ denote the spectra of respectively the maximal unramified extension of $k(\eta)$ and the separable closure of η. Let $\tilde{\mathrm{S}}$ denote the normalization of S in $k(\tilde{\eta})$; and $\tilde{s}$ its closed point; $\tilde{s}$ is the spectrum of the separable closure of $k(s)$ and $\tilde{\mathrm{S}}$ is the strict henselization of S at $\tilde{s}$. Let $\overline{\mathrm{S}}$ denote the normalization of S in $k(\overline{\eta})$: $\overline{\mathrm{S}}$ is the spectrum of a valuation ring with value group $\mathbf{Q}$, generic point $\overline{\eta}$, and closed point $\overline{s}$ with residue field a purely inseparable extension of $k(\tilde{s})$. Let $\rho : \overline{\eta} \to \eta$ denote the chosen geometric generic point of S. We have a commutative diagram

$$\begin{tikzcd}
\overline{s} \arrow[r, "\overline{i}"] \arrow[d] & \overline{\mathrm{S}} \arrow[d] & \overline{\eta} \arrow[l, "\overline{j}"'] \arrow[d] \\
\tilde{s} \arrow[r, "\tilde{i}"] \arrow[d] & \tilde{\mathrm{S}} \arrow[d] & \tilde{\eta} \arrow[l, "\tilde{j}"'] \arrow[d] \\
s \arrow[r, "i"] & \mathrm{S} & \eta. \arrow[l, "j"']
\end{tikzcd}$$

Let $f : X \to S$ and $G := \mathrm{Gal}(\overline{\eta}/\eta)$. Exposé 1 (and Appendix A) suppose S is strictly henselian to begin with, in which case the tilde is superfluous, and define a functor ψ. On the other hand, in Exposé XIII, Deligne defines for (a not necessarily strictly henselian S)

$$R\Psi_\eta(K) := \overline{i}^* R\overline{j}_* K_{\overline{\eta}}, \qquad K \in D^+(X_\eta, \Lambda).$$

Let $\overline{\rho} : \overline{s} \to s$ denote the morphism obtained by base change of $\overline{S} \to S$ along $i : s \to S$ (technically the base change of the latter morphism has its source in the spectrum of a field purely inseparable over $k(\overline{s})$, but the map to s from the spectrum of this larger field factors through our $\overline{\rho}$), and define

$$\psi(K) := \overline{\rho}^* i^* Rj_* \rho_* \rho^* K, \qquad K \in D^b_c(X_\eta, \Lambda).$$

When S is strictly henselian, this coincides with the functor $i^* Rj_* \rho_* \rho^* K$. But now ψ and $R\Psi_\eta$ coincide on an arbitrary henselian trait S: the fact that an integral morphism commutes with all base extension [**SGAA**, VIII 5.6] allows one to write

$$\psi(K) = \overline{\rho}^* i^* Rj_* \rho_* \rho^* K = \overline{\rho}^* i^* \rho_* R\overline{j}_* K_{\overline{\eta}} = \overline{\rho}^* \overline{\rho}_* \overline{i}^* R\overline{j}_* K_{\overline{\eta}} = \overline{i}^* R\overline{j}_* K_{\overline{\eta}} = R\Psi(K).$$

Note that in the case $X = S$, $R\Psi(K)$ just gives the sheaf $K_{\overline{\eta}}$ on the topos $s \times_s \eta \simeq \eta$, and the distinguished triangle defining $R\Psi$ can be written as

$$\mathrm{sp}^* K_s \to K_{\overline{\eta}} \to R\Psi(K) \to .$$

One obtains the isomorphism

$$K \xrightarrow{\sim} R\Gamma(G, \rho_* \rho^* K) \qquad K \in D^b_c(X_\eta, \Lambda)$$

of Appendix A from the same argument, namely by passage to the limit from

$$K \xrightarrow{\sim} R\Gamma(G_i, \rho_{i*} \rho_i^* K) \qquad K \in D^b_c(X_\eta, \Lambda).$$

This isomorphism can be proved after a base change $\eta_i \to \eta$ after which it is literally Shapiro's lemma (compare [**SGA 7**, XIII §1]). (The passage to the limit is justified in light of the isomorphisms

$$H^q(G, A) \xrightarrow{\sim} H^q(G/U, A^U)$$

as U runs over open subgroups of G and A is a discrete G-module (c.f. Serre, *Cohomologie Galoisienne* §2.2).)

Appendix A (i). In light of the above, the identity $i^*\mathrm{R}j_*\mathrm{K} = \mathrm{R}\Gamma(\mathrm{G}, \mathrm{R}\Psi(\mathrm{K}))$ is definitional [**I**, 3.1.3], but perhaps not quite – a detailed argument is written in the note to Morel's article on gluing perverse sheaves. The spectral sequence associated to the perverse canonical filtration is the spectral sequence associated to the filtration of complexes given by the ${}^p\tau_{\leq}$; see [**BBD**, §3.1.5] for the definition of the filtration and `012N` for the formulation of the spectral sequence arising from a filtration of complexes (see also `015X`). Unfortunately when writing down the spectral sequence in Appendix A there is a collision of notation, as a, b are used as indexes in the spectral sequence while $\mathrm{R}\Psi(\mathcal{F})$ still has perverse amplitude $[a, b]$. The perverse amplitude of $i^*j_*\mathcal{F}$ is contained in $[-1, 0]$ by [**BBD**, 4.1.10 (ii)]. The compatibility of $\mathrm{R}\Psi$ with change of trait is *Th. finitude* (3.7).

Appendix A (ii). Y must denote the closed fiber X_s. When $\mathcal{G}$ is supported on the closed fiber, $\mathrm{R}\Psi_\eta(\mathcal{G}) = 0$ and the claim is clear. Therefore we assume the support of $\mathcal{G}$ has nonempty intersection with X_η and $\mathcal{G}$ is a simple perverse sheaf. [**BBD**, 4.3.1 & 4.3.2] gives $\mathcal{G} = j_{!*}j^*\mathcal{G}$, and [**BBD**, 4.1.12] allows us to write the distinguished triangle

$$i^*\mathcal{G} \to \mathrm{R}\Psi(\mathcal{G}_\eta) \to \mathrm{R}\Phi(\mathcal{G}) \to$$

as
$$({}^p\mathrm{H}^{-1}i^*j_*\mathcal{F})[1] \to \mathrm{R}\Psi(\mathcal{F}) \to \mathrm{R}\Phi(\mathcal{G}) \to .$$

This shows that $\mathrm{R}\Psi(\mathcal{G})[-1]$ is a perverse sheaf, and using (i) and (*) obtains

$$\mathrm{R}\Psi(\mathcal{G})[-1] = {}^p\mathrm{H}^{-1}\mathrm{R}\Psi(\mathcal{F})/({}^p\mathrm{H}^{-1}\mathrm{R}\Psi(\mathcal{F}))^{\mathrm{G}}.$$

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5. *Applications de la formule des traces aux sommes trigonométriques*

1.2. Suppose X is locally noetherian and X′ is a connected component of X pointed by a geometric point $a \to X'$. Via Grothendieck's Galois theory, over X′, the A-torsor T can be identified (after a choice e of identity for the stalk of T at a) with the set A with (continuous) left action of $\pi_1(X', a)$ and right action of A.

Given a homomorphism $\tau : A \to B$, we wish to produce a map of sets $\tau : A \to B$ so that $\tau(ta) = \tau(t)\tau(a)$ and give the set B with right action of B a left action of $\pi_1(X', a)$ with respect to which τ is equivariant. The given homomorphism of groups sets up a map of sets compatible with the actions of A and B, and the formula

$$g\tau(t) := \tau(gt)$$

defines a left π_1 action on $\tau(A)$. For $s \in B$ not necessarily in $\tau(A)$,

$$gs := g\tau(e)s = \tau(ge)s$$

extends the left action of π_1 to all of B. The B-torsor $\tau(T)$ is then identified with the set B with right action of B and the given left action of π_1, with respect to which $\tau : T \to \tau(T)$ is equivariant by construction. This discussion depends on the choice of identity for the torsor T (the identity for $\tau(T)$ is obtained from the choice for T and the homomorphism τ) and therefore only defines the torsor $\tau(T)$ up to isomorphism.

Let R (resp. E_λ) be a finite $\mathbf{Z}_\ell$-algebra (resp. finite $\mathbf{Q}_\ell$-algebra). Recall (SGA 5 Exp. VI (α) 1.2.4, (β) 1.4.1, (γ) 1.4.2, (δ) 1.4.3) that the category of

(α) lisse $\mathbf{Z}_\ell$-sheaves
(β) lisse R-sheaves
(γ) lisse $\mathbf{Q}_\ell$-sheaves
(δ) lisse E_λ-sheaves

on a connected locally noetherian scheme X, pointed by a geometric point $a \to X$ is equivalent to the category of

(α) $\mathbf{Z}_\ell$-modules of finite type on which $\pi_1(X) := \pi_1(X, a)$ acts continuously for the ℓ-adic topology (β) R-modules of finite type equipped with a continuous and R-linear action of $\pi_1(X)$ on the underlying $\mathbf{Z}_\ell$-module of finite type

(γ) $\mathbf{Q}_\ell$-vector spaces of finite dimension equipped with a continuous action of $\pi_1(X)$

(δ) E_λ-vector spaces of finite dimension equipped with a continuous and E_λ-linear action of $\pi_1(X)$ on the underlying $\mathbf{Q}_\ell$-vector space.

Given an A-torsor T on X and a finite-dimensional E_λ-vector space V, the data of an E_λ-sheaf $\mathscr{F}$, lisse of rank dim V, together with a morphism of sheaves $\rho : T \to \underline{\mathrm{Isom}}(V, \mathscr{F})$ satisfying $\rho(ta) = \rho(t)\rho(a)$ is the same (after a choice of identity e for the stalk of T at a) as the data of a continuous and E_λ-linear action of $\pi_1(X)$ on the underlying $\mathbf{Q}_\ell$-vector space of V and a π_1-equivariant group homomorphism $\rho : A \to GL(V)$, where the action of π_1 on GL(V) is induced by the action of π_1 on the second factor of Hom(V, V). Given $\rho : A \to GL(V)$ a linear representation of A, there is a unique continuous and E_λ-linear action of π_1 on the $\mathbf{Q}_\ell$-vector space underneath V that makes ρ π_1-equivariant:

$$\begin{aligned} &\pi_1(X) \times V \to V \\ &\qquad (g, v) \mapsto \rho(ge)v \quad \rightsquigarrow \quad \rho(ga) = \rho(gea) = \rho(ge)\rho(a) = g\rho(a). \end{aligned} \tag{†}$$

1.3. Torsors can be discussed in the language of schemes or sheaves; the distinction comes down to whether the torsor is representable as a sheaf, and this distinction motivates the introduction of algebraic spaces. In SGA 1 Exp. V §2 and SGAD Exp. III §0 & Exp. IV §5 the notion of principal homogeneous space is developed; these are the representing objects for certain (sheaf) torsors. There, an A-torsor is called a principal homogeneous space under A. More precisely, given a category $\mathscr{C}$, let $\widehat{\mathscr{C}}$ denote its category of set-valued presheaves $\mathbf{Hom}(\mathscr{C}^\circ, (\mathbf{Set}))$. If A is a $\widehat{\mathscr{C}}$-group acting on X an object of $\widehat{\mathscr{C}}$, X is formally principal homogeneous under A (i.e. an A-pseudo torsor) if the equivalent conditions below are satisfied:

(i) for each object S of $\mathscr{C}$, the set X(S) is empty or principal homogeneous under A(S);

(ii) the morphism of functors $A \times X \to X \times X$ defined setwise by $(a, x) \mapsto (ax, x)$ is an isomorphism.

It amounts to the same to say that the canonical morphism of functors

$$X \times A \to X \times X$$

is an isomorphism. If $\mathscr{C}$ is equipped with a topology, then one says that the S-object X with S-group of operators A is fibered principal homogeneous under A (i.e. is an A-torsor) if it is locally trivial; i.e. there exists a covering family $\{S_i \to S\}$ such that for each i, the S_i-functor $X \times_S S_i$ with S_i-functor-group of operators $A \times_S S_i$ is trivial.

The category $\widehat{\mathscr{C}}$ has a final object $\underline{\mathbf{e}}$ which sends an object of $\mathscr{C}$ to $\{\emptyset\}$, the set with one element. This functor is representable iff $\mathscr{C}$ admits a final object. The 'sections' functor Γ is defined on $\widehat{\mathscr{C}}$ as $\mathrm{Hom}(-, \underline{\mathbf{e}})$ and on $\mathscr{C}$ via $X \mapsto h_X$; if $\mathscr{C}$ admits a final object, this latter functor is isomorphic to $\mathrm{Hom}(e, -)$. SGAD Exp. IV 5.1.2, 5.1.3 observes that X is formally principal homogeneous under A, there is an isomorphism

$$\Gamma(X) \xrightarrow{\sim} \mathrm{Isom}_{\text{A-obj.}}(A, X)$$

of principal homogeneous sets under $\Gamma(A)$; therefore an isomorphism of A-objects

$$X \xrightarrow{\sim} \underline{\mathrm{Isom}}_{\text{A-obj.}}(A, X).$$

The proof is simply that to each section x of X one associates the morphism $A \to X$ defined setwise by $a \mapsto xa$. This implies that an object with group of operators is trivial iff it is formally principal homogeneous and possesses a section.

The algebraic group schemes in the given extension can also be considered as sheaves for the fpqc topology on S. En effet, the surjectivity of π implies that π is faithfully flat, and therefore an fpqc covering.

Proposition (*SGAD Exp. IV 5.1.7.1*). — *Let $\mathscr{C}$ denote a category possessing a final object, stable by fiber products, and equipped with a subcanonical topology $\mathscr{T}$ (such as* ***Sch*** *equipped with fppf, fpqc, étale). Suppose* $\pi : G' \to G$ *is a morphism of $\mathscr{C}$-groups which is covering for the topology $\mathscr{T}$, and* $A = \ker \pi$. *Then* G *represents the quotient sheaf* G'/A, *and* π *is an* A_G*-torsor; i.e.* G' *is an* A*-torsor on* G.

Therefore it makes sense to say 'G' is an A-torsor on G' or that 'the sheaf T of local sections of π, $T = \mathrm{Hom}_G(-, G')$, is an A- torsor on G'; the former represents the latter. The A-torsor G' on G is indeed locally trivial for fpqc: pulling back along the faithfully flat covering π, we find

$$\begin{aligned} G' \times_G A &\to G' \times_G G' \\ (g', a) &\mapsto (g', g'a) \end{aligned}$$

is indeed an isomorphism, as can be checked setwise.

In order to understand how to add torsors, it is instructive to first recall how to add extensions of abelian groups. Suppose

$$0 \to A \to G' \to G \to 0$$
$$0 \to A \to G'' \to G \to 0$$

are exact sequences of abelian groups. G' and G'' are A-torsors in **Set**. The Baer sum $G' + G''$ is constructed from the direct sum of extensions by pushout along addition for A and pullback along the diagonal for G.

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r] \arrow[d, "+"] \arrow[dr, phantom, "\lrcorner"] & G' \oplus G \arrow[r] \arrow[d] & G \oplus G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & +_*(G' \oplus G'') \arrow[r] & G \oplus G \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] & \Delta^* +_* (G' \oplus G'') \arrow[r] \arrow[u] \arrow[ur, phantom, "\llcorner"] & G \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

Here, the square marked with $\llcorner$ is cartesian while the one marked with $\lrcorner$ is cocartesian. The Baer sum of G' and G'' is $\Delta^* +_* (G' \oplus G'')$. Now consider the case of 1.3 where we are given an extension of commutative algebraic group schemes over S, and G' is an A-torsor over G. Pulling back G' along addition for G yields the fiber product $+_G^* G' = G' \times_G G \times_S G$. If X is a scheme over S, to give a morphism over S to $+_G^* G'$ amounts to giving two objects $g_1, g_2 \in G(X)$ and an object in $g \in G'(X)$ such that $\pi g = g_1 + g_2$. This data is equivalent to the data of two objects g_1', g_2' in $G'(X)$ mapping to g_1, g_2, respectively, modulo the relation which considers two such pairs equivalent if their sum is the same. This generalizes pushing out by addition on A to the case of torsors. The pullback of torsors is given on representing objects by fiber product, so that given an S-morphism $f : X \to G$, $f^*T = \mathrm{Hom}_X(-, X \times_{f,\pi} G')$. This torsor is trivial if there is a section $X \to X \times_{f,\pi} G'$ over X; i.e. a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & G' \arrow[d, "\pi"] \\ & G. \end{tikzcd}$$

In other words, the torsor f^*T is defined up to isomorphism by the image of f in $\mathrm{Hom}_S(X, G)/\pi\,\mathrm{Hom}_S(X, G')$.

If $g : X \to G$ is another S-morphism, then g^*T is represented by $X \times_{g,\pi} G'$. The fiber products $X \times_{f,\pi} G'$ and $X \times_{g,\pi} G'$ are A-torsors over X. In analogy with Baer sum, a sum of torsors is given by pushing out by the addition on A followed by pulling back by the diagonal $\Delta_X : X \to X \times_S X$. Since G′ is defined by an extension and both torsors come from pulling back G′ (i.e. T), this is the same as pulling back G′ along the composition of morphisms

$$X \xrightarrow{(f,g)} G \times_S G \xrightarrow{+} G.$$

More explicitly, the morphism $X \to G \times_S G$ factors as

$$X \xrightarrow{\Delta} X \times_S X \xrightarrow{(f\circ \mathrm{pr}_1, g\circ \mathrm{pr}_2)} G \times_S G$$

and therefore pulling back along this morphism corresponds to pulling back (f^*T, g^*T) along the diagonal Δ_X. Of course, the fiber product $G' \times_{G'} (G' \times_S G') \simeq G' \times_S G'$ corresponds to the $A \times A$-torsor (T, T). As we have already seen, pulling back G′ along $G \times_S G \xrightarrow{+} G$ is the quotient of (T, T) by the relation the action of (a_1, a_2) and (a'_1, a'_2) if $a_1 + a_2 = a'_1 + a'_2$. Therefore, $(f + g)^*T$ is the quotient of (f^*T, g^*T) by this same relation, which verifies the formula

$$(f + g)^*T = f^*T + g^*T.$$

The Lang torsor. For the development of the Lang isogeny and its properties, see Borel, *Algebraic Groups* §16. There, he proves that $\mathscr{L}$ is surjective when G^0 is connected. If the group G^0 is smooth (i.e. geometrically reduced), then $\mathscr{L}$ is seen to be étale (as a morphism of schemes) by computing its differential and Milne, *Algebraic Groups*, 1.63.

That the functor $G_0^F = G_0(\mathbf{F}_q)$ is a consequence of the description of F on the set $G_0(\mathbf{F})$; see *Rapport* 1.1.

1.6. If G_0 is a commutative algebraic group defines over $\mathbf{F}_q$, then the norm map

$$N : G_0(\mathbf{F}_{q^n}) \longrightarrow G_0(\mathbf{F}_q) \subset G_0(\mathbf{F}_{q^n})$$

is defined, as F is a group homomorphism. In the case $G_0 = \mathbf{G}_a$, then $\mathbf{G}_a(\mathbf{F}_{q^n}) = \mathbf{F}_{q^n}$ and N coincides with the field *trace* $N(x) = x + x^q + \cdots + x^{q^{n-1}}$, as the group operation is addition. If $G_0 = \mathbf{G}_m$, on the other hand, then $\mathbf{G}_m(\mathbf{F}_{q^n}) = \mathbf{F}_{q^n}^\times$ and N coincides with the field *norm* $N(x) = x \cdot x^q \cdots x^{q^{n-1}}$, as the group operation is multiplication.

1.7. The formulæ $\mathscr{F}(\chi, f_0) = f_0^*\mathscr{F}(\chi, \mathrm{id}_{G_0})$, etc. result from the fact that the pullback of lisse sheaves corresponds to the homomorphism of π_1 induced by f_0; i.e. if $x \longrightarrow X_0$ is a geometric point and $g = f_0(x)$, then both sides of these identities result from the composition of maps

$$\pi_1(X_0, x) \longrightarrow \pi_1(G_0, g) \xrightarrow{L_0} G_0(\mathbf{F}_q) \xrightarrow{\chi^{-1}} E_\lambda^*,$$

where the L_0 indicates the homomorphism of groups defining the Lang torsor $L_0(G_0)$.

The formula that motivates this whole business

$$F_x^* = \chi f_0(x)$$

follows from the hint that the fiber in x of the morphism 1.2.2 commutes with F_x^*. This is true for the following reason: the geometric point $x \in X^F$ and f_0 give homomorphisms

$$\mathrm{Gal}(\mathbf{F}, \mathbf{F}_q) \longrightarrow \pi_1(X_0, x) \longrightarrow \pi_1(G_0, x)$$

and F_x^{*-1} coincides with the image of the Frobenius substitution $\varphi \in \mathrm{Gal}(\mathbf{F}, \mathbf{F}_q)$. Commutation of F_x^* with f_0^* is clear and it remains to show that ρ_x of 1.2.2 is $\pi_1(X, x)$-equivariant. The formula (†) of the note to 1.2 shows this explicitly.

1.8. (ii) The inverse image by f of the the E_λ-sheaf obtained from $\chi^{-1}(L_0(G_0))$ by extension of scalars from $\mathbf{F}_q$ to k is probably better written $f^*\mathscr{F}(\chi)_1$ than $\mathscr{F}(\chi f)$.

1.10. The subtlety in this argument is building a bridge between $\mathbf{C}$ and $\mathbf{F}_q$. Of course, the isomorphisms on singular cohomology with $\mathbf{Q}$ coefficients imply isomorphisms on singular cohomology with $\mathbf{Q}_\ell$ coefficients for all ℓ. The quadric hypersurface X'_0 and hyperplane Y'_0 in $\mathbf{P}_0^{2N}$ over $\mathbf{F}_q$ are each defined by the vanishing of a homogeneous polynomial of degree two and one, respectively, in the ring $\mathbf{F}_q[X_0, X_1, \ldots, X_{2N}]$; let f_0 denote the polynomial defining X'_0. As X'_0 is a nonsingular variety over a perfect field, it is smooth over $\mathbf{F}_q$. Let $m(x)$ be the minimal polynomial for a primitive element of $\mathbf{F}_q$ over $\mathbf{F}_p$; lifting the coefficients of m to $\mathbf{Z}$, m remains irreducible, and defines a finite extension of domains $\mathbf{Z} \to \mathbf{Z}[x]/(m(x)) =: A$ so that $A/(p) \simeq \mathbf{F}_q$. The coefficients of f_0 now admit lifts to A and define a projective quadric X'_S in $\mathbf{P}_S^{2N}$ which is, in particular, flat over $S := \operatorname{Spec} A$, and such that the fiber over (p) is X'_0. Recalling EGA IV 12.2.4, the set of points $s \in S$ such that $(X'_S)_s$ is smooth over $k(s)$ is open. In particular, the fiber over the generic point $\xi \in S$ is smooth over $k(\xi)$, which is a finite extension of $\mathbf{Q}$. The strict henselization $\tilde{A}$ of A at (p) is a regular local ring with spectrum $\tilde{S}$; $X'_{\tilde{S}} \to \tilde{S}$ is proper and smooth, and $k(\tilde{\xi})$ is an algebraic extension of $k(\xi)$. Let $\operatorname{Spec} \mathbf{C} = t \to \tilde{S}$ be a geometric point centered on $\tilde{\xi}$, and put $X := (X'_{\tilde{S}})_t$. The specialization morphism

$$H^*(X', \mathbf{Q}_\ell) \xrightarrow{\sim} H^*(X, \mathbf{Q}_\ell)$$

is an isomorphism (Arcata V 3.1). As X is a projective nonsingular quadric in $\mathbf{P}_{\mathbf{C}}^{2N}$, the comparison theorem between ordinary cohomology and étale cohomology (Arcata V 3.5.1) allow us to apply the transcendental argument to conclude.

1.13. To compute $R^i f_! \mathbf{Q}_\ell$, we may assume Y is the spectrum of a separably closed field and $X = \mathbf{A}^1$. By Poincaré duality and Artin's theorem, $H_c^0(\mathbf{A}^1, \mathbf{Q}_\ell) = 0$. As for H_c^1, the short exact sequence of sheaves on $\mathbf{P}^1$

$$0 \to (\mathbf{Q}_\ell)_{\mathbf{A}^1} \to \mathbf{Q}_\ell \to (\mathbf{Q}_\ell)_\infty \to 0$$

gives rise to a long exact sequence of cohomology

$$0 \to \mathbf{Q}_\ell \to \mathbf{Q}_\ell \to H_c^1(\mathbf{A}^1, \mathbf{Q}_\ell) \to H^1(\mathbf{P}^1, \mathbf{Q}_\ell) = 0 \qquad (\text{as } \operatorname{Pic}^0(\mathbf{P}^1) = 0),$$

verifying $\mathrm{H}^1_c(\mathbf{A}^1, \mathbf{Q}_\ell) = 0$. Evidently $\mathrm{H}^0(\mathbf{A}^1, \mathbf{Q}_\ell) = \mathbf{Q}_\ell$; by Poincaré we conclude

$$\begin{cases} \mathrm{R}^i f_! \mathbf{Q}_\ell = 0 & i \neq 2 \\ \mathrm{R}^2 f_! \mathbf{Q}_\ell = \mathbf{Q}_\ell(-1). \end{cases}$$

2.3*. Leray spectral sequence for cohomology with proper support is a particular case of spectral sequence of composed functors, but the proof that for all composition of morphisms $f = gh : \mathrm{X} \xrightarrow{h} \mathrm{Y} \xrightarrow{g} \mathrm{Z}$ we have 'well-behaved' transitivity isomorphisms between $\mathrm{R}f_!$ and $\mathrm{R}g_!\mathrm{R}h_!$ is not straight forward; it is SGAA Exp. XVII 5.1.8, which also proves that the functors $\mathrm{R}f_!$ are 'way out' and triangulated. The proof of transitivity is formal, relying on §3 of the same exposé, which reduces the problem to the analogous ones for proper morphisms and open immersions, provided one 'compatibility' isomorphism, which is 5.1.6. With transitivity in hand, the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{R}^p g_! \mathrm{R}^q h_!(\mathrm{K}) \Rightarrow \mathrm{R}^{p+1} f_!(\mathrm{K}),$$

valid for K in $\mathrm{D}(\mathrm{X}, \mathscr{A}_\mathrm{X})$ ($\mathscr{A}$ a sheaf of rings), is a consequence of a general spectral sequence written down by Verdier: for all L in $\mathrm{D}(\mathrm{Y}, \mathscr{A}_\mathrm{Y})$

$$\mathrm{E}_2^{pq} = \mathrm{R}^p g_!(\mathscr{H}^q(\mathrm{L})) \Rightarrow \mathrm{R}^{p+q} g_!(\mathrm{L});$$

see *Des catégories dérivées des catégories abéliennes* 4.4.6.

2.4*. Künneth formula in cohomology with proper support is SGA 4 Exp. XVII 5.4.

2.5*. The spectral sequence (2.5.2)* is the spectral sequence associated to a filtration of chain complex. Namely, the complements of the closed subsets in the filtration on X gives rise to a filtration on X by open subsets $\mathrm{U}_p \subset \mathrm{U}_{p+1}$. To calculate cohomology with proper support, fix a compactification $j : \mathrm{X} \hookrightarrow \overline{\mathrm{X}}$. In what follows, we write $\mathscr{F}$ for the sheaf and its various inverse images. Let $j_p : \mathrm{U}_p \hookrightarrow \overline{\mathrm{X}}$ and $i_{p+1} : \mathrm{U}_{p+1} - \mathrm{U}_p = \mathrm{X}_p - \mathrm{X}_{p+1} \hookrightarrow \overline{\mathrm{X}}$ be the immersions. The filtration $j_{p!}\mathscr{F} \subset j_{p+1!}\mathscr{F}$ on $j_!\mathscr{F}$ has successive quotients isomorphic to $i_{p+1!}\mathscr{F}$. In light of the spectral sequence (2.3.1)*, the spectral sequence associated to a filtered injective resolution of $j_!\mathscr{F}$ (Stacks `05TT`) gives rise (Stacks `015W`) to the spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^{p+q}(\overline{\mathrm{X}}, i_{p+1!}\mathscr{F}) = \mathrm{H}_c^{p+q}(\mathrm{X}_p - \mathrm{X}_{p+1}, \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(\overline{\mathrm{X}}, j_!\mathscr{F}) = \mathrm{H}_c^{p+q}(\mathrm{X}, \mathscr{F}).$$

Remark. Let C be a curve of finite type over an algebraically closed field k, P the set of rational points of the curve, $\mathscr{F}$ the constant sheaf with value A on C, and $x \in C(k)$. The fiber of $P_*P^*\mathscr{F}$ is not $\mathscr{F}_x$; rather, it can be identified with functions from P to A continuous for the discrete topology on P, modulo the relation that $f \sim g$ if $f(x) = g(x)$ and f and g disagree on only finitely many points.

2.6*. SGAA Exp. V §3 introduces the Čech complex and the Cartan-Leray spectral sequence associated to a covering, which is simply the spectral sequence of the composition of functors, where the first is inclusion of sheaves into presheaves and the second is $\check{H}^0$ (it is shown that the $\check{H}^i$ associated to the Čech complex are indeed right derived functors of $\check{H}^0$ on presheaves). The condition on the morphisms in the covering is simply that fibered products are representable, which is true in the category of schemes. However, this is not done for cohomology with support. In the introduction to SGAA Exp. Vbis, the Leray spectral sequence is discussed for an open covering and for a locally finite closed covering, and this guides SGAA Exp. XVII 6.2.8–6.2.10, which discusses in full detail the construction of the so-called 'extraordinary' spectral sequence (2.6.2)*. One can also obtain the more 'ordinary' spectral sequence of (2.6.1)* from the argument there by replacing the trace morphism $u_!u^* \to \mathrm{id}$ in the case of $u : Y \to X$ separated, étale, surjective, and finite type with the unit of adjunction $\mathrm{id} \to u_*u^*$ when u arises from a finite covering by closed subschemes. Then instead of getting a left resolution of the sheaf, one gets a right resolution, as in the usual Čech resolution.

Given $(X_i)_{i\in I}$ a finite covering by closed subschemes, let $u : Y = \coprod_i X_i := Y \to X$,

$$Y_n = \underbrace{Y \times_X \cdots \times_X Y}_{n+1 \text{ times}},$$

and $u_n : Y_n \to X$. The sheaves $\mathscr{F}_n = u_{n*}u_n^*\mathscr{F}$ form, via the units of adjunction $\mathscr{F} \to u_{n*}u_n^*\mathscr{F}$, a simplicial sheaf coaugmented by $\mathscr{F}$. As discussed in SGAA Exp. Vbis, if $\mathscr{C}^*(\mathscr{F})$ denotes an injective resolution of $\mathscr{F}$, the double complex

$$\mathscr{F} \to (u_{p*}\mathscr{C}^q(u_p^*(\mathscr{F})))_{p,q}$$

defines a resolution of $\mathscr{F}$ by injective sheaves.

$$\begin{tikzcd} \mathrm{Y} \arrow[rr, "u"] \arrow[dr, "f"'] & & \mathrm{X} \arrow[dl, "g"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

If f_n is the projection of Y_n to $\operatorname{Spec} k$, then $\mathrm{R}f_{n!} = \mathrm{R}g_! u_{n!} = \mathrm{R}g_! u_{n*}$. Applying the functor $g_!$ to the double complex and filtering by semi-simplicial degree gives rise to the Čech spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{R}^q f_{p!}(u_p^* \mathscr{F}) \Rightarrow \mathrm{R}^{p+q} g_! \mathscr{F};$$

this is (2.6.1)* for cohomology with support.

The problem when taking the X_i to be open is that $u_{n*} \neq u_{n!}$ and there is no longer a coaugmentation map $\mathrm{id} \to u_{n!}u_n^*$. The trace morphism, however, provides an augmentation map $u_{n!}u_n^* \to \mathrm{id}$ and one proceeds by resolving to the left to produce the extraordinary Čech resolution and spectral sequence

$$\mathrm{E}_1^{-p,q} = \mathrm{R}^q f_{p!}(u_p^* \mathscr{F}) \Rightarrow \mathrm{R}^{-p+q} g_! \mathscr{F};$$

this is (2.6.2)*. For more on the extraordinary Čech resolution, see SGAA Exp. XVII 6.2.8–6.2.10.

Bibliography

[Sommes trig.] *Application de la formule des traces aux sommes trigonométriques* dans SGA $4\frac{1}{2}$.

6. *Transformation de Fourier*

As a warm-up, let's recall some Galois theory from SGA 1 in connection with the beginning of *Sommes trig*. All references in this paragraph are to SGA 1 Exposé V. The Lang isogeny for $\mathbf{G}_{a,\mathbf{F}_q}$ is written

$$0 \longrightarrow \mathbf{F}_q \longrightarrow \mathbf{G}_a \xrightarrow{x^q - x} \mathbf{G}_a \longrightarrow 0$$

and is a revêtement étale called the Artin-Schreier revêtement. The sheaf of sections defines an $\mathbf{F}_q$-torsor. The Galois group of the Artin-Schreier revêtement is therefore $\mathbf{F}_q$, as a connected torsor under a finite group G has Galois (= automorphism) group G. Just as in the theory of fields, the choice of words 'Galois group' in place of 'automorphism group' is reserved for Galois objects in the Galois category. Remark 5.11 characterizes the Galois objects in a Galois category $\mathscr{C}$ equipped with fiber functor F as the connected torsors X under a finite group G (torsor = principal homogeneous space). The implications go as follows: an object X is galoisian if it is connected, not isomorphic to $\emptyset_{\mathscr{C}}$, the initial object of $\mathscr{C}$ ($\Leftrightarrow$ $\mathrm{F(X)} \neq \emptyset$), and $\mathrm{Aut\,X}$ is transitive ($\Leftrightarrow$ simply transitive) (N° 4, f) & 5.4). X is a torsor under the group opposite Aut X iff F(X) is a torsor under the group opposite Aut X; i.e. Aut X acts simply transitively. Therefore X is a connected torsor under the group opposite Aut X. On the other hand, suppose X is a torsor under G, i.e. G acts on X on the right and on F(X) simply transitively, yielding a natural injection from G into the group opposite Aut X. As G acts simply transitively, Aut X acts transitively. If moreover X is connected, then N° 4 f) gives that Aut X acts transitively iff it acts simply transitively, showing that the injection above is actually an isomorphism between G and the group opposite Aut X. This is justification for the fact 'In a Galois category, a connected torsor under a finite group G is a Galois object with Galois group G.'

1.1.1.5. En vue de l'additivité de t (0.9), l'énoncé $t_{\mathrm{R}f_*\overline{\mathbf{Q}}_{\ell,\mathrm{X}}}(\infty) = 1 - q$ résulte de la pureté relative pour $\mathbf{A}^1 \overset{j}{\hookrightarrow} \mathbf{P}^1 \hookleftarrow \infty$ (Arcata V 3.4) en ce que

$$\begin{cases} j_*\mathbf{Q}_\ell = \mathbf{Q}_\ell \\ \mathrm{R}^1 j_*\mathbf{Q}_\ell = \mathbf{Q}_\ell(-1)_\infty \\ \mathrm{R}^q j_*\mathbf{Q}_\ell = 0 \text{ pour } q \geq 2. \end{cases}$$

1.1.3. The action of Frobenius on the fiber is clearer in *Sommes trig.* 1.5, 1.6 since Deligne's notation for Frobenius is clear and consistent.

The rigidification (1.1.3.1) is a consequence of the fact that there is a distinguished element of $\mathrm{L}^{-1}(1)$, namely 1.

The trivialization (1.1.3.2) depends on the construction of $\mathscr{L}_\chi$ from a torsor defined by an extension. In the setting of *Sommes trig.* 1.3, the A-torsor T is defined as the sheaf of local sections of an extension π. As H represents $m^*\mathrm{T}$ in the commutative diagram with exact rows ($\ulcorner$ = cartesian and m denotes the group law)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{A}\times\mathrm{A} \arrow[r] \arrow[d, "m"] & \mathrm{G}'\times\mathrm{G}' \arrow[r, "\pi\times\pi"] \arrow[d] & \mathrm{G}\times\mathrm{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{A} \arrow[r] \arrow[d, equal] & \mathrm{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{G}\times\mathrm{G} \arrow[r] \arrow[d, "m"] & 0 \\
0 \arrow[r] & \mathrm{A} \arrow[r] & \mathrm{G}' \arrow[r, "\pi"] & \mathrm{G} \arrow[r] & 0,
\end{tikzcd}$$

in the language of torsors there is a canonical isomorphism

$$m(\mathrm{T}\times\mathrm{T}) = m^*\mathrm{T}.$$

The structure of group on G provides commutative diagrams

$$
\begin{tikzcd}
\mathrm{G} & & & & \mathrm{G} \\
\mathrm{G}\times * \arrow[u, "\sim"] \arrow[rr, "\mathrm{id}\times 1"] \arrow[drr, "\sim"'] & & \mathrm{G}\times\mathrm{G} \arrow[ull, "\mathrm{pr}_1"'] \arrow[urr, "\mathrm{pr}_2"] \arrow[d, "m"] & & * \times \mathrm{G} \arrow[u, "\sim"'] \arrow[ll, "1\times\mathrm{id}"'] \arrow[dll, "\sim"] \\
& & \mathrm{G} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(\mathrm{G},1) & & & & \pi_1(\mathrm{G},1) \\
\pi_1(\mathrm{G},1) \arrow[u, "\sim"] \arrow[rr] \arrow[drr, "\sim"'] & & \pi_1(\mathrm{G}\times\mathrm{G},(1,1)) \arrow[ull, "\mathrm{pr}_1"'] \arrow[urr, "\mathrm{pr}_2"] \arrow[d, "m"] & & \pi_1(\mathrm{G},1) \arrow[u, "\sim"'] \arrow[ll] \arrow[dll, "\sim"] \\
& & \pi_1(\mathrm{G},1) & &
\end{tikzcd}
$$

The diagram

$$
\begin{tikzcd}
\pi_1(\mathrm{G}\times\mathrm{G},(1,1)) \arrow[r, "\mathrm{pr}_1\times\mathrm{pr}_2"] & \pi_1(\mathrm{G},1)\times\pi_1(\mathrm{G},1) \arrow[r, "m"] & \pi_1(\mathrm{G},1) \\
& \circlearrowright & \circlearrowright \\
& \mathrm{A}\times\mathrm{A} & \mathrm{A}
\end{tikzcd}
$$

gives the action of $\pi_1(\mathrm{G}\times\mathrm{G},(1,1)$ on the torsors $\mathrm{T}\times\mathrm{T}$ and $m(\mathrm{T}\times\mathrm{T}) = m^*\mathrm{T}$. Pushing by χ^{-1} gives a representation of $\pi_1(\mathrm{G}\times\mathrm{G},(1,1))$ on $\overline{\mathbf{Q}}_\ell$ which is induced by

$$(g_1, g_2) \mapsto \chi^{-1}(g_1)\otimes\chi^{-1}(g_2),$$

demonstrating explicitly an isomorphism

$$m^*\mathscr{L}_\chi \simeq \mathrm{pr}_1^*\mathscr{L}_\chi \otimes \mathrm{pr}_2^*\mathscr{L}$$

and hence a trivialization of $\mathscr{D}_2(\mathscr{L}_\chi)$ compatible with the identification of the fibers of $\mathscr{D}_2(\mathscr{L}_\chi)$ and $\overline{\mathbf{Q}}_\ell$ at $(1,1)$.

(1.1.3.3) = *Sommes trig.* 1.7.7 (see note to *Sommes trig.* 1.6)

(1.1.3.4) = *Sommes trig.* 2.7*.

The remark (1.1.3.7) can be summed up by the morphism of torsors induced by the commutative diagram below, the projective limit of which defines $\varprojlim_{\mathrm{I}(k)} \mu_{\mathrm{N}}(k)$:

$$
\begin{tikzcd}
1 \arrow[r] & \mu_{\mathrm{NM}}(k) \arrow[r] \arrow[d, "{[\mathrm{M}]}"] & \mathbf{G}_{m,k} \arrow[r, "{[\mathrm{NM}]}"] \arrow[d, "{[\mathrm{M}]}"] & \mathbf{G}_{m,k} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mu_{\mathrm{N}}(k) \arrow[r] & \mathbf{G}_{m,k} \arrow[r, "{[\mathrm{N}]}"] & \mathbf{G}_{m,k} \arrow[r] & 1.
\end{tikzcd}
$$

As $\mathbf{F}_q^\times \simeq \mathbf{Z}/(q-1)$, $\mathrm{I}(\mathbf{F}_q) = \{n \in \mathbf{N}_{>0} : n|q-1\}$.

1.2.1. The pairing $\langle\ ,\ \rangle : \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \to \mathbf{G}_{a,k}$ should rather land in $\mathbf{G}_{a,\mathrm{S}}$. See *Sommes trig.* 1.7 c) to make sense of what this does to $\mathscr{L}_\psi$.

To be maximally pedantic, if $q = p^n$, with the notation of (1.1.3.3) we should write ψ_n where ψ_q appears in the definition of $\hat{t}$.

1.2.2. Locally $\mathrm{S} = \operatorname{Spec} \mathrm{A}$, $\mathrm{E} = \operatorname{Spec} \mathrm{A}[t_1, \ldots, t_r]$, $\mathrm{E} \times_{\mathrm{S}} \mathrm{E} = \operatorname{Spec} \mathrm{A}[t_1, \ldots, t_r, t'_1, \ldots, t'_r]$, and the addition $\mathrm{E} \times_{\mathrm{S}} \mathrm{E} \to \mathrm{E}$ is given by $t_i \mapsto t_i + t'_i$ while $[-1]$ is given by $t_i \mapsto -t_i$. More to the point, writing $\mathscr{E} \simeq \mathbf{Spec}\,\mathrm{Sym}(\mathscr{E})$ with $\mathrm{Sym}(\mathscr{E})$ the symmetric algebra on a locally free sheaf $\mathscr{E}$, then addition of sections gives addition on E, and vice versa, as the sheaf of local sections of E coincides with $\mathscr{E}^\vee$.

(1.2.2.1) Identifying E with E″ via $e \mapsto \langle e,\ \rangle = -a(e)$, the diagram becomes the base change by $\pi' : \mathrm{E}' \to \mathrm{S}$ of addition on E″,

$$\begin{tikzcd}
 & \mathrm{E}'' \times_{\mathrm{S}} \mathrm{E}'' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}"] \arrow[rr, "{(e''_1,e''_2)\mapsto e''_1+e''_2}"] & & \mathrm{E}'' \\
\mathrm{E}'' \arrow[dr] & & \mathrm{E}'' \arrow[dl] & \\
 & \mathrm{S} & &
\end{tikzcd}$$

and the isomorphism

$$\mathrm{pr}_{12}^* \mathscr{L}(\langle\ ,\ \rangle) \otimes \mathrm{pr}_{23}^* \mathscr{L}(\langle\ ,\ \rangle) = \alpha^* \mathscr{L}(\langle\ ,\ \rangle)$$

becomes (1.1.3.2) on the nose.

$$\begin{aligned}
\mathscr{F}' \circ \mathscr{F}(\mathrm{K}) &= \mathrm{R}\,\mathrm{pr}''_!(\mathrm{pr}'^*(\mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}^*(\mathrm{K}) \otimes \mathscr{L}(\langle\ ,\ \rangle))) \otimes \mathscr{L}(\langle\ ,\ \rangle))[2r]\\
&\simeq \mathrm{R}(\mathrm{pr}'' \circ \mathrm{pr}_{23})_!((\mathrm{pr} \circ \mathrm{pr}_{13})^*(\mathrm{K}) \otimes \mathrm{pr}_{13}^* \mathscr{L}(\langle\ ,\ \rangle) \otimes \mathrm{pr}_{23}^* \mathscr{L}(\langle\ ,\ \rangle))[2r]\\
&\simeq \mathrm{R}(\mathrm{pr}'' \circ \mathrm{pr}_{23})_!((\mathrm{pr} \circ \mathrm{pr}_{13})^*(\mathrm{K}) \otimes \alpha^* \mathscr{L}(\langle\ ,\ \rangle))[2r]\\
&\simeq \mathrm{R}(\mathrm{pr}'' \circ \mathrm{pr}_{13})_!((\mathrm{pr} \circ \mathrm{pr}_{13})^*(\mathrm{K}) \otimes \alpha^* \mathscr{L}(\langle\ ,\ \rangle))[2r]\\
&\simeq \mathrm{R}\,\mathrm{pr}''_!(\mathrm{pr}^*\,\mathrm{K} \otimes \mathrm{R}\,\mathrm{pr}_{13!}\,\alpha^* \mathscr{L}(\langle\ ,\ \rangle))[2r]\\
&\simeq \mathrm{R}\,\mathrm{pr}''_!(\mathrm{pr}^*\,\mathrm{K} \otimes \beta^* \mathrm{R}\,\mathrm{pr}''_!\, \mathscr{L}(\langle\ ,\ \rangle))[2r].
\end{aligned}$$

In (1.2.2.2), the first equation of the proof is missing a π'^*; it should read

$$\mathscr{F}(\pi^*\mathrm{L}[r]) = \pi'^*\mathrm{L} \otimes \mathrm{R}\,\mathrm{pr}'_!\, \mathscr{L}(\langle\ ,\ \rangle)[2r].$$

The rest of the proof of (1.2.2.2) is straightforward. It allows us to write

$$\begin{aligned}
\mathscr{F}' \circ \mathscr{F}(\mathrm{K}) &\simeq \mathrm{R}\,\mathrm{pr}''_!(\mathrm{pr}^*\,\mathrm{K} \otimes \beta^* \mathscr{F}'(\overline{\mathbf{Q}}_{\ell,\mathrm{E}'}[r]))\\
&\simeq \mathrm{R}\,\mathrm{pr}''_!(\mathrm{pr}^*\,\mathrm{K} \otimes \beta^* \sigma''_* \overline{\mathbf{Q}}_{\ell,\mathrm{S}}(-r))\\
&\simeq a_* \mathrm{K}(-r),
\end{aligned}$$

as $\beta^*\sigma''_*$ is supported precisely on the locus where $e'' = a(e)$, and after base change along $\mathrm{E} \times_{a,\mathrm{id}} \mathrm{E}'' \to \mathrm{E} \times_{\mathrm{S}} \mathrm{E}''$, which commutes with $\mathrm{pr}''_!$, we find $\mathrm{R}\,\mathrm{pr}''_!\,\mathrm{pr}^*\,\mathrm{K} = a_*$.

$$\begin{tikzcd}
 & \mathrm{E} \times_{a,\mathrm{id}} \mathrm{E}'' \arrow[rr] \arrow[dl] \arrow[dr] & & \mathrm{E} \times_{\mathrm{S}} \mathrm{E}'' \arrow[dl] \arrow[dr] & \\
\mathrm{E} \arrow[dr, "a"'] & & \mathrm{E}'' \arrow[dl, "\mathrm{id}"] \arrow[dr, "\pi''"'] & & \mathrm{E} \arrow[dl, "\pi"] \\
 & \mathrm{E}'' & & \mathrm{S} &
\end{tikzcd}$$

Remark. Identifying $\mathrm{E} \simeq (\mathrm{E}^\vee)^\vee$ in the canonical way, the involutivity can be written

$$\mathscr{F}' \circ \mathscr{F}(\mathrm{K}) \simeq [-1]_* \mathrm{K}(-r),$$

and the inverse image under $[-1]$ of this identity is $\mathscr{F}'_{\psi^{-1}} \circ \mathscr{F}(\mathrm{K}) \simeq \mathrm{K}(-r)$, since $[-1]^* \mathscr{F}'_\psi = \mathscr{F}'_{\psi^{-1}}$ in light of the fact that $[-1]^*\psi(\langle e'', e' \rangle) = \psi(\langle -e, e' \rangle) = \psi^{-1}(\langle e, e' \rangle)$.

(1.2.2.4) The adjunction upon which the equality rests is

$$\langle f(s), \varphi\rangle_2 = \langle s, f'^{*}\varphi := \varphi \circ f\rangle_1,$$

as pullback by f' on a section φ of E'_2 gives, by definition, the element of E'_1 which, to a section s of E_1, applies f then φ. We then write

$$\begin{aligned}
\mathscr{F}_2(\mathrm{R}f_!\mathrm{K}_1) &= \mathrm{R}\,\mathrm{pr}'_{2!}(\mathrm{pr}_2^*(\mathrm{R}f_!\mathrm{K}_1) \otimes \mathscr{L}(\langle\ ,\ \rangle))[r_2]\\
&\simeq \mathrm{R}\,\mathrm{pr}'_{2!}(\mathrm{R}(f\times 1)_!(\mathrm{pr}_1^*\,\mathrm{K}_1) \otimes \mathscr{L}(\langle\ ,\ \rangle)_2)[r_2]\\
&\simeq \mathrm{R}(\mathrm{pr}'_2 \circ (f\times 1))_!(\mathrm{pr}_1^*\,\mathrm{K}_1 \otimes (f\times 1)^*\mathscr{L}(\langle\ ,\ \rangle)_2)[r_2]\\
&\simeq \mathrm{R}(\mathrm{pr}'_2 \circ (f\times 1))_!((\mathrm{pr}_1 \circ (1\times f'))^*\mathrm{K}_1 \otimes (1\times f')^*\mathscr{L}(\langle\ ,\ \rangle)_1)[r_2]\\
&\simeq \mathrm{R}\,\mathrm{pr}'_{2!}(1\times f')^*\mathrm{R}(f\times 1)_!(\mathrm{pr}_1^*\,\mathrm{K}_1 \otimes \mathscr{L}(\langle\ ,\ \rangle)_1)[r_2]\\
&\simeq f'^*\mathrm{R}\,\mathrm{pr}'_{1!}(\mathrm{pr}_1^*\,\mathrm{K}_1 \otimes \mathscr{L}(\langle\ ,\ \rangle)_1)[r_2]\\
&\simeq f'^*\mathscr{F}_1(\mathrm{K}_1)[r_2 - r_1].
\end{aligned}$$

(1.2.2.5) If $(\pi')'$ is the morphism of dual bundles $\mathrm{S} \to \mathrm{E}''$ induced by π', then $(\pi')'^*a_* \simeq \sigma^*$. To make sense of the morphism $(\pi')' : \mathrm{S} \to \mathrm{E}''$, you have to unpack what it means to consider S as a vector bundle of rank 0 over S. The only section of S is the zero section, as at all points of S, as a rank 0 vector bundle returns the vector space $\{0\}$ at every point of S. In other words, we dispose of a tautological isomorphim $\mathrm{S} \xrightarrow{\sim} \mathrm{S} \times_k * = \mathrm{S} \times \{0\}$. Thinking about $\pi' : \mathrm{E}' \to \mathrm{S}$ as sending $(s, e') \mapsto (s, 0)$, the map $(\pi')' : \mathrm{S} \to \mathrm{E}''$ turns the 0 section of S into a section of E″, i.e. the one which to a section s' of E′ returns the 0 section of S. Therefore $(\pi')'$ is the embedding by zero section $0 : \mathrm{S} \hookrightarrow \mathrm{E}''$; as $-0 = 0$, the zero section of E″ corresponds under the isomorphism a to the zero section of E, which sees $(\pi')'^*a_* \simeq \sigma^*$.

(1.2.2.7) For the isomorphism

$$\mathscr{F}(\mathrm{K}_1 \boxtimes_{\mathrm{S}} \mathrm{K}_2) \simeq \mathscr{F}(\mathrm{K}_1) \boxtimes_{\mathrm{S}} \mathscr{F}(\mathrm{K}_2),$$

perhaps it helps to write the commutative diagram

$$\begin{tikzcd}
\mathrm{E} & \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \arrow[l] \arrow[r] & \mathrm{E} \\
\mathrm{E}\times_{\mathrm{S}}\mathrm{E}' \arrow[u] \arrow[d] & (\mathrm{E}\times_{\mathrm{S}}\mathrm{E})\times_{\mathrm{S}}(\mathrm{E}'\times_{\mathrm{S}}\mathrm{E}') \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[u] \arrow[d] & \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \arrow[u] \arrow[d] \\
\mathrm{E}' & \mathrm{E}'\times_{\mathrm{S}}\mathrm{E}' \arrow[l] \arrow[r] & \mathrm{E}'.
\end{tikzcd}$$

If one imagines this diagram drawn on the xy-plane in 3D with the center at $(0,0,0)$, places S at $(0,0,1)$, and connects every node of this diagram to this S, every row and column of the 3×3 above will then form a cartesian square with this S. The equality of Fourier transform then follows from the Künneth formula in light of the isomorphism

$$\mathrm{pr}_1^*\mathscr{L}(\langle\ ,\ \rangle_{\mathrm{E}_1})\otimes\mathrm{pr}_2^*\mathscr{L}(\langle\ ,\ \rangle_{\mathrm{E}_2})\simeq\mathscr{L}(\langle\ ,\ \rangle_{\mathrm{E}_1\oplus\mathrm{E}_2}),$$

which is an obvious consequence of the fact that given vector bundles $\mathrm{E}_1,\mathrm{E}_2$,

$$\langle\ ,\ \rangle_{\mathrm{E}_1\oplus\mathrm{E}_2}=\langle\ ,\ \rangle_{\mathrm{E}_1}+\langle\ ,\ \rangle_{\mathrm{E}_2}.$$

The character ψ carries this additive identity to a multiplicative one, hence the $\otimes$.

To go from here to the stated isomorphism

$$\mathscr{F}(\mathrm{K}_1*\mathrm{K}_2)\simeq\mathscr{F}(\mathrm{K}_1)\otimes\mathscr{F}(\mathrm{K}_2)[-r],$$

after applying (1.2.2.4) in the given way, one is faced with the problem of justifying

$$s'^*(\mathscr{F}(\mathrm{K}_1)\boxtimes_{\mathrm{S}}\mathscr{F}(\mathrm{K}_2))\simeq\mathscr{F}(\mathrm{K}_1)\otimes\mathscr{F}(\mathrm{K}_2).$$

It is useful to reason by adjunction, writing the following adjoint pair of diagrams.

$$\begin{tikzcd}
 & \mathrm{E}' \arrow[d, "s'"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathrm{E}'\times_{\mathrm{S}}\mathrm{E}' \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
\mathrm{E}' & & \mathrm{E}'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{E} & \\
 & \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \arrow[u, "s"'] & \\
\mathrm{E}\times_{\mathrm{S}}\{0\} \arrow[ur, "\sigma_1"'] \arrow[uur, bend left, "\mathrm{id}"] & & \{0\}\times_{\mathrm{S}}\mathrm{E} \arrow[ul, "\sigma_2"] \arrow[uul, bend right, "\mathrm{id}"']
\end{tikzcd}$$

The point is that given e, e' sections of E, E' respectively,

$$\langle e, (\mathrm{pr}_i \circ s')(e')\rangle = \langle \sigma_i(e), s'(e')\rangle = \langle (s \circ \sigma_i)(e), e'\rangle = \langle e, e'\rangle.$$

Ergo,

$$s'^*(\mathscr{F}(\mathrm{K}_1) \boxtimes_{\mathrm{S}} \mathscr{F}(\mathrm{K}_2)) \simeq (\mathrm{pr}_1 \circ s')^* \mathscr{F}(\mathrm{K}_1) \otimes_{\mathrm{S}} (\mathrm{pr}_2 \circ s')^* \mathscr{F}(\mathrm{K}_2)) \simeq \mathscr{F}(\mathrm{K}_1) \otimes \mathscr{F}(\mathrm{K}_2).$$

1.2.3. Quelques détails supplémentaires pour les exemples donnés suivent.

(1.2.3.1) With $\sigma'_{\mathrm{F}} : \mathrm{S} \to \mathrm{F}'$,

$$\mathscr{F}(i_* \overline{\mathbf{Q}}_{\ell,\mathrm{F}}[s]) \simeq i'^* \mathscr{F}(\overline{\mathbf{Q}}_{\ell,\mathrm{F}}[s])[r-s] \simeq i'^* \sigma'_{\mathrm{F}*}(\overline{\mathbf{Q}}_{\ell,\mathrm{S}}(-s))[r-s];$$

now use proper base change for the cartesian square

$$\begin{tikzcd} \mathrm{E}' \arrow[d, "i'"] & \mathrm{F}^{\perp} \arrow[l, "i^{\perp}"'] \arrow[d, "\pi'"] \\ \mathrm{F}' & \mathrm{S}. \arrow[l, "\sigma'_{\mathrm{F}}"] \end{tikzcd}$$

(1.2.3.2) A few useless words: $e_* \overline{\mathbf{Q}}_{\ell,\mathrm{S}}$ is the sheaf on E that is supported precisely on the section e of E. Restricted to the closed subscheme which is the image of e, $e_* \overline{\mathbf{Q}}_{\ell,\mathrm{S}}$ is constant with value $\overline{\mathbf{Q}}_{\ell}$. Therefore $(e_* \overline{\mathbf{Q}}_{\ell,\mathrm{S}}) \boxtimes_{\mathrm{S}} \mathrm{K}$ has support contained in $e \times_{\mathrm{S}} \mathrm{E}$, and restricted to this closed subscheme, s is an isomorphism with inverse $\mathrm{E} \xrightarrow{\sim} (-e) \times_{\mathrm{S}} \mathrm{E} \xrightarrow{s} \mathrm{E} \xrightarrow{\sim} e \times_{\mathrm{S}} \mathrm{E}$. En effet, τ_e factors as $\mathrm{E} \xrightarrow{\sim} e \times_{\mathrm{S}} \mathrm{E} \xrightarrow{s} \mathrm{E}$.

After applying (1.2.2.7), one uses (1.2.3.1) to compute

$$\mathscr{F}(e_* \overline{\mathbf{Q}}_{\ell,\mathrm{S}}) \simeq e_*^{\perp} \overline{\mathbf{Q}}_{\ell,\mathrm{E}'}[r] \simeq \mathscr{L}(\langle e, \ \rangle)[r].$$

(1.2.3.3) The isomorphism α gives rise to a nondegenerate bilinear form $\mathrm{B} : \mathrm{E} \times_{\mathrm{S}} \mathrm{E} \to \mathbf{G}_{a,\mathrm{S}}$ via $\mathrm{B}(e_1, e_2) = \langle e_1, \alpha(e_2)\rangle$, and α is symmetric if B is; equivalently,

$$\langle \alpha^{-1}(e'), \alpha(e)\rangle = \langle e, e'\rangle.$$

This allows the easy verification of the identity

$$q(e) + \langle e, 2e'\rangle = q(e + \alpha^{-1}(e')) - q'(e'). \tag{†}$$

The cartesian square

$$\begin{tikzcd}
\mathrm{E}\times\mathrm{E}' \arrow[r, "\mathrm{id}\times[2]"] \arrow[d] & \mathrm{E}\times\mathrm{E}' \arrow[d] \\
\mathrm{E}' \arrow[r, "{[2]}"] & \mathrm{E}'
\end{tikzcd}$$

and proper base change gives

$$[2]^*\mathscr{F}(\mathscr{L}(q)) \simeq \mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}^*\mathscr{L}(q)\otimes\mathscr{L}(\langle\ ,2\ \rangle)),$$

where here we write $\mathscr{L}(\langle\ ,2\ \rangle$ for $(\mathrm{id}\times[2])^*\mathscr{L}(\langle\ ,\ \rangle)$. Let f denote the composition of maps in the diagram

$$\mathrm{E}\times\mathrm{E}' \xrightarrow{\mathrm{id}\times\alpha^{-1}} \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \xrightarrow{s} \mathrm{E}.$$

We have the following correspondences between functions and sheaves on $\mathrm{E}\times_{\mathrm{S}}\mathrm{E}'$.

$$\begin{aligned}
&q(e+\alpha^{-1}(e')) \longleftrightarrow f^*\mathscr{L}(q)\\
&-q'(e') \longleftrightarrow \mathscr{L}(-q')\\
&q(e) \longleftrightarrow \mathrm{pr}^*\mathscr{L}(q)\\
&\langle e,2e'\rangle \longleftrightarrow (\mathrm{id}\times[2])^*\mathscr{L}(\langle\ ,\ \rangle) =: \mathscr{L}(\langle\ ,2\ \rangle).
\end{aligned}$$

Disposing of this dictionary, the identity (†) gives an isomorphism of sheaves on $\mathrm{E}\times_{\mathrm{S}}\mathrm{E}'$

$$\mathrm{pr}^*\mathscr{L}(q)\otimes\mathscr{L}(\langle\ ,2\ \rangle) \simeq \mathrm{pr}'^*\mathscr{L}(-q')\otimes f^*\mathscr{L}(q)).$$

Therefore the stated isomorphism rests on showing that

$$\mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}'^*\mathscr{L}(-q)\otimes f^*\mathscr{L}(q)) \simeq \mathscr{L}(-q')\otimes\pi'^*\mathrm{R}\pi_!\mathscr{L}(q).$$

The following diagram is commutative with cartesian squares.

$$\begin{tikzcd}
\mathrm{E}\times_{\mathrm{S}}\mathrm{E}' \arrow[r, "\mathrm{id}\times\alpha^{-1}"] \arrow[d, "\mathrm{pr}'"] \arrow[rr, bend left, "f"] \arrow[dr, phantom, "\ulcorner"] & \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \arrow[r, "s"] \arrow[d, "\mathrm{pr}_2"] \arrow[dr, phantom, "\ulcorner"] & \mathrm{E} \arrow[d, "\pi"] \\
\mathrm{E}' \arrow[r, "\alpha^{-1}"] \arrow[rr, bend right, "\pi'"'] & \mathrm{E} \arrow[r, "\pi"] & \mathrm{S}
\end{tikzcd}$$

To see that the right square is cartesian, note that the isomorphism

$$\begin{aligned}\beta : \mathrm{E}\times_{\mathrm{S}}\mathrm{E} &\longrightarrow \mathrm{E}\times_{\mathrm{S}}\mathrm{E}\\ (e_1, e_2) &\longmapsto (e_1 - e_2, e_2)\end{aligned}$$

induces an isomorphism of cartesian squares

$$\begin{tikzcd} \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\beta"] \arrow[dd, bend right=60, "\mathrm{pr}_2"'] & \mathrm{E} \arrow[d, "\mathrm{id}"'] \arrow[dd, bend left=60, "\pi"] \\ \mathrm{E}\times_{\mathrm{S}}\mathrm{E} \arrow[r, "s"] \arrow[d, "\mathrm{pr}_2"] & \mathrm{E} \arrow[d, "\pi"'] \\ \mathrm{E} \arrow[r, "\pi"] & \mathrm{S}. \end{tikzcd}$$

By the projection formula and proper base change, we conclude

$$\mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}'^*\mathscr{L}(-q)\otimes f^*\mathscr{L}(q)) \simeq \mathscr{L}(-q')\otimes\mathrm{R}\,\mathrm{pr}'_!\,f^*\mathscr{L}(q) \simeq \mathscr{L}(-q')\pi'^*\mathrm{R}\pi_!\mathscr{L}(q).$$

(1.2.3.4) Let $p : \mathrm{G} \to \mathrm{S}$ denote the structure morphism and its various base extensions along π, π'. Let $f = (\mathrm{pr}_{\mathrm{G}}, m)^{-1}$, with transpose $f' = (\mathrm{pr}_{\mathrm{G}}, m')$. The diagram

$$\begin{tikzcd} \mathrm{G}\times_{\mathrm{S}}\mathrm{E}' \arrow[r, "f'"] \arrow[rr, bend right, "m'"'] & \mathrm{G}\times_{\mathrm{S}}\mathrm{E}' \arrow[r, "p"] & \mathrm{E}' \end{tikzcd}$$

commutes, so that

$$m'^*\mathscr{F}(\mathrm{K}) = f'^*p^*\mathscr{F}(\mathrm{K}) \simeq f'^*\mathscr{F}(p^*\mathrm{K}) \simeq \mathscr{F}(f_!p^*\mathrm{K}).$$

where in the first isomorphism we have used (1.2.2.9) and in the second (1.2.2.4). The commutative diagram

$$\begin{tikzcd} \mathrm{G}\times_{\mathrm{S}}\mathrm{E} \arrow[r, "f"] \arrow[rrr, bend left, "p"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{G}\times_{\mathrm{S}}\mathrm{E} \arrow[r, "(\mathrm{pr}_{\mathrm{G}},m)"] \arrow[rr, bend right, "m"'] & \mathrm{G}\times_{\mathrm{S}}\mathrm{E} \arrow[r, "p"] & \mathrm{E} \end{tikzcd}$$

together with the isomorphism $\mathrm{L} \xrightarrow{\sim} f_*f^*\mathrm{L} = f_!f^*\mathrm{L}$ for all L in $\mathrm{D}^b_c(\mathrm{G} \times_{\mathrm{S}} \mathrm{E}, \overline{\mathbf{Q}}_\ell)$ and the hypothesis $m^*\mathrm{K} \simeq \mathrm{M} \boxtimes_{\mathrm{S}} \mathrm{L}$ let us write

$$\mathscr{F}(f_!p^*\mathrm{K}) \simeq \mathscr{F}(m^*\mathrm{K}) \simeq \mathscr{F}(\mathrm{M} \boxtimes_{\mathrm{S}} \mathrm{L}).$$

Letting pr, pr′ denote $\mathrm{E} \xleftarrow{\mathrm{pr}} \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \xrightarrow{\mathrm{pr}'} \mathrm{E}'$ and their base extensions by p, the commutative diagrams below have cartesian diamonds marked.

$$\begin{tikzcd}
 & & \mathrm{G} \times_{\mathrm{S}} \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] \arrow[dd, phantom, "\wedge"] & & \\
 & \mathrm{G} \times_{\mathrm{S}} \mathrm{E} \arrow[dl, "p"'] \arrow[dr, "\pi"] & & \mathrm{G} \times_{\mathrm{S}} \mathrm{E}' \arrow[dl, "\pi'"'] \arrow[dr, "p"] & \\
\mathrm{E} & & \mathrm{G} & & \mathrm{E}'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathrm{G} \times_{\mathrm{S}} \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \arrow[dll, "\mathrm{pr}'"'] \arrow[dd, "\mathrm{pr}'"] \arrow[drr, "\ulcorner p"] & & \\
\mathrm{G} \times_{\mathrm{S}} \mathrm{E}' \arrow[dd, equal] & & & & \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \arrow[dd, "\mathrm{pr}'"] \\
 & & \mathrm{G} \times_{\mathrm{S}} \mathrm{E}' \arrow[dll, "\mathrm{id}"] \arrow[drr, "p"'] \arrow[dd, phantom, "\wedge"] & & \\
\mathrm{G} \times_{\mathrm{S}} \mathrm{E}' \arrow[drr] & & & & \mathrm{E}' \arrow[dll] \\
 & & \mathrm{E}' & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{M} \arrow[d, dash] & & \mathrm{G} \times_{\mathrm{S}} \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \arrow[dl, "\mathrm{pr}'"'] \arrow[dr, "p"] \arrow[dd, phantom, "\wedge"] & & \mathscr{L}(\langle\ ,\ \rangle) \arrow[dl, dash] \\
\mathrm{G} & \mathrm{G} \times_{\mathrm{S}} \mathrm{E}' \arrow[l, "\pi'"] \arrow[dr] & & \mathrm{E} \times_{\mathrm{S}} \mathrm{E}' \arrow[dl] \arrow[dr] & \mathrm{L} \arrow[d, dash] \\
 & & \mathrm{E}' & & \mathrm{E}
\end{tikzcd}$$

The projection formula and proper base change find

$$\begin{aligned}
\mathscr{F}(\mathrm{M} \boxtimes_{\mathrm{S}} \mathrm{L}) &\simeq \mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}^*(\mathrm{M} \boxtimes_{\mathrm{S}} \mathrm{L}) \otimes \mathscr{L}(\langle\ ,\ \rangle)) \\
&\simeq \mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}^*\,\pi^*\mathrm{M} \otimes \mathrm{pr}^*\,p^*\mathrm{L} \otimes \mathscr{L}(\langle\ ,\ \rangle)) \\
&\simeq \mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}'^*\,\pi'^*\mathrm{M} \otimes p^*(\mathrm{pr}^*\,\mathrm{L} \otimes \mathscr{L}(\langle\ ,\ \rangle))) \\
&\simeq \pi'^*\mathrm{M} \otimes \mathrm{R}\,\mathrm{pr}'_!(p^*(\mathrm{pr}^*\,\mathrm{L} \otimes \mathscr{L}(\langle\ ,\ \rangle))) \\
&\simeq \pi'^*\mathrm{M} \otimes p^*\mathscr{F}(\mathrm{L}) = \mathrm{M} \boxtimes_{\mathrm{S}} \mathscr{F}(\mathrm{L}).
\end{aligned}$$

1.2.3.5. As $E_1 \times_S E' \simeq E_1 \times_{S_1} E'_1 \simeq E \times_S E'_1$, the diagram below has cartesian diamonds.

$$\begin{tikzcd}
 & & S_1 & & \\
 & & E_1 \times_{S_1} E'_1 \arrow[dll, "\mathrm{pr}_1"'] \arrow[d, "f"] \arrow[drr, "\mathrm{pr}'_1"] & & \\
E_1 \arrow[uurr] \arrow[dr, "f_E"] & & E \times_S E' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] & & E'_1 \arrow[uull] \arrow[dl, "f_{E'}"] \\
 & E \arrow[dr] & & E' \arrow[dl] & \\
 & & S & &
\end{tikzcd}$$

In light of the fact that $\mathscr{L}(\langle\ ,\ \rangle)$ on $E_1 \times_{S_1} E'_1$ coincides with the inverse image under f of $\mathscr{L}(\langle\ ,\ \rangle)$ on $E \times_S E'$, proper base change and the projection formula for f give

$$\begin{aligned}\mathscr{F}(Rf_{E!}K_1) &\simeq R\,\mathrm{pr}'_!(Rf_!\,\mathrm{pr}_1^*\,K_1 \otimes \mathscr{L}(\langle\ ,\ \rangle)) \simeq R(\mathrm{pr}' \circ f)_!(\mathrm{pr}_1^*\,K_1 \otimes f^*\mathscr{L}(\langle\ ,\ \rangle)) \\ &\simeq R(f_{E'} \circ \mathrm{pr}'_1)_!(\mathrm{pr}_1^*\,K_1 \otimes \mathscr{L}(\langle\ ,\ \rangle)) \simeq Rf_{E'!}\mathscr{F}_1(K_1).\end{aligned}$$

1.3.1–1.3.2. I would like to advocate for something of a shortcut through these sections.

t-exactness of $\mathscr{F}$. Laumon deduces the t-exactness of $\mathscr{F}$ from the fact that the 'forget supports' map is an isomorphism. However, the following direct argument (lifted from the appendix to the reprinted Astérisque **100**) is immediate. As in the remark to the note to (1.2.2.1), $\mathscr{F}_{\psi^{-1}} \circ \mathscr{F}_\psi(K) = K(-r)$. As $\mathscr{F}_\psi$ is the composition of exact functors $\mathrm{pr}^*[n]$, $\mathscr{L}(\langle\ ,\ \rangle)$ and the left t-exact functor $\mathrm{pr}_!$ (BBD 4.1.2), it is left t-exact. But it is also left adjoint to its inverse, and this inverse is also left t-exact since up to a Tate twist it coincides with $\mathscr{F}_{\psi^{-1}}$. Therefore $\mathscr{F}_\psi$ is also right t-exact (BBD 1.3.17 (iii)).

'Forget supports'. The theorem (1.3.1.1) in Laumon's paper states that for all K in $D_c^b(E, \overline{\mathbf{Q}}_\ell)$, the 'forget supports' map

$$R\,\mathrm{pr}'_!(\mathrm{pr}^*\,K \otimes \mathscr{L}(\langle\ ,\ \rangle)) \to R\,\mathrm{pr}'_*(\mathrm{pr}^*\,K \otimes \mathscr{L}(\langle\ ,\ \rangle))$$

is an isomorphism. Laumon refers the reader to his paper with Katz, where they give an involved geometric proof that ends up yielding more. But Verdier gave the first proof of this isomorphism, and his proof is very short and completely formal. It can be found in Katz's 1988 Séminaire Bourbaki talk 'Travaux de Laumon.'

As for the proof of (1.3.2.1), surely the stated isomorphism should read

$$\mathrm{R}\underline{\mathrm{Hom}}(\mathscr{F}_\psi(\mathrm{K}), \pi'^!\mathrm{L}) \simeq \mathrm{R}\,\mathrm{pr}'_*(\mathrm{pr}^!(\mathrm{R}\underline{\mathrm{Hom}}(\mathrm{K}, \pi^!\mathrm{L})) \otimes \mathscr{L}_{\psi^{-1}}(\langle\ ,\ \rangle)).$$

1.4.1.

$$\begin{array}{ccc} \mathrm{A}\times\mathrm{A}' & \xrightarrow{\mathrm{pr}'} & \mathrm{A}' \\ \big\downarrow{\scriptstyle \alpha\times\alpha'} & & \big\downarrow{\scriptstyle \alpha'} \\ \mathrm{D}\times\mathrm{D}' & \xrightarrow{\overline{\mathrm{pr}}'} & \mathrm{D}' \end{array}$$

$$\begin{aligned} \alpha'_!\mathscr{F}(\mathrm{K}) &= \mathrm{R}\overline{\mathrm{pr}}'_*(\alpha\times\alpha')_!(\mathrm{pr}^*\,\mathrm{K}\otimes\mathscr{L}(xx'))[1] \\ &= \mathrm{R}\overline{\mathrm{pr}}'_*((\alpha\times\alpha')_!\,\mathrm{pr}^*\,\mathrm{K}\otimes\overline{\mathscr{L}}(xx'))[1] \\ &= \mathrm{R}\overline{\mathrm{pr}}'_*(\overline{\mathrm{pr}}^*\alpha_!\mathrm{K}\otimes\overline{\mathscr{L}}(xx'))[1]. \end{aligned}$$

1.4.2. On a curve X with $j : \mathrm{U} \hookrightarrow \mathrm{X}$ a dense smooth open with complement F and A a lisse sheaf on U, A[1] is perverse on U and Verdier's formula [**BBD**, 2.2.4] gives

$$j_{!*}\mathrm{A}[1] = \tau^{\mathrm{F}}_{<0}\mathrm{R}j_*\mathrm{A}[1] = j_*\mathrm{A}[1].$$

The simple perverse sheaves on a curve come as (a) i_* of an irreducible $\overline{\mathbf{Q}}_\ell$-sheaf on a closed point or (b) from $j_{!*}[1]$ of an irreducible sheaf on a dense open. (T_1) takes care of (a), but we must verify that a K of type (b) is either (T_2) or (T_3). Given a K of type (T_2),

$$\mathrm{K}|\overline{\mathrm{A}} = (\mathrm{pr}_{\overline{\mathrm{A}}})_*(\mathscr{L}(x.\overline{s}')\otimes\mathrm{pr}^*_{\overline{s}'}\,\mathrm{F}')[1]$$

where $\overline{s}'$ denotes the geometric fiber of s', a discrete set on which F′ is a constant sheaf, so that $\mathrm{K}|\overline{\mathrm{A}}[-1]$ has all of its constituents (as a lisse sheaf [**Weil II**, 1.1.6]) isomorphic to $\mathscr{L}(x.a')$ for some $a' \in \overline{k}$.

Lemma. — Suppose $\mathrm{K} = j_{!*}\mathrm{F}[1] = j_*\mathrm{F}[1]$ *for* F *irreducible lisse on dense* $\mathrm{U} \subset \mathrm{A}$*, and* $\mathrm{K} = \mathscr{F}'(\mathrm{K}')$*, where* $\mathrm{K}' = j'_*\mathrm{F}'[1]$*,* F′ *irreducible lisse on dense* $\mathrm{U}' \subset \mathrm{A}'$*. Then* K *is* (T_3).

Proof. — Let ε denote $\operatorname{Spec}\overline{k} \to \operatorname{Spec} k$ and its various extensions. The constituents of $\varepsilon^*\mathrm{K}'$ coincide with $j'_*[1]$ of the constituents of the lisse sheaf $\varepsilon^*\mathrm{F}'$ (exact sequences of lisse sheaves give rise to distinguished triangles concentrated in degree 0, apply triangulated functor $j'_{!*}[1]$ followed by ${}^p\mathrm{H}^0$); in particular they are all of type (b). As $\mathscr{F}$ induces an equivalence of perverse sheaves on A and A′, $\mathscr{F}$ is *a fortiori* exact (likewise for $\mathscr{F}'$), so that the constituents of $\varepsilon^*\mathrm{K}'$ coincide with those of $\mathscr{F} \circ \mathscr{F}'(\varepsilon^*\mathrm{K}')$; in particular, they are still of type (b), and this implies that none of the constituents of $\mathscr{F}'(\varepsilon^*\mathrm{K}')$ are of type (T_2). As the formation of Fourier transform commutes with any base change (1.2.2.9), this implies the same for the constituents of $\varepsilon^*\mathscr{F}'(\mathrm{K}') = \varepsilon^*\mathrm{K}$; i.e. that none are isomorphic to $\mathscr{L}(x.a')[1]$ for some $a' \in \overline{k}$. In light of [**BBD**, 4.3.2] or the lemma in the note to 4.3.2 below, which says that if F is a lisse sheaf on a normal connected curve, the unit of adjunction $\mathrm{F} \to j_*j^*\mathrm{F}$ is an isomorphism, this implies that K is of type (T_3): if $\varepsilon^*\mathrm{F}$ had a constituent isomorphic to $\mathscr{L}(x.a')|\overline{\mathrm{U}}$, by the above $\varepsilon^*\mathrm{K}$ would have a constituent isomorphic to $\overline{j}_*(\mathscr{L}(x.a')|\overline{\mathrm{U}})[1] \simeq \mathscr{L}(x.a')[1]$, where $\overline{j} = \varepsilon^* j_* : \overline{\mathrm{U}} \hookrightarrow \overline{\mathrm{A}}$. □

Corollary. — *$\mathscr{F}$ exchanges* (T_3) *and* (T'_3).

Proof. — Given a simple perverse sheaf K′ of type (T'_3), $\mathscr{F}'(\mathrm{K}')$ is simple and is not (T_1) by (1.4.2.1 (i)), therefore must be of type (b); i.e. $\mathscr{F}'(\mathrm{K}')$ satisfies the hypotheses of the lemma, so is (T_3). □

Corollary (*Dichotomy*). — *An irreducible lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on a dense open* $\mathrm{U} \hookrightarrow \mathrm{A}$

(α) *has every constituent of $\varepsilon^*\mathcal{F}$ isomorphic to $\mathscr{L}(x.a')|\overline{\mathrm{U}}$ for various $a' \in \overline{k}$, or*
(β) *has no constituent of $\varepsilon^*\mathcal{F}$ isomorphic to $\mathscr{L}(x.a')|\overline{\mathrm{U}}$, for any $a' \in \overline{k}$.*

Remark. In analogy with the Fourier transform on function spaces on **R**,

$$\begin{aligned} \text{constant functions} &\leftrightarrow (\mathrm{T}_2) \\ \mathrm{L}^2(\mathbf{R}) &\leftrightarrow (\mathrm{T}_3) \\ \text{point masses} &\leftrightarrow (\mathrm{T}_1). \end{aligned}$$

Remark. It is tempting to observe that if we were in the abelian category of constructible sheaves (perverse of pervervity $p = 0$) shifted by 1, of course $j_!\mathrm{A} \hookrightarrow j_*\mathrm{A}$, although in the category of perverse sheaves for the middle perversity, $j_*\mathrm{A}[1]$ is simple. In this category,

$$\ker(j_!\mathrm{A}[1] \to j_*\mathrm{A}[1]) = i_*\mathrm{H}^0 i^*\mathrm{R}j_*\mathrm{A} = i_*\mathrm{R}^1 i^! j_!\mathrm{A} \qquad \text{placed in degree } 0,$$

where i denotes the immersion of the complement [**BBD**, 4.1.2]. The point is that although $i^! j_!\mathrm{A} = 0$, $\mathrm{R}^1 i^! j_!\mathrm{A}$ vanishes iff $j_*\mathrm{A}$ extends to a lisse sheaf on X. Assuming it doesn't, $j_!\mathrm{A}[1]$ is not simple, as it admits a nontrivial subobject $i_*\,{}^p\mathrm{H}^0\mathrm{R}i^! j_!\mathrm{A}[1] =: i_*\,{}^p i^! j_!\mathrm{A}[1]$; this is nothing other than $i_*\mathrm{R}^1 i^! j_!\mathrm{A}$ placed in degree 0, and coincides with the largest sub-object of $j_!\mathrm{A}[1]$ in the essential image (via ${}^p i_*$) of the category of perverse sheaves on S (for the middle perversity – this is simply the category of constructible sheaves on the finite set S) [**BBD**, 1.4.25].

2.1.1. A first curiosity: are the conventions (0.3) *en rigeur*? Evidently T isn't essentially of finite type over k, but perhaps it is implicit that it is the spectrum of a ring ind-étale over a k-algebra of finite type. Does the inclusion $k\{\pi\} \subset \mathrm{R} \subset k[[\pi]]$ require that $k \subset \mathrm{R}$ by assumption, or as in the complete case, does a coefficient field exist automatically for R? The answer is that we must assume that $k \subset \mathrm{R}$, as the following stupid example shows.

Let $k_1 = \mathbf{F}_q(t)$, $\mathrm{A}_1 := k_1[\pi]_{(\pi)}$, the local ring at 0 of $\mathbf{A}^1_{k_1}$, and put

$$\mathrm{A}_n := \mathrm{A}_1[x_n]/(x_n^{p^n} - (\pi + t))$$
$$\mathrm{A} := \varinjlim_n \mathrm{A}_n.$$

For each n, A_n is a d.v.r. with uniformizer π. For $m < n$ the map goes $x_m \mapsto x_n^{p^{n-m}}$. Let A^h denote the henselization of A; both A and A^h have the perfect closure $k_1^{p^{-\infty}}$ of k_1 as residue field, but neither contain $k_1^{p^{-\infty}}$.

Remark. As a partial converse, if A is an *excellent* henselian d.v.r. with perfect residue field, then its completion $\hat{\mathrm{A}}$ contains a *canonical* coefficient field, and Artin's approximation theorem gives that A contains a coefficient field.

As R is equicharacteristic, the inertia admits this simple description; c.f. note to Weil II, 1.7.11.

One must always repair to *Corps Locaux* Ch. IV for the ultra-mystical 'upper numbering' filtration on I. In Proposition 3 of §1, it is claimed that $s(f) - f$ has all its coefficients divisible by $s(y) - y$. If we let $\mathfrak{p}_{K'}$ denote the maximal ideal of $A_{K'}$, the definition of $i_{G/H}$ means that $s(y) - y$ is of order $i_{G/H}(s)$ in $A_{K'}$; i.e. $(s(y) - y) = \mathfrak{p}_{K'}^{i_{G/H}(s)}$. Lemma 1 then shows that all the coefficients of $s(f) - f$ have order $\geq i_{G/H}(s)$, hence are divisible by $s(y) - y$.

In light of *Corps Locaux* IV §3 Prop. 14 & Rmk. 1, if L/K is an infinite Galois extension with Galois group G, one defines $G^v := \varprojlim G(L'/K)^v$ as L′ runs over the set of finite Galois sub-extensions of L. This description shows that G^v is a compact subgroup of G, hence closed in G, hence also in the compact open subgroup $I = G_0 = G^0$, (provided of course $v \geq 0$). It also shows that G^v is normal, as it is a projective limit of the normal groups $G(L'/K)^v$ (*Corps Locaux* IV Prop. 1). Left continuity

$$G^v = \bigcap_{w<v} G^w$$

amounts to the statement that if $s \in G$ is not in G^v, then $s \notin G^w$ for some $w < v$. An element $s \in G$ belongs to G^v if for every finite Galois subextension $L \supset L' \supset K$ with $\mathrm{Gal}(L'/K) = H$, $i_{G/H}(s) \geq \psi(v)$, so it will suffice to show that if there is some L′ as above with $i_{G/H}(s) < \psi(v)$, then there is some $w < v$ such that $i_{G/H}(s) < \psi(w)$. As ψ is continuous and increasing, this is trivial.

Laumon considers the induced filtration on $I = G_0 = G^0$. The filtration is separated,

$$\bigcap_{\lambda \geq 0} I^{(\lambda)} = \{1\},$$

as the same is true for G/H for every normal open subgroup H of G (*Corps Locaux* IV Prop. 1).

It's clear that $I^{(\lambda\,+)} \subset I^{(\lambda)}$ as $I^{(\lambda)}$ is closed. But why $I^{(0\,+)} = P$? On the level of a finite Galois extension L′ of K, *Corps Locaux* IV §2 explains that over a perfect field of characteristic p, G_1 is a p-group and the quotient G_0/G_1 is sent isomorphically by the inertia character to a subgroup of the group of roots of unity of the residue field of $\overline{L}'$; this is the tame inertia, and G_1 is the wild inertia. It is necessary to switch to the upper numbering filtration in order for the filtration to play well with quotients, and as the index $(G_0 : G_1)$ increases (corresponding to more tame inertia in the extension L'), $\varphi(1)$ approaches 0 from the right. This means that (provided the maximal tamely ramified extension of K is not finite over K_r, the maximal unramified extension of K) for every $\varepsilon > 0$ and $s \in I^{(0+\varepsilon)}$ there exists an extension L' of K with $1/(G_0 : G_1) < \varepsilon$, so that $s \notin G_1$, $G := \mathrm{Gal}(L'/K)$. So P coincides with the completion of $\cup I^{(0+\varepsilon)}$, which coincides with the closure of $\cup I^{(0+\varepsilon)}$ in I.

2.1.2. (2.1.2.2) The kernel of the representation of P is closed and of finite index, hence open, hence by *Corps Locaux* IV Prop 14. & Prop. 1, $I^{(\lambda)}$ acts trivially for $\lambda \gg 0$. The corollary (2.1.2.3) is Maschke's theorem. In the definition of *pente*, $\mathbf{R}_+ := \mathbf{R}_{\geq 0}$, and λ may equal 0; $\lambda = 0$ iff W is trivial. If H denotes the kernel of the representation $P \to GL(V)$, then P/H is a finite group and $(P/H)^v = P^v H/H$ (*Corps Locaux* IV Prop. 14). The slope λ coincides with the largest real number v such that $(P/H)^v = \{1\}$; in the vocabulary of *Corps Locaux* IV §3 Rmk. 1, this is the largest break in the filtration on P/H. The canonical slope decomposition (2.1.2.4) of V is a decomposition as I- or G-module since if W is a simple P-submodule of V of slope λ and $g \in G$, then gW is still simple as P-module and still of slope λ, as the groups $I^{(v)}$ are normal subgroups of G, for all $v \geq 0$. Therefore V_λ is preserved by G.

(2.1.2.8) To see that $L_\psi(1/\pi)$ has slope 1, it will suffice to show that if $\Gamma := \mathrm{Gal}(\eta'/\eta)$, then $\Gamma = \Gamma_1$ and $\Gamma_2 = \{1\}$. It would follow that $\varphi(1) = 1$ for this extension so that $\Gamma^1 = \Gamma_1 = \Gamma$ and $\Gamma^{1+\varepsilon} = \{1\}$ for all $\varepsilon > 0$. With the criterion of *Corps Locaux* IV §2 Prop. 5, it would suffice to show that for all $s \in \Gamma$,

$$s(\pi')/\pi' \equiv 1 \mod (\pi')$$
$$s(\pi')/\pi' \not\equiv 1 \mod (\pi')^2.$$

As $s(\pi')/\pi' = 1/(1 + \alpha\pi')$ for $\alpha \in \mathbf{F}_p$,

$$s(\pi')/\pi' \equiv 1 - \alpha\pi' \mod (\pi')^2.$$

2.1.4. Schur's lemma gives that every simple tame I-module has rank 1. The action of $\mathrm{Gal}(\overline{k}/k)$ on $\hat{\mathbf{Z}}(1)(\overline{k})$ by inner automorphisms coincides with the action of Galois on roots of unity; c.f. Stacks tag `0BU5`.

2.1.5. The equivalence of the three conditions can be seen as follows: (i) trivially implies (ii) and (iii) as V^{I} (resp. V_{I}) is the largest subobject (resp. quotient) on which I acts trivially. The decomposition $\mathrm{V} = \oplus \mathrm{V}_\lambda$ permits us to assume $\mathrm{V} = \mathrm{V}_\lambda$. If $\lambda > 0$, then by definition $\mathrm{V}_\lambda^{\mathrm{I}} = 0$ and moreover P acts nontrivially. Restricting the action of P to a simple P-submodule W of V_λ, one finds $\mathrm{W}_{\mathrm{P}} = 0$, hence $(\mathrm{V}_\lambda)_{\mathrm{P}} = 0$ and *a fortiori* $(\mathrm{V}_\lambda)_{\mathrm{I}} = 0$. Moreover, V_λ has no nontrivial subquotient on which I acts trivially, as such a subquotient would be a direct sum of simple P-modules of slope λ, so (i), (ii), and (iii) are automatically verified and we consider $\mathrm{V} = \mathrm{V}_0$, a tame G-module. The existence of a geometrically constant subquotient of V as G-module implies the same as I-module, and therefore we need only show that (ii)⇔(iii) and the combination implies that there exists no I-module subquotient of the tame I-module V. As $\mathrm{T} := \mathrm{I}/\mathrm{P} \simeq \hat{\mathbf{Z}}(1)(\bar{k})$ is procyclic with topological generator, say, t, and the representation V of T is continuous,

$$0 \to \mathrm{V}^{\mathrm{T}} \to \mathrm{V} \xrightarrow{t-1} \mathrm{V} \to \mathrm{V}_{\mathrm{T}} \to 0$$

is exact, and shows that $\mathrm{V}^{\mathrm{T}} = 0 \Leftrightarrow \mathrm{V}_{\mathrm{T}} = 0$. Given T-submodules

$$\mathrm{V} = \mathrm{V}_0 \supset \mathrm{V}_1 \supset \mathrm{V}_2 \supset 0,$$

$\mathrm{V}_1^{\mathrm{T}} = 0 \Rightarrow (\mathrm{V}_1)_{\mathrm{T}} = 0$ for each i, so if $\mathrm{V}_1/\mathrm{V}_2$ is T-invariant, the quotient $\mathrm{V}_1 \to \mathrm{V}_1/\mathrm{V}_2$ factors through $(\mathrm{V}_1)_{\mathrm{T}} = 0$, and $\mathrm{V}_1/\mathrm{V}_2 = 0$. So, V has no T-invariant subquotient.

Duality $\mathrm{V} \mapsto \mathrm{V}^\vee$ sends $\mathscr{G}_{(0,\infty[}$ into itself as $(\mathrm{V}^\vee)^{\mathrm{I}} = (\mathrm{V}_{\mathrm{I}})^\vee$.

2.2.1. Of course, the various functions are extended additively with sign, so that e.g. for a perverse sheaf K on X, $s_x(\mathrm{K}) \le 0$ for $x \in |\mathrm{X}|$.

(2.2.1.1) Let $x \overset{i}{\hookrightarrow} \mathrm{X} \overset{j}{\hookleftarrow} \mathrm{X} - x$. There is a distinguished triangle ($i^! = \mathrm{R}i^!$ etc.)

$$i^!\mathrm{K} \to i^*\mathrm{K} \to i^*j_*j^*\mathrm{K} \to$$

As K is perverse, $\mathscr{H}^{-1}(i^!\mathrm{K}) = 0$ and $\mathscr{H}^i(j^*\mathrm{K}) = 0$ for $j \neq -1$. Representing $j^*\mathrm{K}$ by a complex I of injectives in degrees ≥ -1

$$0 \to \cdots \to 0 \to \mathrm{I}_{-1} \xrightarrow{d^{-1}} \mathrm{I}_0 \to \mathrm{I}_1 \to \cdots$$

we have $\mathscr{H}^{-1}(j^*\mathrm{K}) = \ker d^{-1}$. As j_* is left exact,

$$\mathscr{H}^{-1}(j_*j^*\mathrm{K}) = \mathscr{H}^{-1}(j_*\mathrm{I}) = \ker j_*(d^{-1}) = j_*(\ker d^{-1}) = j_*\mathscr{H}^{-1}(j^*\mathrm{K}) = j_*j^*\mathscr{H}^{-1}(\mathrm{K}),$$

so that

$$0 \to i^*\mathscr{H}^{-1}(\mathrm{K}) \to i^*j_*(j^*\mathscr{H}^{-1}(\mathrm{K}))$$

is exact, proving $\mathscr{H}^{-1}(\mathrm{K})_{\overline{x}} \subset j_*(j^*\mathscr{H}^{-1}(\mathrm{K}))_{\overline{x}}$ and the inequality $r(\mathrm{K}) \leq r_x(\mathrm{K})$, as $r(\mathscr{H}^0(\mathrm{K})) = 0$, and showing that $\mathscr{H}^{-1}(\mathrm{K})$ is lisse at x iff I_x acts trivially on $\mathscr{H}^{-1}(\mathrm{K})_{\overline{x}}$ iff $r_x(\mathscr{H}^{-1}(\mathrm{K})) = r(\mathscr{H}^{-1}(\mathrm{K}))$; of course $\mathscr{H}^0(\mathrm{K})_{\overline{x}} = 0$ iff $r_x(\mathscr{H}^0(\mathrm{K})) = 0$. As $r_x(\mathrm{K}) = r_x(\mathscr{H}^0(\mathrm{K})) - r_x(\mathscr{H}^{-1}(\mathrm{K}))$ and $r(\mathrm{K}) = -r(\mathscr{H}^{-1}(\mathrm{K}))$, $r_x(\mathrm{K}) \geq r(k)$ with equality iff (i) holds. Of course, s_x measures wild ramification and so (ii)$\Leftrightarrow$(iii) trivially. See Reprise.

2.2.2. The tame quotient $\pi_1(\xi, \overline{\xi}) \to \pi_1(\xi, \overline{\xi})^{\mathrm{mod}}$ has not actually been defined; 'tame quotient' in (2.1.1) meant $\mathrm{I} \twoheadrightarrow \mathrm{I}/\mathrm{P}$. This tame quotient corresponds to $\mathrm{G} \twoheadrightarrow \mathrm{G}/\mathrm{P}$.

(2.2.2.1) It suffices to show $\overline{i}_*^{\mathrm{mod}}$ is an isomorphism in light of the short exact sequence of π_1 (SGA 1 6.11) which expresses the fundamental group as extension of $\mathrm{Gal}(\overline{k}/k)$ by the geometric fundamental group.

(2.2.2.2) This mysterious theorem is found in Katz, *Local-to-global extensions of representations of fundamental groups,* where he also proves a cohomological formula for the ℓ-adic Swan representation. In light of Reprise below, some cursory analysis of the meaning of this theorem can be made. The discussion doesn't change if one replaces k by $\overline{k}$. First of all, $\mathrm{Gal}(\overline{\xi}/k(u))$ surjects onto $\pi_1(\mathbf{G}_{m,k}, \overline{\xi})$, and we can describe the kernel in terms of the monodromy at all geometric closed points of $\mathbf{G}_{m,k}$.

$$\mathrm{G} := \mathrm{Gal}(\overline{\xi}/k(u)) \twoheadrightarrow \pi_1(\mathbf{G}_{m,k}, \overline{\xi}) \twoheadrightarrow \pi_1(\mathbf{G}_{m,k}, \overline{\xi})^{\mathrm{mod}.\infty} \twoheadrightarrow \pi_1(\mathbf{G}_{m,k}, \overline{\xi})^{\mathrm{mod}}$$

With the notation (2.2.1), the kernel of the first map is topologically generated by $\{\mathrm{I}_x\}_{x \in \mathbf{G}_{m,k}}$. The kernel of the second map is topologically generated by P_∞. The kernel of the third map is topologically generated by P_0. (Both $\mathrm{P}_\infty, \mathrm{P}_0 \subset \mathrm{G}$.) The injection $\pi_1(\xi, \overline{\xi}) \hookrightarrow \mathrm{G}$ corresponds to the Galois extension $k(\xi)/k(u)$.

That $\pi_1(\xi,\overline{\xi})$ injects into $\pi_1(\mathbf{G}_{m,k},\overline{\xi})^{\mathrm{mod}.\infty}$ corresponds simply to the statement that membership in the latter group puts no condition on the monodromy at 0. An object in the latter Galois category corresponds to a finite cover of $\mathbf{P}^1 - \{0\}$ which is étale away from ∞ and tamely ramified at infinity. The injection $\pi_1(\xi,\overline{\xi}) \hookrightarrow \pi_1(\mathbf{G}_{m,k},\overline{\xi})^{\mathrm{mod}.\infty}$ corresponds to the statement that the constraints on monodromy at all closed points of $\mathbf{P}^1$ other than 0 imposed by membership in $\pi_1(\mathbf{G}_{m,k},\overline{\xi})^{\mathrm{mod}.\infty}$ put in effect 'no constraints' on the monodromy at 0.

The isomorphism $\pi_1(\xi\otimes_k \overline{k},\overline{\xi})^{\mathrm{mod}} \simeq \pi_1(\mathbf{G}_{m,\overline{k}},\overline{\xi})^{\mathrm{mod}}$ is the statement that the Galois category of étale covers of the generic point of $(\mathbf{A}^1_{\overline{k}})_{(0)}$ tamely ramified at the closed point is equivalent to the category of étale covers of $\mathbf{G}_{m,\overline{k}}$ which are tamely ramified at 0 and ∞. The proof given notes that both are isomorphic to $\hat{\mathbf{Z}}(1)(\overline{k})$, but this fact for $\mathbf{G}_{m,\overline{k}}$ is not proved in (1.1.3.7), which introduces the Kummer coverings of $\mathbf{G}_{m,\overline{k}}$. What's needed is Grothendieck's comparison theorem for curves (SGA 1 XIII 2.12), which immediately shows that the tame fundamental group of $\mathbf{P}^1_{\overline{k}} - \{\infty\}$ is trivial (this is the automorphism group of a fiber functor on the Galois category consisting of finite covers of $\mathbf{P}^1_{\overline{k}}$ étale over $\mathbf{A}^1_{\overline{k}}$ and tame at ∞) and that $\pi_1(\mathbf{G}_{m,\overline{k}},\overline{\xi})^{\mathrm{mod}}$ is freely generated by a generator of the tame inertia at 0, hence is indeed isomorphic to $\hat{\mathbf{Z}}(1)(\overline{k})$. In light of this, we know that the Kummer coverings of (1.1.3.7) do exhaust the set of finite maps to $\mathbf{P}^1_k$ étale over $\mathbf{G}_{m,\overline{k}}$ and tamely ramified at 0 and ∞.

Intermezzo I: Katz-Gabber extensions. The paper is by Katz, *Local-to-global extensions of representations of fundamental groups*.

(1.2.3) En effet, when you pull E back to $\mathrm{X}\otimes_{\mathrm{K}}\mathrm{K}^{\mathrm{sep}}$, the only monodromy is geometric. The map $\pi_1(\mathrm{X}\times_{\mathrm{K}}\mathrm{K}^{\mathrm{sep}},\overline{x}) \to \mathrm{Aut}(\mathrm{E}(\overline{x}))$ is continuous with open kernel U corresponding to the Galois G-torsor Z, where of course $\mathrm{G} = \pi_1(\mathrm{X}\times_{\mathrm{K}}\mathrm{K}^{\mathrm{sep}},\overline{x})/\mathrm{U}$. Pulling back to Z kills the monodromy on E (i.e. E 'splits'). Now the point is that in the exact sequence ($\overline{\mathrm{X}} := \mathrm{X}\times_{\mathrm{K}}\mathrm{K}^{\mathrm{sep}}$)

$$e \to \pi_1(\overline{\mathrm{X}},\overline{x}) \to \pi_1(\mathrm{X},\overline{x}) \to \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K}) \to e, \tag{\dagger}$$

the surjection is a quotient map, *a fortiori* open. Let V be defined by the exact sequence

$$e \to \mathrm{V} \to \pi_1(\mathrm{X},\overline{x}) \to \mathrm{Aut}(\mathrm{E}(\overline{x})) \to e$$

and let V' be the image of V in $\mathrm{Gal}(K^{\mathrm{sep}}/K)$; V' is open and we let the finite Galois extension K' correspond to any such containing V' and over which Z is defined..

(1.3) A unique p-Sylow is the same as a normal p-Sylow. A normal p-Sylow in a group G is characteristic as it is the unique subgroup with its order (any p-subgroup of G is contained in it).

(1.3.2) In proof of 1) $\Rightarrow$ 4), the statement that 'the unique open normal subgroup of $\pi_1(\mathbf{G}_{m,L}, \bar{x})$ of index $N \geq 1$ prime to p is the one corresponding to the Nth power covering [N] of $\mathbf{G}_{m,L}$ by itself' is easily seen: denoting such a subgroup by U, U corresponds to a finite étale connected torsor T with group $\pi_1(\mathbf{G}_{m,L}, \bar{x})/U$, and the homomorphism from this group into the group opposite $\mathrm{Aut}\,T = \mathrm{Aut}\,T(\bar{x})$ is an isomorphism [**SGA1**, 5.11]. So T is a cover of $\mathbf{G}_{m,L}$ tamely ramified at 0 and ∞ since $p \nmid |\mathrm{Aut}\,T(\bar{x})|$, and as in the note to (2.2.2), $\pi_1(\mathbf{G}_{m,L}, \bar{\xi})^{\mathrm{mod}} \simeq \hat{\mathbf{Z}}(1)(L)$, so the statement is obvious.

Given a special $E \to \mathbf{G}_{m,K}$, the preimage of the unique p-Sylow of the geometric monodromy of E in $\pi_1(\mathbf{G}_{m,K^{\mathrm{sep}}}, \bar{x})$ is an open subgroup with index N_1 prime to p; by the above claim, $[N_1]^*E$ has geometric monodromy a p-group and is still at worst tamely ramified at 0 so that for some N_2, $[N_2]^*[N_1]^*E$ extends to an étale cover of $\mathbf{A}^1_K$; now let $N = N_1N_2$.

The proof of 4) $\Rightarrow$ 7) is self-evident after you observe that

$$\begin{tikzcd} & \mathbf{G}_{m,K} \arrow[dl, "{[N]}"'] \arrow[dr, "\mathrm{Trans}_b"] & \\ \mathbf{G}_{m,K} \arrow[dr, "\mathrm{Trans}_a"'] & & \mathbf{G}_{m,K} \arrow[dl, "{[N]}"] \\ & \mathbf{G}_{m,K} & \end{tikzcd}$$

commutes, since the left composition corresponds to $T \mapsto aT \mapsto aT^N$ as [N] is K-linear, while the right composition corresponds to $T \mapsto T^N \mapsto (bT)^N = aT^N$.

(1.4) When Katz writes 'an action of G on E covering its action on $\mathbf{G}_{m,K'}$,' for example, 'covering' means, in the language of [**SGA 7**, XIII 1.1], 'compatible with.' Moreover the action of G on E should be continuous.

In the proof of the main theorem, once reduced to $(\mathrm{N}, \mathrm{K}') = (1, \mathrm{K})$ over the field K', there is 'nothing to prove' when $p = 1$ since in this case the words 'monodromy group a p-group' means 'trivial monodromy group,' and so putting

$\mathscr{A}$:= category of finite étale coverings of $\mathbf{A}^1_{\mathrm{K}'}$ with trivial monodromy, *and*

$\mathscr{B}$:= category of finite étale coverings of $\operatorname{Spec} \mathrm{K}'((\mathrm{T}^{-1}))$ with trivial monodromy,

both $\mathscr{A}$ and $\mathscr{B}$ are isomorphic to the category of finite étale coverings of $\operatorname{Spec} \mathrm{K}'$: in the case of $\mathscr{A}$, this follows from the exact sequence (†) in light of the fact that $\pi_1(\mathbf{P}^1_{\mathrm{K}^{\mathrm{sep}}}) = 1$; in the case of $\mathscr{B}$, this follows from the same exact sequence and the observation that $\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1}))$ is the function field of the strict henselization $\mathbf{P}^1_{(\infty)}$ of $\mathbf{P}^1_{\mathrm{K}'}$ at ∞; in both cases, the condition 'trivial mondromy' means that the revêtement étale extends over ∞ (resp. the closed point of the strict henselization), and $\pi_1(\mathbf{P}^1_{\mathrm{K}^{\mathrm{sep}}}) = 1 = \pi_1(\mathbf{P}^1_{(\infty)})$ as any connected revêtement étale of the spectrum of a strictly henselian ring is trivial.

(1.4.2) If R is a ring, the idempotents of $\mathrm{R}[[x]]$ are in bijection with those of R, for, given an idempotent $f \in \mathrm{R}[[x]]$, write $f(x) = r_0 + f_1(x)$ with $f_1(x)$ a power series with constant term 0; r_0 must be idempotent and if $f_1(x) \neq 0$ it has a nonzero term of lowest degree, say $r_n x^n$. Then

$$r_0 + f_1(x) = f(x) = f(x)^2 = r + 2r_0 f_1(x) + f_1(x)^2$$

so that $2r_0 r_n = r_n$ and therefore also $2r_0 r_n = 2r_0^2 r_n = r_0 r_n$ so $r_0 r_n = 0$, contradicting $r_n \neq 0$.

Using this fact, we find that the idempotents of R are in bijection with those of $\mathrm{R}[\mathrm{T}]$; writing $\mathrm{R}((\mathrm{T}^{-1})) = \mathrm{R}[\mathrm{T}][[\mathrm{T}^{-1}]]$ finds that the idempotents of $\mathrm{R}((\mathrm{T}^{-1}))$ are in bijection with those of $\mathrm{R}[\mathrm{T}]$ and so also with those of R.

(1.4.4) It's helpful to recall *Arcata* II (2.1).

(1.4.5) To see $\mathscr{P}$ is surjective on $\mathrm{T}^{-1}\mathrm{R}[[\mathrm{T}^{-1}]]$, pick a power series in T^{-1} with no constant term $b = \sum_{i>0} b_i \mathrm{T}^{-i}$. To hit b with an element c of $\mathrm{T}^{-1}\mathrm{R}[[\mathrm{T}^{-1}]]$, we proceed in the usual way: given b at step $i > 0$ with $b_j = 0$ for $j < i$ and $c_1, \ldots, c_j$ already fixed, put $c_i := b_i$ and replace b by $b - \mathscr{P}(c_i \mathrm{T}^{-i})$. Then $c := \sum_{i>0} c_i \mathrm{T}^{-i}$ has $\mathscr{P}(c) = b$.

(1.4.6) Recall Serre, *Cohomologie Galoisienne* 2.3 Prop. 8 for

$$\mathrm{H}^*(\varprojlim \mathrm{G}_i, \varinjlim \mathrm{A}_i) = \varinjlim \mathrm{H}^*(\mathrm{G}_i, \mathrm{A}_i).$$

(1.4.8) As tensor product commutes with colimits, the assertion about $\mathrm{K}^{\mathrm{sep}} \otimes_{\mathrm{K}} \mathrm{K}((\mathrm{T}^{-1}))$ is clear from the isomorphism $\mathrm{K}' \otimes_{\mathrm{K}} \mathrm{K}((\mathrm{T}^{-1})) \xrightarrow{\sim} \mathrm{K}'((\mathrm{T}^{-1}))$ which in turn follows from the basic fact that a finitely generated module M over a noetherian local ring A with maximal ideal $\mathfrak{m}$ has $\hat{\mathrm{M}} \simeq \mathrm{M} \otimes_{\mathrm{A}} \hat{\mathrm{A}}$ for the $\mathfrak{m}$-adic topology; in our case $\mathrm{A} = \mathrm{K}[\mathrm{T}^{-1}]$, $\mathrm{M} = \mathrm{K}'[\mathrm{T}^{-1}]$, and the isomorphism of fields above is the statement over the generic fiber; i.e.

$$\mathrm{K}' \otimes_{\mathrm{K}} \mathrm{K}[[\mathrm{T}^{-1}]] = \mathrm{K}'[\mathrm{T}^{-1}] \otimes_{\mathrm{K}[\mathrm{T}^{-1}]} \mathrm{K}[[\mathrm{T}^{-1}]] \xrightarrow{\sim} \mathrm{K}'[[\mathrm{T}^{-1}]] \quad \textit{and}$$
$$\mathrm{K}' \otimes_{\mathrm{K}} \mathrm{K}((\mathrm{T}^{-1})) = \mathrm{K}'[\mathrm{T}^{-1}] \otimes_{\mathrm{K}[\mathrm{T}^{-1}]} \mathrm{K}((\mathrm{T}^{-1})) \xrightarrow{\sim} \mathrm{K}'((\mathrm{T}^{-1})).$$

We see that the field $\mathrm{K}'((\mathrm{T}^{-1}))$ is the fraction field of the henselian d.v.r. $\mathrm{K}'[[\mathrm{T}^{-1}]]$ and the colimit of henselian local rings along local ring homomorphisms is henselian local, it suffices to observe that the colimit is also along étale, *a fortiori* unramified ring maps that the colimit is a henselian a d.v.r.

That the fraction field of a henselian d.v.r. has the same Galois theory as that of its completion is discussed in the proof of the lemma in note to (2.4.1). This fact can also be obtained from the structure we have found for the Galois group: the unramified and tame parts are identical, and the free pro-p quotients can be shown to be mapped isomorphically: it amounts to showing that the map on étale $\mathrm{H}^1(-, \mathbf{Z}/p)$ from the (spectra of the) maximal pro-p Galois extension of $\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1}))$ to that of $\mathrm{K}^{\mathrm{sep}} \otimes_{\mathrm{K}} \mathrm{K}((\mathrm{T}^{-1}))$ is an isomorphism.

(1.4.10) The point is that the map of π_1s of (1.4.7) factors as

$$\pi_1(\mathrm{Spec}(\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1})), \bar{x}_1) \to \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \bar{y}_1)(\textit{tame at 0}) \twoheadrightarrow \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \bar{y}_1)(\textit{special})$$

by functoriality of the inverse image. This composite is an isomorphism and its inverse, preceded by the (tame-at-0)-to-(special) quotient, provides the retraction. Why is the retraction unique? Identifying $\pi_1(\mathrm{Spec}(\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1})), \bar{x}_1)$ with its image under the canonical injection, retractions are in bijection with complements to $\pi_1(\mathrm{Spec}(\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1})), \bar{x}_1)$:

closed normal subgroups N of $\pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0})$ such that

$$\begin{aligned}&\mathrm{N} \cap \pi_1(\mathrm{Spec}(\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1})), \overline{x}_1) = \{1\} \quad \textit{and}\\ &\mathrm{N} \cdot \pi_1(\mathrm{Spec}(\mathrm{K}^{\mathrm{sep}}((\mathrm{T}^{-1})), \overline{x}_1) = \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0}).\end{aligned}$$

Recall that $\pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{special})$ is the maximal pro-'group with unique p-Sylow subgroup' quotient of $\pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0})$. In the language of *Profinite Groups* by Ribes & Zalesskii, the class $\mathscr{C}$ of finite groups with unique p-Sylow subgroup is a *variety* of finite groups, hence *a fortiori* a *formation* of finite groups. Putting

$$\begin{aligned}\mathrm{R}_{\mathscr{C}} := \bigcap\{\mathrm{N} : \mathrm{N} \text{ open normal subgroup of } \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0}),\\ \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0})/\mathrm{N} \in \mathscr{C}\},\end{aligned}$$

(3.4) of *op. cit.* says that $\mathrm{R}_{\mathscr{C}}$ is a characteristic subgroup, the sequence

$$1 \to \mathrm{R}_{\mathscr{C}} \to \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0}) \to \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{special}) \to 1$$

is exact, and given any closed normal subgroup K of $\pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0})$ such that $\pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0})/\mathrm{K}$ is pro-$\mathscr{C}$, then K is a subgroup of $\mathrm{R}_{\mathscr{C}}$. This last point implies that the only N as above is $\mathrm{R}_{\mathscr{C}}$, hence the retraction is unique.

Since specialness is geometric, the following diagram of profinite groups and continuous homomorphisms is commutative with exact rows and columns (compare 1.3.3).

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathrm{R}_{\mathscr{C}} \arrow[r, equal] \arrow[d] & \mathrm{R}_{\mathscr{C}} \arrow[d] & & \\
1 \arrow[r] & \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{tame at 0}) \arrow[r] \arrow[d] & \pi_1(\mathbf{G}_{m,\mathrm{K}}, \overline{y}_1)(\textit{tame at 0}) \arrow[r] \arrow[d] & \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\mathbf{G}_{m,\mathrm{K}^{\mathrm{sep}}}, \overline{y}_1)(\textit{special}) \arrow[r] \arrow[d] & \pi_1(\mathbf{G}_{m,\mathrm{K}}, \overline{y}_1)(\textit{special}) \arrow[r] \arrow[d] & \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K}) \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

In more words, clearly the kernel of the tame-to-special quotient for K contains that for K^{sep} (the functor of restriction to geometric monodromy corresponds to the functor 'reciprocal image along $\operatorname{Spec} K^{sep} \to \operatorname{Spec} K$'); to see it is no larger, note that the special coverings of G_{m,K_i} are special coverings of $G_{m,K}$ for all Galois extensions K_i of K; therefore an element of the kernel must fix all the special coverings of G_{m,K_i} for each i, and $K^{sep} = \cup_i K_i$.

This diagram implies the uniqueness of the retraction for K restricting to the given one for K^{sep}. In more words, as before, retractions of

$$\pi_1(\mathbf{G}_{m,K}, \overline{y}_1)(\textit{tame at 0}) \twoheadrightarrow \pi_1(\mathbf{G}_{m,K}, \overline{y}_1)(\textit{special})$$

are in bijection with closed normal subgroups N satisfying the same conditions as before with K^{sep} replaced by K and $\overline{x}_1$ replaced by $\overline{x}$. The condition that the retraction must restrict to the given one for K^{sep} implies that N must contain $R_{\mathscr{C}}$. But as seen in the diagram above, $N = R_{\mathscr{C}}$. Therefore, as before, there is only one possible choice for N.

(1.6.4) Finitely presented since constructible – *Th. finitude*.

(1.6.6) See Katz, *Gauss Sums, Kloosterman Sums, and Monodromy Groups* (2.1.1) & (2.3.3).

(1.6.8) There are 3 claims to verify: (1) that the isomorphism class of a projective $\mathbf{Z}_\ell[G]$-module can be deduced by character; (2) that $\chi(\mathbf{A}^1_K, -) = \chi_c(\mathbf{A}^1_K, -)$, and (3) that $\chi_c(\mathbf{G}_{m,K}, \mathscr{F}^{can}) = -\operatorname{swan}_\infty(\mathscr{F})$.

(1) Referring to Serre, *Linear Representations of Finite Groups*, (14.4) gives that for a mixed-characteristic local field K with valuation ring A with residue field $A/\mathfrak{m} =: k$ and G a finite group, every projective $A[G]$-module is a direct sum of projective indecomposable $A[G]$-modules unique up to isomorphism so that two projective $A[G]$-modules are isomorphic iff their images in the Grothendieck group of the category of projective $A[G]$-modules coincide. (16.1) gives that two projective $A[G]$-modules P, P' are isomorphic if the $K[G]$-modules $P \otimes K, P' \otimes K$ are. Let $R_K(G)$ denote the Grothendieck group of finite-dimensional $K[G]$-modules. (14.5) gives that $\langle E, F\rangle = \dim \operatorname{Hom}^G(E, F)$ on $K[G]$-modules E, F induces a bilinear form

$$R_K(G) \times R_K(G) \to \mathbf{Z}$$

and (14.6) shows that it is compatible with extension of field $K \subset K'$, which induces an injection $R_K(G) \to R_{K'}(G)$. Over a large enough field (e.g. the algebraic closure of K), the above bilinear form is nondegenerate and induces an isomorphism of $R_K(G)$ onto its dual. The identity

$$\mathrm{Hom}_G(E, F) \simeq \check{E} \otimes_G F$$

and the fact that the Swan representation is self-dual [**SGA5**, Exp. X (3.8) & (4.4)] show that indeed it suffices to show that

$$\dim_{\overline{\mathbf{Q}}_\ell}(H^1(\mathbf{A}^1_K, j_!\mathscr{F}^{\mathrm{can}})) = \mathrm{swan}_\infty(\mathscr{F}),$$

where the latter is the Swan conductor $b(M)$ of Serre (19.3), in light of the isomorphism

$$H^1(\mathbf{A}^1_K, j_!((\mathrm{Reg}_{G;\mathbf{Z}_\ell})^{\mathrm{can}})) \otimes_{\mathbf{Z}_\ell[G]} M \xrightarrow{\sim} H^1(\mathbf{A}^1_K, j_!\mathscr{F}^{\mathrm{can}}).$$

(2) The equality $\chi(X, \mathscr{F}) = \chi_c(X, \mathscr{F})$ for a proper smooth curve X over an algebraically closed field k was proven by Grothendieck [**SGA5**, Exp. X (7.12)], and later by Laumon for any X separated and finite type over k algebraically closed in his 1981 article *Comparaison de caractéristiques d'Euler-Poincaré en cohomologie l-adique*.

(3) The formula $\chi_c(\mathbf{G}_{m,K}, \mathscr{F}^{\mathrm{can}}) = -\,\mathrm{swan}_\infty(\mathscr{F})$ can be deduced from the Grothendieck-Ogg-Šavarevič formula as appears in Laumon (2.2.1.2). Laumon's Swan conductor coincides with Serre's Swan conductor $b(M)$ after passing from upper to lower numbering

$$s(M) = \sum_{\Lambda \in \Lambda(M)} \lambda . r(V_\lambda) = \int_0^\infty \mathrm{codim}\, M^{G^\lambda}\, d\lambda$$
$$= \int_0^\infty \frac{\mathrm{codim}\, M^{G_r}}{[G_0 : G_r]}\, dr = \sum_{i=1}^\infty \frac{\mathrm{codim}\, M^{G_i}}{[G_0 : G_i]} = b(M).$$

Incidentally, if $a(M)$ denotes the Artin conductor, putting

$$t(M) := \int_{-1}^0 \mathrm{codim}\, M^{G^\lambda}\, d\lambda = \int_{-1}^0 \frac{\mathrm{codim}\, M^{G_r}}{[G_0 : G_r]}\, dr = \mathrm{codim}\, M^{G_0} = \mathrm{codim}\, M^{I_\infty}$$

allows us to write $a(M) = t(M) + s(M)$, expressing the Artin conductor as the sum of the tame conductor $t(M)$ and the Swan (or wild) conductor $s(M)$.

2.3.1. (2.3.1.1) The negative sign before χ_c is due to the shift $[1]$ in the formula for Fourier transform. The contribution in the formula of Grothendieck-Ogg-Šafarevič from $a_\infty(\mathrm{K})$ is $r(\mathrm{K}) - s(\mathrm{F}_{\overline{\eta}_\infty} \otimes \mathscr{L}(x.s')_{\overline{\eta}_\infty})$ and of course $r_\infty(\mathrm{K}') = 0$ as our K on $\mathbf{P}^1$ is obtained as extension by zero of K on A. The Swan conductor s_∞ doesn't care about the stalk of K at ∞ but only about the restriction of K to the generic point η_∞ of the henselian trait $(\mathbf{P}^1_k)_{(\infty)}$. As $\mathscr{L}$ is locally constant on A, $a_s(\mathrm{K} \otimes \mathscr{L}(x.s')) = a_s(\mathrm{K})$ for $s \in \mathrm{S}$.

To understand $\mathscr{L}(x.a')_{\overline{\eta}_\infty}$ when $a' \neq 0$, let's again write the Artin-Schreier covering

$$0 \to \mathbf{F}_p \to \mathbf{G}_{a,k} \xrightarrow{t^p - t} \mathbf{G}_{a,k} \to 0.$$

The induced map $k[t_1] \to k[t_2]$ on coordinate rings is $t_1 \mapsto t_2^p - t_2$, as can be readily seen by remembering what F^* and $+$ do on the coordinate ring of $\mathbf{G}_{a,k}$. The Artin-Schreier sheaf $\mathscr{A}$ is the sheaf of local sections of this covering. It is an $\mathbf{F}_p$-torsor on $\mathbf{G}_{a,k} = \operatorname{Spec} k[t_1]$. As the above map on coordinates makes t_2 integral over $k[t_1]$ with equation of integral dependence $t_2^p - t_2 = t_1$, $\mathscr{A}$ is represented over $\mathbf{G}_{a,k} = \operatorname{Spec} k[t_1]$ by the revêtement étale $\operatorname{Spec} k[t_1, t_2]/(t_2^p - t_2 - t_1)$. Pushing $\mathscr{A}$ by the character ψ^{-1} gives $\mathscr{L}_\psi$ but it is conceptually easier to keep working with $\mathscr{A}$. The pullback of $\mathscr{A}$ to A via $\mathrm{A} \times a' \to \mathrm{A} \times \mathrm{A}' \to \mathbf{G}_a$ corresponds to the revêtement étale of $\mathrm{A} = \operatorname{Spec} k[x]$ given by $\operatorname{Spec} k[x,t]/(t^p - t - xa')$, this is the one-parameter family of Artin-Schreier coverings

$$t^p - t = xa', \qquad a' \in \overline{k}.$$

Let $\mathbf{F} = \overline{\mathbf{F}}_p$. Given $x \in \mathrm{A}(\mathbf{F}), x' \in \mathrm{A}'(\mathbf{F})$, we can consider x, x' as elements of $\mathbf{F}_q$ for some $q = p^n$, $\mathscr{L}(xx') = \psi^{-1}(\mathrm{N}_{\mathbf{F}_q/\mathbf{F}_p}(xx'))$. When k is merely a perfect field of characteristic p, $\mathscr{L}(x.a')$ corresponds to the push of the above $\mathbf{F}_p$-torsor $\mathscr{A}$ by $\psi^{-1} : \mathbf{F}_p \to \overline{\mathbf{Q}}_\ell^\times$. The sheaf $\mathscr{A}$ corresponds to the representation of $\mathrm{Gal}(\eta'/\eta)$ discussed in the note to (2.1.2.8); likewise $\mathscr{L}(x.a')_{\overline{\eta}_\infty}$ corresponds to $\mathrm{L}_\psi(1/xa')$; here xa' corresponds to π and indeed $1/xa'$ is a uniformizer for the strictly henselian trait $(\mathbf{P}^1_{\overline{k}})_{(\infty)}$. So $\mathscr{L}(x.a')_{\overline{\eta}_\infty}$ has slope 1.

If there is an $a'_1 \in \overline{k}$ such that

$$((\mathrm{F}_{\overline{\eta}_\infty})_1 \otimes \mathscr{L}(x.a'_1)_{\overline{\eta}_\infty})^{\mathrm{I}_\infty^{(1)}} \neq 0,$$

then for all $a'_2 \neq a'_1 \in \overline{k}$,

$$((\mathrm{F}_{\overline{\eta}_\infty})_1 \otimes \mathscr{L}(x.a'_2)_{\overline{\eta}_\infty})^{\mathrm{I}_\infty^{(1)}} = ((\mathrm{F}_{\overline{\eta}_\infty})_1 \otimes \mathscr{L}(x.a'_1)_{\overline{\eta}_\infty} \otimes \mathscr{L}(x.(a'_2 - a'_1))_{\overline{\eta}_\infty})^{\mathrm{I}_\infty^{(1)}} = 0,$$

as $\mathscr{L}(x.(a'_2 - a'_1))_{\overline{\eta}_\infty}$ has slope 1.

Formula (i): in light of the above discussion, (2.1.2.7) gives that for almost all $a' \in \overline{k}$,

$$s(\mathrm{F}_{\overline{\eta}_\infty} \otimes \mathscr{L}(x.a')_{\overline{\eta}_\infty}) = r((\mathrm{F}_{\overline{\eta}_\infty})_{[0,1[}) + r((\mathrm{F}_{\overline{\eta}_\infty})_1) + s((\mathrm{F}_{\overline{\eta}_\infty})_{]1,\infty[}).$$

By (2.2.1.1), $r(\mathrm{K}') = r_{s'}(\mathrm{K}')$ for all but the finitely many $s' \in \mathrm{S}'$. Combining these two facts with the Grothendieck-Ogg-Šafarevič formula for $r_{s'}(\mathrm{K}')$ gives

$$\begin{aligned} r(\mathrm{K}') &= r(\mathrm{F}_{\overline{\eta}_\infty}) + \sum_{s\in\mathrm{S}} \deg(s).a_s(\mathrm{K}) - (r((\mathrm{F}_{\overline{\eta}_\infty})_{[0,1[}) + r((\mathrm{F}_{\overline{\eta}_\infty})_1) + s((\mathrm{F}_{\overline{\eta}_\infty})_{]1,\infty[})) \\ &= \sum_{s\in\mathrm{S}} \deg(s).a_s(\mathrm{K}) + r((\mathrm{F}_{\overline{\eta}_\infty})_{]1,\infty[}) - s((\mathrm{F}_{\overline{\eta}_\infty})_{]1,\infty[}). \end{aligned}$$

The difference between formulæ (i) and (ii) is in $s((\mathrm{F}_{\overline{\eta}_\infty})_1 \otimes \mathscr{L}(x.s')_{\overline{\eta}_\infty})$; in the generic formula (i) we could discard the finitely many geometric points of A' where this differs from $r((\mathrm{F}_{\overline{\eta}_\infty})_1)$. Therefore,

$$r_{s'}(\mathrm{K}') - r(\mathrm{K}') = -s((\mathrm{F}_{\overline{\eta}_\infty})_1 \otimes \mathscr{L}(x.s')_{\overline{\eta}_\infty}) + r((\mathrm{F}_{\overline{\eta}_\infty})_1).$$

The difference between formulæ (i) and (iii) is that in (i) we applied (2.1.2.7) while in (iii) we cannot, so that

$$\begin{aligned} r(\mathrm{K}')_{0'} - r(\mathrm{K}') &= -s(\mathrm{F}_{\overline{\eta}_\infty}) + (r((\mathrm{F}_{\overline{\eta}_\infty})_{[0,1[}) + r((\mathrm{F}_{\overline{\eta}_\infty})_1) + s((\mathrm{F}_{\overline{\eta}_\infty})_{]1,\infty[})) \\ &= -s((\mathrm{F}_{\overline{\eta}_\infty})_{[0,1[}) + r((\mathrm{F}_{\overline{\eta}_\infty})_{[0,1[}). \end{aligned}$$

Intermezzo II: SGA 7 Exposé XIII. §1 In the definition of a compatible action of G on a sheaf of sets $\mathcal{F}$ on $\overline{\mathrm{Y}}$, to an étale $a : \mathrm{U} \to \overline{\mathrm{Y}}$ and $g \in \mathrm{G}$ we must associate an isomorphism

$$\begin{array}{ccc} \overline{\mathrm{Y}} \times_{u(g),a} \mathrm{U} \simeq \mathrm{U} & \xrightarrow{u(g)} & \mathrm{U} \\ \downarrow a \quad \ulcorner & & \downarrow a \\ \overline{\mathrm{Y}} & \xrightarrow{u(g)} & \overline{\mathrm{Y}} \end{array} \rightsquigarrow \mathcal{F}(\mathrm{U} \xrightarrow{u(g)} \mathrm{U} \xrightarrow{a} \overline{\mathrm{Y}}) \xrightarrow{\sigma(g)(\mathrm{U})} \mathcal{F}(\mathrm{U} \xrightarrow{a} \overline{\mathrm{Y}})$$

so as to induce morphisms of sheaves $\sigma(g) : u(g)^*\mathcal{F} \to \mathcal{F}$ in a compatible way so that $\sigma(gh) = \sigma(g)\sigma(h)$. In effect, there is an obvious choice: $\mathcal{F}(u(g))$, and this choice explains why, given $\mathcal{G}$ on Y, the action of $\mathrm{Gal}(\overline{k}, k)$ on $\overline{\mathcal{G}}$ by transport of structure is compatible with the action of the same group on $\overline{\mathrm{Y}}$ (action 'by transport of structure' means the above action). *A priori* the compatible action of G on $\mathcal{F}$ need not even factor through u.

(1.2.7) Let $f : \mathrm{S}' \to \mathrm{S}$ be a surjective morphism of henselian traits. Then $\eta' \mapsto \eta, s' \mapsto s$, and we can choose $\overline{s}', \overline{s}, \overline{\eta}', \overline{\eta}$ so that $\overline{s}' \to s$ factors through $\overline{s}$ and likewise $\overline{\eta}' \to \overline{\eta} \to \eta$ so that the diagram below commutes.

$$\begin{array}{ccc} \mathrm{Y} \times_s \overline{s}' & \xrightarrow{f} & \mathrm{Y} \times_s \overline{s} \\ \big\downarrow{\scriptstyle \overline{s}'} & & \big\downarrow{\scriptstyle \overline{s}} \\ \mathrm{Y} \times_s s' & \xrightarrow{f} & \mathrm{Y} \end{array}$$

This implies that for $\mathcal{F}$ a sheaf on Y, $f^*\mathcal{F}_{\overline{s}} := f^*\overline{s}^*\mathcal{F} = \overline{s}'^* f^*\mathcal{F}$, so that $\mathrm{Gal}(\overline{s}'/s')$ acts on $f^*\mathcal{F}_{\overline{s}}$ *via* the homomorphism

$$\mathrm{Gal}(\overline{s}'/s') \to \mathrm{Gal}(\overline{s}/s) \qquad \text{induced by the restriction of } f \text{ to } s';$$

likewise, $\mathrm{Gal}(\overline{\eta}'/\eta')$ acts on $f^*\mathcal{F}_{\overline{\eta}}$ *via* the homomorphism

$$\mathrm{Gal}(\overline{\eta}'/\eta') \to \mathrm{Gal}(\overline{\eta}/\eta) \qquad \text{induced by the restriction of } f \text{ to } \eta'.$$

(1.3) Typo: $\overline{\mathrm{X}} := \mathrm{X} \times_{\mathrm{S}} \overline{\mathrm{S}}$. Recall that $\overline{\mathrm{S}}$ is the normalization of S in $\overline{\eta}$, which is the spectrum of a strictly henselian valuation ring with value group **Q** so that the stalk at the closed point of $\overline{\mathrm{S}}$ coincides with global sections; its separably closed residue field may be an inseparable extension of $k(s)$, but we take $\overline{s}$ to denote the spectrum of this separably closed field, which can be considered as the closed point of $\overline{\mathrm{S}}$ or, by light abuse of notation (0.2.4), as defining a geometric point of S.

It is perhaps worth comparing $i^* j_* j^*$ with $\overline{i}^*\overline{j}_*\overline{j}^*$ in the definition of Ψ_η. Let $\mathcal{F}$ be a sheaf on $\mathrm{Y} \to \mathrm{S}$; then $\mathcal{F}_{\overline{\eta}}$ carries action of $\mathrm{Gal}(\overline{\eta}/\eta)$. In effect, $i^* j_* j^*\mathcal{F}$ is $\mathcal{F}_{\overline{\eta}}^{\mathrm{I}}$, while $\overline{i}^*\overline{j}_*\overline{j}^*\mathcal{F} = \mathcal{F}_{\overline{\eta}}$ endowed with continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$ compatible with the action of the latter on $\mathrm{Y}_{\overline{s}}$. In galoisian terms of (1.2.2), this is the observation that $\mathrm{Gal}(\overline{\eta}/\overline{\eta}) = \{e\}$.

In (1.3.6.2), the ψ is omitted on the left hand side because it induces an equivalence (see note to (2.1.8) below). The $X_s \times_s \eta$-component of both sides is readily computed from the definitions, keeping in mind that $\psi_{\overline{\eta}}$ is a sheaf on $X_{\overline{s}}$; namely,

$$\Psi_\eta(f_*\mathscr{F}) = \overline{i}^*\overline{j}_*(f_*(F)_{\overline{\eta}}) = \overline{i}^*\overline{j}_*\Gamma(X_{\overline{\eta}}, \mathscr{F}),$$

which, as $\mathrm{Gal}(\overline{\eta}/\eta)$-module, is simply $\Gamma(X_{\overline{\eta}}, \mathscr{F})$ (sitting on $\overline{s}$). With $f : X_{\overline{s}} \to \overline{s}$,

$$f_*\Psi_\eta(\mathscr{F}) = \Gamma(X_{\overline{s}}, \Psi_{\overline{\eta}}(\mathscr{F})).$$

(1.3.8) The commutativity of $f_!$ with change of base does not require that f be quasi-finite, only separated and locally of finite type [**SGAA**, Exp. XVII 6.1.4]. The morphism $f_!\overline{j}_* \to \overline{j}_* f_!$ follows from the definition of $f_!$ [**SGAA**, Exp. XVII 6.1.2] and the observation that the sections of $\overline{j}_* f_!$ over X' are the sections of $\overline{j}_* f_*$ over $X_{\overline{\eta}}$ with support proper over $X'_{\overline{\eta}}$ while the sections of $f_!\overline{j}_*$ are the same, only they must now have support proper over the larger space $\overline{X}'$. Therefore the map is induced simply by the inclusion $\overline{\eta} \hookrightarrow \overline{S}$. When f is finite it is proper and all these maps are isomorphisms.

(1.3.9) Commute with direct image $\rightsquigarrow$ [**SGAA**, Exp. XVIII 3.1.12.3].

$\overline{i}^* f^! \to f^!\overline{i}^*$ $\rightsquigarrow$ [**SGAA**, Exp. XVIII 3.1.14.2].

f étale $\rightsquigarrow$ [**SGAA**, Exp. XVIII 3.1.8].

(1.3.10) The definition of this arrow is clarified with a diagram in [**Th. finitude**, 3.7], where the derived version of this map is shown to be an isomorphism, provided you assume $\mathscr{F}$ is of torsion prime to the residual characteristic of S.

(1.4) The business of securing the injectivity of φ' is the usual trick (SGA V Exp. XV p. 479). Namely, the cone of $\mathrm{sp}^* K_s$ (where here $\mathrm{sp} = \mathrm{sp} \circ j$ in the sense of (1.2.2)) is the cone of the identity map on this complex; i.e. the complex $C(\mathrm{sp}^* K_s)^i := (\mathrm{sp}^* K_s)^i \oplus (\mathrm{sp}^* K_s)^{i-1}$ with differential $d^i(x, y) = (d^i x, x - d^{i-1} y)$. The identity map on this complex is homotopic to 0 via the homotopy $h(x, y) = (y, 0)$, so it is acyclic and $K'_\eta = K_\eta \oplus C(\mathrm{sp}^* K_s)$ is homotopic to K_η. Replacing φ by $\varphi' := (\varphi, \mathrm{id}, 0)$ (as $\mathrm{sp}^* K_s$ coincides with $K_{\overline{s}}$ with action of $\mathrm{Gal}(\overline{\eta}/\eta)$ factoring through $\mathrm{Gal}(\overline{s}/s)$ so that φ' remains equivariant), φ' is now a termwise split injection with termwise splitting given by projection to the second factor. In (1.4.2.2) sp again coincides with $\mathrm{sp} \circ j$ (1.2.2). En termes imagés, $\Phi(\mathrm{K})$ keeps track of the sections of K_η which do not come from residual extension of sections of K_s; bref, the discrepancy between K_η and K_s.

(2.1.2) 'l'image réciproque $\mathrm{F}_{\overline{\eta}}$ est acyclique' ⇝ [**SGAA**, VII 5.7].

(2.1.3) S_{nr} is a strictly henselian trait coinciding with the normalization of S in η_{nr}.

Claim: $\overline{\mathrm{X}}_{(\overline{x})} \simeq \mathrm{X}_{(\overline{x})} \times_{\mathrm{S}_{\mathrm{nr}}} \overline{\mathrm{S}}$. As $\overline{\mathrm{X}} = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}_{\mathrm{nr}} \times_{\mathrm{S}_{\mathrm{nr}}} \overline{\mathrm{S}}$ and $\mathrm{X}_{(\overline{x})}$ coincides with the strict henselization of $\mathrm{X} \times_{\mathrm{S}} \mathrm{S}_{\mathrm{nr}}$ at $\overline{x}$, $\mathrm{X}_{(\overline{x})} \times_{\mathrm{S}_{\mathrm{nr}}} \overline{\mathrm{S}}$ is pro-étale over $\overline{\mathrm{X}}$. On the other hand, let η' be a finite separable extension of η_{nr} and S' the normalization of S_{nr} in η'. Then $\mathrm{X}_{(\overline{x})} \times_{\mathrm{S}_{\mathrm{nr}}} \mathrm{S}'$ is finite over $\mathrm{X}_{(\overline{x})}$, so splits as a disjoint union of henselian local ring spectra indexed by the points in the closed fiber $\overline{x} \times_{\mathrm{S}_{\mathrm{nr}}} \mathrm{S}'$. As the closed fiber of the map $\mathrm{S}' \to \mathrm{S}_{\mathrm{nr}}$ is radicial, the map $\overline{x} \times_{\mathrm{S}_{\mathrm{nr}}} \mathrm{S}' \to \overline{x}$ is injective and hence we see that $\overline{x} \times_{\mathrm{S}_{\mathrm{nr}}} \mathrm{S}'$ is one point (Stacks tag `01S2`) so that $\mathrm{X}_{(\overline{x})} \times_{\mathrm{S}_{\mathrm{nr}}} \mathrm{S}'$, and hence by passage to the limit $\mathrm{X}_{(\overline{x})} \times_{\mathrm{S}_{\mathrm{nr}}} \overline{\mathrm{S}}$ too, are strictly henselian local (Stacks tag `04GI`). It is now immediate that the generic fiber of the strict henselization of $\overline{\mathrm{X}}$ at $\overline{x}$ coincides with

$$(\mathrm{X}_{(\overline{x})} \times_{\mathrm{S}_{\mathrm{nr}}} \overline{\mathrm{S}})_{\overline{\eta}} = \mathrm{X}_{(\overline{x})} \times_{\eta_{\mathrm{nr}}} \overline{\eta}.$$

(2.1.5) Considering $\mathscr{F}$ as concentrated in degree 0, then so too is $\mathrm{sp}^* i^* \mathscr{F}$. We write the long exact sequence of cohomology associated to the stalk at $\overline{x}$ (technically the stalk at the point $(\overline{x}, \overline{\eta})$ of the topos $\mathrm{X}_s \times_s \eta$) of the distinguished triangle (2.1.2.4)

$$0 \to \mathscr{F}_{\overline{x}} \xrightarrow{\sim} (\overline{j}_* \overline{j}^* \mathscr{F})_{\overline{x}} \to \mathrm{R}^0\Phi(\mathscr{F}) \to 0 \to \mathrm{R}^1\Psi(\mathscr{F}_\eta)_{(\overline{x},\overline{\eta})} \to \mathrm{R}^1\Phi(\mathscr{F}) \to \cdots$$

with the second arrow an isomorphism as $\mathscr{F}$ is lisse (c.f. Reprise). (2.1.3) gives that

$$\mathrm{R}^i\Psi(\mathscr{F}_\eta)_{(\overline{x},\overline{\eta})} \simeq \mathrm{H}^i(\mathrm{X}_{(\overline{x})} \times_{\eta_{\mathrm{nr}}} \overline{\eta}, \mathscr{F})$$

and the note to (2.1.3) above shows that $\mathrm{X}_{(\overline{x})} \times_{\eta_{\mathrm{nr}}} \overline{\eta}$ is a variety of vanishing cycles of f at the point $\overline{x}$ in the sense of Arcata V 1.3 (more properly it should be called a variety of nearby cycles); as f is smooth hence (universally) locally acyclic (Arcata V 2.1) and the restriction of $\mathscr{F}$ to $\mathrm{X}_{(\overline{x})}$ is constant, this same definition 1.3 gives that $\mathrm{R}\Psi(\mathscr{F})_{(\overline{x},\overline{\eta})}$ is connective (i.e. acyclic in degrees > 0) so that $\mathrm{R}\Phi(\mathscr{F})_{(\overline{x},\overline{\eta})} = 0$.

In the case of a complex K in $D^+(X, \Lambda)$ with f smooth and the $\mathscr{H}^i(K)$ lisse, we can reduce to the previous paragraph with the help of the spectral sequence

$$E_{II}^{p,q} = R^p\Psi_\eta(\mathscr{H}^q(K_s)) \Rightarrow \mathscr{H}^{p+q}\Psi_\eta(K_s)$$

which finds that the map $\mathrm{sp}^*\, i^*K \to R\Psi_\eta(K_s)$ is a quasi-isomorphism.

See also [**Th. finitude**, 2.12].

(2.1.6) Rf_* is described as a functor $D^+(Y \times_s S, \Lambda) \to D^+(Y' \times_s S, \Lambda)$.

c) f quasi-fini $\Rightarrow$ $f_!$ exact $\rightsquigarrow$ [**SGAA**, XVII 6.1.4].

d) f quasi-fini $\Rightarrow$ $Rf^!$ is the right derived functor of $f^!$ $\rightsquigarrow$ [**SGAA**, XVIII 3.1.8 (i)].

(2.1.7) Climbing to the highest heights of pedantry, to derive (1.3.6.1), start by representing our K in $D^+(X, \Lambda)$ by a complex of injectives I_1; as f_* preserves injectives, $R\Psi\ Rf_*K = \Psi\ f_*I_1$. By (1.3.6.1) we find a morphism of complexes $\Psi\ f_*I_1 \to f_*\ \Psi I_1$. Taking a quasi-isomorphism $\Psi I_1 \to I_2$ into a complex of injectives I_2 composes to give

$$R\Psi\ Rf_*K = \Psi\ f_*I_1 \to f_*\ \Psi I_1 \to f_*I_2 = Rf_*\ R\Psi K.$$

The same method gives (2.1.7.2) and the first arrow in the below for (2.1.7.3).

$$Rf_!\ R\Psi = R\tilde{f}_*\ j_!\ R\Psi \to R\tilde{f}_*\ R\Psi\ j_! \to R\Psi\ R\tilde{f}_*\ j_! = R\Psi\ Rf_!.$$

For f quasi-finite, $f^!$ is right adjoint to the exact functor $f_!$ and so preserves injectives.

Deligne shows [**Th. finitude**, 3.7] that the formation of nearby cycles $R\Psi$ commutes with change of trait; i.e. the base-change morphism (2.1.7.5) is an isomorphism (for sheaves of torsion prime to the residual characteristic of S). (Confusingly, he calls $R^i\Psi_\eta$ vanishing cycles. This is surely because, according to Illusie, *Grothendieck and vanishing cycles*, when Grothendieck introduced the functors $R\Psi$, $R\Phi$, he called both these 'functors of vanishing cycles.')

(2.1.8) Typo: K belongs in $D^+(X, \Lambda)$. When $X' = S$, Ψ induces an equivalence of sheaves on S with sheaves on the topos $s \times_s S$ (1.2.2 b)) and is therefore omitted. The $X_s \times_s \eta$-component of the left-hand side of (2.1.8.1) is

$$R\,\Psi_\eta\ Rf_*K = \bar{i}^*\bar{j}_*\bar{j}^*Rf_*K = \bar{i}^*\bar{j}_*(Rf_*K)_{\bar{\eta}} = \bar{i}^*\bar{j}_*R\Gamma(X_{\bar{\eta}}, K).$$

As in the note to (1.3) above, as $\mathrm{Gal}(\overline{\eta}/\eta)$-module, $\overline{i}^*\overline{j}_*\mathrm{R}\Gamma(\mathrm{X}_{\overline{\eta}},\mathrm{K}) = \mathrm{R}\Gamma(\mathrm{X}_{\overline{\eta}},\mathrm{K})$, but the left-hand side can be considered as sitting on $\overline{s}$ while the right-hand side sits on $\overline{\eta}$. This is pedantry. In any event, the $\mathrm{X}_s \times_s \eta$-component of the right-hand side of (2.1.8.1) coincides with the right-hand side of (2.1.8.3) in light of (2.1.6.2), which in itself is pedantic and simply recognizes that the object $\mathrm{R}\Psi_{\overline{\eta}}(\mathrm{K})$ sits on $\mathrm{X}_{\overline{s}}$, and we take the stalk at the point $(\overline{x},\overline{\eta})$ of $\mathrm{X}_s \times_s \eta$ as in (2.1.3). In other words, with $f : \mathrm{X}_{\overline{s}} \to \overline{s}$,

$$\mathrm{R}f_*\,\mathrm{R}\Psi_\eta(\mathrm{K}) = \mathrm{R}f_*\overline{i}^*\overline{j}_*(\mathrm{K}_{\overline{\eta}}) = \mathrm{R}\Gamma(\mathrm{X}_{\overline{s}},\overline{i}^*\overline{j}_*(\mathrm{K}_{\overline{\eta}}) = \mathrm{R}\Gamma(\mathrm{X}_{\overline{s}},\mathrm{R}\Psi_{\overline{\eta}}(\mathrm{K})).$$

The long exact sequence (2.1.8.9) exists when f is proper because in that case (2.1.8.3) is an isomorphism.

2.3.2. The translation of the note to (2.2.1.1) into $\mathrm{G}_{s'}$-modules is that

$$\mathscr{H}^{-1}(\mathrm{K}'_{\overline{s}'}) \subset \mathrm{F}'^{\mathrm{I}_{s'}}_{\overline{\eta}_{s'}},$$

confirming the exactness on the left of the exact sequence of $\mathrm{G}_{s'}$-modules.

(2.3.2.1) (i) The point is that the Artin-Schreier $\mathbf{F}_p$-torsor $\mathscr{A}$ (c.f. note to 2.3.1) is trivialized by the base change $\mathbf{G}_{a,k} \xrightarrow{t^p-t} \mathbf{G}_{a,k}$ by the Lang isogeny for $\mathbf{G}_{a,k}$, which is a revêtement étale. Pulling back to $\mathrm{A} \times_k \mathrm{A}' \to \mathbf{G}_{a,k}$ trivializes $\mathscr{L}(x.x')$ over this revêtement étale and reduces to the stated theorem (see also note to (2.1.5) in Intermezzo II).

(In galoisian terms, if $\overline{x}$ is a geometric point of $\mathbf{G}_{a,k}$, $\mathscr{L}_\psi$ is defined by

$$\pi_1(\mathbf{G}_{a,k},\overline{x}) \to \mathbf{F}_p \xrightarrow{\psi^{-1}} \overline{\mathbf{Q}}_\ell^\times$$

The Artin-Schreier covering $\mathbf{G}_{a,k} \xrightarrow{t^p-t} \mathbf{G}_{a,k}$ corresponds to the open subgroup of $\pi_1(\mathbf{G}_{a,k},\overline{x})$ which coincides with the kernel of this representation.)

(ii) Use (1.4.1.1) and [**SGA 7**, Exp. XIII 2.1.7.1] for $\mathrm{R}\Psi$; proper base change for $\mathrm{sp}^*\,i^*$ and (TR3) gives the desired isomorphism

$$\begin{array}{ccccccc}
\mathrm{sp}^*\,i^*\mathrm{R}\overline{\mathrm{pr}}'_* & \longrightarrow & \mathrm{R}\Psi_{\eta_{s'}}\mathrm{R}\overline{\mathrm{pr}}'_* & \longrightarrow & \mathrm{R}\Phi\mathrm{R}\overline{\mathrm{pr}}'_* & \longrightarrow & \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow & & \\
\mathrm{R}\overline{\mathrm{pr}}'_*\,\mathrm{sp}^*\,i^* & \longrightarrow & \mathrm{R}\overline{\mathrm{pr}}'_*\mathrm{R}\Psi_{\eta_{s'}} & \longrightarrow & \mathrm{R}\overline{\mathrm{pr}}'_*\mathrm{R}\Phi & \longrightarrow &
\end{array}$$

Let $\mathrm{M} := \overline{\mathrm{pr}}^*(\alpha_!\mathrm{K}) \otimes \overline{\mathscr{L}}(x.x')[1]$. As $\mathrm{R}\overline{\mathrm{pr}}'_* \,\mathrm{sp}^* = \mathrm{sp}^*\,\mathrm{R}\overline{\mathrm{pr}}'_*$, this diagram computes to

$$\begin{tikzcd}
\mathrm{sp}^*\,\mathrm{K}'_{\overline{s}'} \arrow[r] \arrow[d, "\sim"] & \mathrm{R}\Psi_{\eta_{s'}}(\mathrm{K}') \arrow[r] \arrow[d, "\sim"] & \mathrm{R}\Phi(\mathrm{K}') \arrow[r] \arrow[d] & {} \\
\mathrm{sp}^*\,\mathrm{R}\Gamma(\mathrm{D}_{\overline{s}'}, i^*(\mathrm{M})) \arrow[r] & \mathrm{R}\Gamma(\mathrm{D}_{\overline{s}'}, \mathrm{R}\Psi_{\overline{\eta}_{s'}}(\mathrm{M})) \arrow[r] & \mathrm{R}\Phi(\mathrm{M})_{(\overline{\infty},\overline{s}')} \arrow[r] & {}
\end{tikzcd}$$

where here $\mathrm{R}\Gamma(\mathrm{D}_{\overline{s}'}, i^*(\mathrm{M})) = \mathrm{R}\overline{\mathrm{pr}}'_*(i^*\mathrm{M})_{\overline{s}'}$ for $\overline{\mathrm{pr}}' : \mathrm{D}\times_k s' \to s'$. En effet, (i) gives that $\mathrm{R}\Phi(\mathrm{M})$ is supported on $\infty\times_k \eta_{s'}$ (the topos; literally as a sheaf it sits on $\infty\times_k \overline{s}' \hookrightarrow \mathrm{D}_{\overline{s}'}$); if k denotes the inclusion of this point into $\mathrm{D}\times_k \eta_{s'}$, $\mathrm{R}\Phi(\mathrm{M}) = k_*\mathrm{L}$ so that $\mathrm{R}\overline{\mathrm{pr}}'_*\mathrm{R}\Phi(\mathrm{M}) = \mathrm{R}\Gamma(\mathrm{D}_{\overline{s}'}, k_*\mathrm{L}) = \mathrm{L}_{(\overline{\infty},\overline{s}')}$. Compare with [**SGA 7**, Exp. XIII 2.1.8] and note to (2.1.8) in Intermezzo II.

(2.3.2.3) = (2.3.2.1) (iii) + the exact sequence at the top of the page + (2.3.1.1).

(2.3.3.1) Typos: in the first displayed equation of (ii) the stalk is at $(\overline{s},\overline{\infty}')$ not $(\overline{s},\overline{\infty})$. In the first displayed equation of the proof obviously it is $\mathrm{D}\times_k\overline{\infty}'$ not $\mathrm{D}\otimes_k\overline{\infty}'$.

In the proof of (iii), the point is that as the support of $\mathrm{R}\Psi_{\overline{\eta}_{\infty'}} = \mathrm{R}\Phi_{\overline{\eta}_{\infty'}}$ is contained in $(\mathrm{S}\cup\infty)\times_k \eta_{\infty'}$. Let M_s denote the restriction of $\mathrm{R}\Psi_{\overline{\eta}_{\infty'}}(\overline{\mathrm{pr}}^*(\alpha_!\mathrm{K})\otimes\overline{\mathscr{L}}(x.x')[1])$ to $s\times_k\overline{\infty}'$ (it is the fiber at $\overline{\eta}_{\infty'}$ of a sheaf on the topos $s\times_k\eta_{\infty'}$, hence literally a sheaf on $s\times_k\overline{\infty}'$), and put $s : s\times_k \mathrm{D}'_{(\infty')} \hookrightarrow \mathrm{D}\times_k\mathrm{D}'_{(\infty')} \xrightarrow{\mathrm{pr}'} \mathrm{D}'_{(\infty')}$ $(s\in\mathrm{S}\cup\infty)$. Then

$$\mathrm{R}^{-1}\Gamma(\mathrm{D}\times_k\overline{\infty}', \mathrm{R}\Psi_{\overline{\eta}_{\infty'}}(\overline{\mathrm{pr}}^*(\alpha_!\mathrm{K})\otimes\overline{\mathscr{L}}(x.x')[1])) = \bigoplus_{s\in\mathrm{S}\cup\infty} s_{\overline{\infty}'*}(\mathrm{M}_s)$$

with the notation of [**SGA 7**, 2.1.6]. Fix some $s\in\mathrm{S}\cup\infty$. $s_{\infty'} : s\times_k\infty' \to \infty'$ is a revêtement étale that realizes $\mathrm{G}_{s\times_k\infty'} \hookrightarrow \mathrm{G}_{\infty'}$ as an open subgroup, and $s\times_k\overline{\infty}'$ is topologically a disjoint union of copies of $\overline{\infty}'$ indexed by the (algebraic) geometric points centered on s (there are $\deg(s)$ many). As $\mathrm{G}_{\infty'}$-module,

$$s_{\overline{\infty}'*}(\mathrm{M}_s) = \mathrm{G}_{\infty'}\otimes_{\mathrm{G}_{s\times_k\infty'}}\mathrm{M}_s =: \mathrm{Ind}_{\mathrm{G}_{s\times_k\infty'}}^{\mathrm{G}_{\infty'}}(\mathrm{M}_s).$$

The proof of (i) is given by (1.3.1.2), the proof of which is relegated to the paper of Katz-Laumon *Transformation de Fourier et Majoration de Sommes Exponentiels* and is discussed next.

Intermezzo III: Katz-Laumon (2.4). All notation and references in this section are to *Transformation de Fourier et Majoration de Sommes Exponentiels* unless otherwise noted. Let us first discuss how their proof of (2.4.1) gives the theorem (2.4.4) about universal strong local acyclicity. With the notation of the proof of (2.4.1), and letting M throughout denote constant sheaf with value the A-module M, the translation trick allows us to replace $\mathbf{P}^1 \times \mathbf{A}^1$ by $\mathbf{P}^1 \times \mathbf{G}_m$. The claim is trivial away from $\infty \times \mathbf{G}_m$. It suffices to show that $\widetilde{\mathrm{pr}}_1$ is universally locally acyclic rel. $\mathrm{M} \otimes^{\mathrm{L}} j_! \mu^* \mathscr{L}_\psi$. By (2.4.2), this will be true iff the same is true rel. $\mathrm{M} \otimes^{\mathrm{L}} \tilde{f}_* \mathrm{A}_{\tilde{\mathrm{X}}}$. By the projection formula for $\tilde{f}$ & Leray, this is the same as $\widetilde{\mathrm{pr}}_1 \circ \tilde{f}$ being universally locally acyclic rel. M. As proven, there is a neighborhood of $\infty \times \mathbf{G}_m$ over which $\widetilde{\mathrm{pr}}_1 \circ \tilde{f}$ factors as a surjective radicial morphism followed by a smooth morphism; hence by universal local acyclicity of a smooth morphism, $\widetilde{\mathrm{pr}}_1 \circ \tilde{f}$ is universally locally acyclic rel. any locally constant sheaf, in particular rel. M.

In the proof itself, the formula for $[a]^* \mu^* \mathscr{L}_\psi$ depends on [**Sommes trig.**, 1.7.1]. The extension of the Artin-Schreier covering $\tilde{\mathrm{X}}$ is defined as the finite covering of $\mathbf{P}^1 \times \mathbf{G}_m$ defined by $\mathrm{X}_0 \mathrm{T}_1^q - \mathrm{X}_0 \mathrm{T}_1 \mathrm{T}_0^{q-1} = \mathrm{X}_1 \mathrm{T}_0^q y$. It suffices to verify the various properties of $\tilde{\mathrm{X}}$ when $\mathrm{S} = \mathbf{F}_q$. Note that $\tilde{\mathrm{X}}$ is integral as $t^q - t - xy$ doesn't have a root over $k(x, y)$, and $\mathbf{F}_q$ acts transitively on the roots by $t \mapsto t + \gamma$, $\gamma \in \mathbf{F}_q$. Pick any point $\operatorname{Spec} \mathbf{F}_q(y)$ of $\mathbf{G}_{m, \mathbf{F}_q}$, corresponding to a choice of $y \in \overline{\mathbf{F}}_q^\times$ or y as a transcendental generator of $\mathbf{F}_q(y)/\mathbf{F}_q$. On one hand, the point $\infty \in \mathbf{P}^1_{\mathbf{F}_q(y)}$ has one point of $\tilde{\mathrm{X}} \times_{\mathbf{G}_m} \operatorname{Spec} \mathbf{F}_q(y)$ lying over it, since when $\mathrm{X}_0 = 0$, T_0 is nilpotent. On the other, $\mathrm{T}_0 = 0$ implies $\mathrm{X}_0 = 0$ as then T_1 is a unit. Therefore T_0 generates the maximal ideal of the local ring of the point of $\tilde{\mathrm{X}} \times_{\mathbf{G}_m} \operatorname{Spec} \mathbf{F}_q(y)$ lying over $\infty \times_{\mathbf{F}_q} \mathbf{F}_q(y)$, which shows that every point of $\tilde{f}^{-1}(\infty \times_{\mathbf{F}_q} \mathbf{F}_q(y))$ is totally ramified over $\infty \times_{\mathbf{F}_q} \mathbf{F}_q(y)$, and also shows that regardless of whether or not we extend scalars to $\mathbf{F} := \overline{\mathbf{F}}_q$, the following is true: The points of $\tilde{f}^{-1}(\infty \times \mathbf{G}_m)$ are in bijection with the points of $\infty \times \mathbf{G}_m$, hence also with the points of $\mathbf{G}_m$; given a point $a \in \mathbf{G}_m$, the maximal ideal $\mathfrak{m}$ of the corresponding point of $\tilde{\mathrm{X}}$ is generated by $1 + \operatorname{codim} a$ elements: if a is closed, $\mathfrak{m}$ is generated by T_0 and the minimal polynomial of y, and if a is the generic point, $\mathfrak{m}$ is generated by T_0 only. This implies that $\tilde{f}^{-1}(\infty \times_{\mathbf{F}_q} \mathbf{G}_{m, \mathbf{F}_q})$ is geometrically regular over $\mathbf{F}_q$, hence, in light of the lemma in the note to [**BBD**, 2.2.1.0], smooth over $\mathbf{F}_q$.

Checking that $\tilde{X}$ is étale over $\mathbf{A}^1 \times \mathbf{G}_m$ amounts to showing that for any choice of $x \in \mathbf{F}$, $y \in \mathbf{F}^\times$, the closed subscheme $Q \hookrightarrow \mathbf{P}^1_{\mathbf{F}_q(x,y)}$ cut out by $T_1^q - T_1 T_0^{q-1} = T_0^q xy$ is étale over $\mathbf{F}_q(x, y)$. The subscheme Q has empty intersection with $T_0 = 0$, and away from $T_0 = 0$, Q is defined by the separable polynomial $t^q - t = xy$, and as above, $\mathbf{F}_q$ acts transitively on the roots, so that Q either is the disjoint union of q copies of Spec $k(x, y)$ or is the spectrum of a separable field extension of $k(x, y)$. In any event: étale, and $\tilde{f}^{-1}(\mathbf{A}^1 \times \mathbf{G}_m)$ is smooth over $\mathbf{F}_q$.

The proof of (2.4.2) hits a snag when it is only assumed, as stated, that there exists a nontrivial additive character $\psi : \mathbf{F}_q \to A^\times$. The representation $f_* A_X$ coincides with the regular $A[\mathbf{F}_q]$-module $A[\mathbf{F}_q]$. If we write multiplicatively, this is the ring $A[x]/(x^q - 1)$ acting on itself as a free module of rank 1. The point is that of course the separable polynomial $x^q - 1$ need not split into linear factors over A if it has one root distinct from 1 in A. What is needed is a primitive root of this polynomial; i.e. a root which is not a root of any polynomial $x^{p^m} - 1$, $m < n$ (suppose $q = p^n$). (Actually, $x^q - 1$ splits as a product of n polynomials each of degree p, as $(x^{p^m} - 1) | (x^{p^{m+1}} - 1)$.) The existence of such a root in $A^\times$ implies that $A[\mathbf{F}_q]$ decomposes completely into q A-modules of rank 1 indexed by the q distinct characters $\mathbf{F}_q \to A^\times$; conversely, if such a root fails to exist, there will be irreducible factors of $x^q - 1$ of degree $d > 1$ corresponding to irreducible factors of $A[\mathbf{F}_q]$ free of rank d as A-module.

With this proviso, the decomposition of $f_* A_X$ is achieved, and all that is left to note for the decomposition of $\tilde{f}_* A_{\tilde{X}}$ is that the stalk over a point of the base of the latter are free A-modules of rank which coincides with the number of points in the geometric fiber [**SGAA**, VIII 5.5], and we have shown that every geometric point of $\infty \times \mathbf{G}_m$ has precisely one point of $\tilde{X}$ lying over it, so that the stalk of $\tilde{f}_* A_{\tilde{X}}$ is free of rank 1 at every geometric point of $\infty \times \mathbf{G}_m$. As the direct image of a constant sheaf on the generic point of a normal integral scheme is constant (c.f. remark at the end of Reprise),

$$\tilde{f}_* A_{\tilde{X}} = \tilde{f}_* j'_* A_X = j_* f_* A_X = A_{\mathbf{P}^1 \times \mathbf{G}_m} \oplus \Big(\bigoplus_{\psi' \neq 1} j_* \mu^* \mathscr{L}_{\psi'} \Big),$$

and by rank considerations we must have $j_* \mu^* \mathscr{L}_{\psi'} = j_! \mu^* \mathscr{L}_{\psi'}$ for all nontrivial ψ.

To see this last fact directly, observe that given a geometric point $\overline{x}$ centered on a point $x \in \tilde{\mathrm{X}}$ with image $y \in \infty \times \mathbf{G}_m$, and letting $\overline{y}$ denote the image of $\overline{x}$, let $\tilde{\mathrm{X}}_{(\overline{x})}$, resp. $\mathrm{Y}_{(\overline{y})}$ denote the strict henselizations of $\tilde{\mathrm{X}}$, resp. $\mathrm{Y} = \mathbf{P}^1 \times \mathbf{G}_m$ at $\overline{x}$ and $\overline{y}$, respectively. They are regular local rings. The map on generic points $\eta_{\overline{x}} \longrightarrow \eta_{\overline{y}}$ is still given by $t^q - t = xy$, as this still gives a finite extension of degree q of $\mathrm{Y}_{(\overline{y})}$ so that if ξ denotes the generic point of $\mathbf{G}_m$, the uniformizer of $\mathrm{Y}_{(\overline{y})} \times_{\mathbf{G}_m} \xi$ is the q^{th} power of that of $\mathrm{X}_{(\overline{x})} \times_{\mathbf{G}_m} \xi$. It follows that provided $\psi' \neq 1$, $\mathrm{Gal}(\overline{\eta}_y/\eta_{\overline{y}})$ acts nontrivially on the reciprocal image of $\mathscr{L}_{\psi'}$ to $\mathrm{Y}_{(\overline{y})}$, as $\mathrm{Gal}(\eta_{\overline{x}}/\eta_{\overline{y}})$ does. As the stalk of $j_*\mathscr{L}_{\psi'}$ at $\overline{y}$ coincides with the invariants of the $\mathrm{Gal}(\overline{\eta}_y/\eta_{\overline{y}})$-module A, $j_*(\mathscr{L}_{\psi'})_{\overline{y}} = 0$.

In the last paragraph of the proof, we can do all this assuming $\mathrm{S} = \mathbf{F}_q$. Fix coordinates (τ, y) on $\tilde{f}^{-1}(\mathrm{U})$ via

$$\mathbf{F}_q[\tau, y, y^{-1}] \to \mathbf{F}_q[\tau, \xi, y, y^{-1}]/(\xi(1 - \tau^{q-1}) - \tau^q y).$$

The fiber of this map over the closed subscheme of $\mathbf{A}^1 \times \mathbf{G}_m$ defined by τ^{q-1} is empty. Therefore (τ, y) factors through the complement $\mathrm{U}' \times \mathbf{G}_m$ of this closed subscheme, which corresponds to inverting $(1 - \tau^{q-1})$, after which the above map on functions becomes an isomorphism. As $\xi = \tau^q y/(1 - \tau^{q-1})$ in (τ, y) coordinates, $\tilde{\mathrm{pr}}_1 \circ \tilde{f}$ takes the stated form. The following diagram commutes.

$$\begin{tikzcd}
 & & \tilde{f}^{-1}(\mathrm{U}) \arrow[lld, bend right, "\tilde{\mathrm{pr}}_1 \circ \tilde{f}"'] \arrow[d, "(\tau,y)"', "\wr"] \\
\mathbf{A}^1 & \mathrm{U}' \times \mathbf{G}_m \arrow[l, "\upsilon"'] & \mathrm{U}' \times \mathbf{G}_m \arrow[l, "\kappa"']
\end{tikzcd}$$

$$\mathbf{F}_q[\xi] \xrightarrow{\xi \mapsto \tau' y'} \mathbf{F}_q[\tau', y', y'^{-1}, (1 - \tau'^{q-1})^{-1}] \xrightarrow{(\tau', y') \mapsto (\tau^q, y/(1-\tau^{q-1}))} \mathbf{F}_q[\tau, y, y^{-1}, (1 - \tau^{q-1})^{-1}]$$

The map $\upsilon : \mathrm{U}' \times \mathbf{G}_m \to \mathbf{A}^1$ is smooth since the coordinate ring of the former can be written as $\mathrm{S} := \mathbf{F}_q[\xi, y', a, b]/(ay' - 1, b(1 - \xi a) - 1)$, and

$$\det \begin{pmatrix} y' & -b\xi \\ 0 & 1 - \xi a \end{pmatrix} = y'(1 - \xi a)$$

is invertible in S with inverse ab, so that $\mathbf{F}_q[\xi] \to \mathrm{S}$ is standard smooth and hence smooth (Stacks tags `00T6` & `00T7`).

To verify that κ is radicial and surjective, it suffices to show κ is bijective and induces purely inseparable residual extensions. This can be done by verifying that for each $x \in \mathrm{U}' \times \mathbf{G}_m$, the scheme-theoretic fiber $\kappa^{-1}(x)$ is isomorphic to the spectrum of an artinian local ring with purely inseparable residual extension.

Picking a point x amounts to choosing a field $\mathbf{F}_q(\alpha, \beta)$ such that $\alpha^{q-1} \neq 1$; as this is trivially satisfied when α is transcendental and Frobenius is an isomorphism $\mathbf{F}_q(\alpha)$ when α is algebraic, $\alpha^{q-1} \neq 1$ is equivalent to the condition that $\alpha \notin \mathbf{F}_q$. The fiber $\kappa^{-1}(x)$ in this case is isomorphic to the spectrum of the finite $\mathbf{F}_q(\alpha, \beta)$-algebra

$$\begin{aligned} \mathrm{A} &:= \mathbf{F}_q(\alpha, \beta)[\tau, y, y^{-1}, (1-\tau^{q-1})^{-1}]/(\tau^q - \alpha, y - \beta(1-\tau^{q-1})) \\ &\simeq \mathbf{F}_q(\alpha, \beta)[\tau, (1-\tau^{q-1})^{-1}]/(\tau^q - \alpha); \end{aligned}$$

α (resp. β) is algebraic over $\mathbf{F}_q$ iff x does not map to the generic point of U' (resp. $\mathbf{G}_m$). If α is transcendental then

$$\mathrm{A} \simeq \mathbf{F}_q(\alpha, \beta)[\tau]/(\tau^q - \alpha)$$

is a field which is purely inseparable over $\mathbf{F}_q(\alpha, \beta)$. If α is algebraic then Frobenius induces an isomorphism of $\mathbf{F}_q(\alpha)$ and there exists a $\sqrt[q]{\alpha} \in \mathbf{F}_q(\alpha)$ so that $\tau^q - \alpha = (\tau - \sqrt[q]{\alpha})^q$; A is therefore isomorphic to a localization of a local Artinian ring with trivial residual extension, and we need check only that $\kappa^{-1}(x)$ is nonempty; i.e. that $\mathrm{A} \neq 0$. This is true iff $1 - \tau^{q-1}$ is not nilpotent, in which case it's already a unit in A. As $\alpha \notin \mathbf{F}_q$ by assumption and $\alpha \neq \sqrt[q]{\alpha}$ so that $(\sqrt[q]{\alpha})^{q-1} \neq 1$ and $1 - \tau^{q-1}$ indeed doesn't go to zero under $\tau \mapsto \sqrt[q]{\alpha}$.

2.4.1. Back to Laumon. The point is that π/π', π'/π, and $1/\pi\pi'$ determine three distinct rational maps $\mathrm{T} \times_k \mathrm{T}' \to \mathbf{G}_{a,k}$ (regular over $\mathrm{T} \times \eta'$, $\eta \times \mathrm{T}'$, $\eta \times \eta'$, respectively), and $\mathscr{L}_\psi(\pi/\pi')$, $\mathscr{L}_\psi(\pi'/\pi)$, and $\mathscr{L}_\psi(1/\pi\pi')$ are the reciprocal images of $\mathscr{L}_\psi$ defined over these open loci of $\mathrm{T} \times \mathrm{T}$ via these various maps.

By way of preface, it would seem from the discussion below that T implicitly is assumed to coincide with $\operatorname{Spec} k\{x\}$. Certainly it would seem that Néron desingularization should not be necessary for the result (2.4.2.1) cited as 'immediate,' or a somewhat esoteric result about the Galois group of generic points of henselian d.v.r.s to connect (2.4.2.2) to the discussion in (2.2.2). Perhaps there is an implicit assumption I've missed, or far simple ways to obtain these results with no assumption on T other than $k\{\pi\} \subset \mathrm{R} \subset k[[x]]$. As remarked in the note to (2.1.1), one must assume $k \subset \mathrm{R}$; in contrast with an equicharacteristic complete d.v.r., an equicharacteristic henselian d.v.r. need not contain a coefficient field.

(2.4.2.1) As (i), (ii), and (iii) are interchangeable, we dissect (i) as an example. The map $\pi' : \eta' \to \eta_{\infty'}$ is specified by $\pi' \mapsto 1/x'$. As $1/x'$ is a uniformizer for the local ring of D′ at ∞′ (and also its henselization $\mathrm{D}'_{(\infty')}$), from the data of $\pi' \mapsto 1/x'$ we get a morphism $\mathrm{T}' \to \operatorname{Spec} \mathcal{O}_{\mathrm{D}',\infty'}$ and by the universal property of henselization (Raynaud, *Anneaux Locaux Henséliens*, VIII Déf. 1) a unique factorization of this morphism to a commutative triangle

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{\mathrm{D}',\infty'} & \mathrm{D}'_{(\infty')} \arrow[l] \\ & \mathrm{T}' \arrow[u, "\pi'"'] \arrow[ul] \end{tikzcd}$$

which recovers π' on the generic fiber. Note that $\mathrm{R}\Psi_{\eta_{\infty'}} = \mathrm{R}\Phi_{\eta_{\infty'}}$ as in (2.3.3.1). It is asserted that the base-change morphism

$$(\pi \times \pi')^* \mathrm{R}\Psi_{\eta_{\infty'}} \to \mathrm{R}\Psi_{\eta'}(\pi \times \pi')^*$$

is an isomorphism. As $\pi \times \pi' = (1 \times \pi') \circ (\pi \times 1)$, we first deal with $(1 \times \pi')^*$, which is the morphism induced by change of trait $\pi' : \mathrm{T}' \to \mathrm{D}'_{(\infty')}$ and is an isomorphism by [**Th. finitude**, 3.7]. On the other hand, letting $\mathrm{V}_0 := k[\pi]_{(\pi)}$ and $\mathrm{T}_0 := \operatorname{Spec} \mathrm{V}_0$, the morphism $\pi : \mathrm{T} \to \mathrm{A}$ factors as $\mathrm{T} \xrightarrow{\pi_1} \mathrm{T}_0 \to \mathrm{A}$ where the second morphism is ind-étale and therefore commutes with base change by passage to the limit. It doesn't seem 'immediate' to me that

$$(\pi_1 \times 1)^* \mathrm{R}\Psi_{\eta'} \to \mathrm{R}\Psi_{\eta'}(\pi_1 \times 1)^*$$

is an isomorphism. The cleanest way I know to show this is to invoke Néron desingularisation in the form [**SGA 7**, I 0.5.1]. In the paragraphs following the statement of this lemma (0.5.1), it's explained how the conditions of the lemma are satisfied in the case of $\pi_1 : \mathrm{T} \to \mathrm{T}_0$. The residual extension is trivial in our case; to see that the extension η/η_0

is separable, it suffices to show that $k(\eta)/k(\pi)$ is separable. For this one need only show that $k(\pi^{1/p}) \otimes_{k(\pi)} k(\eta) \simeq k(\eta)[t]/(t^p - \pi)$ is separable. Assuming $p > 1$, this is true iff π does not have a p^{th} root in $k(\eta)$, which is true as π is a uniformizer. Néron says that π_1 is a limit of smooth morphisms and therefore commutes with $\mathrm{R}\Psi$.

On a related note, I know of a funny way to show the more general

Lemma. — *Let* $f : \mathrm{T} \to \mathrm{T}_0$ *be a surjective morphism of henselian traits and suppose*

(α) *the special fiber* $\mathrm{T} \times_{\mathrm{T}_0} s_0$ *is reduced,*
(β) $k(s_0) \subset k(s)$ *is purely inseparable, and*
(γ) *the maximal purely inseparable extension of the completion* $\widehat{k(\eta_0)}$ *is dense in* $\widehat{k(\eta_1)}$.

Then for every scheme $\mathrm{X}_0 \to \mathrm{T}_0$, f *induces an equivalence of topoi* $\widetilde{\mathrm{X}}_{\text{ét}} \to \widetilde{(\mathrm{X}_0)}_{\text{ét}}$, *where* $\mathrm{X} := \mathrm{X}_0 \times_{\mathrm{T}_0} \mathrm{T}$.

This applies in the situation where $\mathrm{T}_0 = \operatorname{Spec} \mathrm{R}_0$ is equicharacteristic and R_0 contains a coefficient field mapping isomorphically onto its residue field, which is assumed perfect, as assumed in (2.4.1), because of the inclusion $k\{\pi\} \subset \mathrm{R}_0 \subset k[[\pi]]$.

Proof. — By (2.4.1–2.4.3) in Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. IHES 78, reciprocal image $\mathrm{L} \mapsto \mathrm{L} \otimes_{k(\eta_0)} k(\eta)$ induces an equivalence of categories of finite separable extensions of $k(\eta_0)$ and $k(\eta)$, and hence induces an isomorphism $\mathrm{Gal}(\overline{\eta}/\eta) \xrightarrow{\sim} \mathrm{Gal}(\overline{\eta}/\eta_0)$. The same is true for the closed fiber (replacing η by s) if we restrict to finite Galois extensions by e.g. Stacks tag `030M`. Recall [**SGAA**, IV 9.5.4], which says in our case that the category of sheaves on $\mathrm{X} \to \mathrm{T}$ is equivalent to the category of triples

$$(\mathcal{F}_s, \mathcal{F}_\eta, \varphi : \mathcal{F}_s \to i^* j_* \mathcal{F}_\eta) \qquad \text{where } \eta \xhookrightarrow{j} \mathrm{T} \xhookleftarrow{i} s,$$

($\mathcal{F}_s$ a sheaf on $\mathrm{X} \times_{\mathrm{T}} s$, $\mathcal{F}_\eta$ a sheaf on $\mathrm{X} \times_{\mathrm{T}} \eta$) via the functor $\mathcal{F} \mapsto (\mathcal{F}_s, \mathcal{F}_\eta, \varphi)$ where

$$\varphi : \mathcal{F}_s = i^* \mathcal{F} \to i^* j_* j^* \mathcal{F} = \mathrm{sp}_* \mathcal{F}_\eta,$$

with quasi-inverse the functor which associates to $(\mathcal{F}_s, \mathcal{F}_\eta, \varphi)$ the object W defined by the cartesian square

$$\begin{array}{ccc} W & \longrightarrow & j_*\mathcal{F}_\eta \\ \downarrow & \ulcorner & \downarrow \\ i_*\mathcal{F}_s & \xrightarrow{i_*\varphi} & i_*i^*j_*\mathcal{F}_\eta. \end{array}$$

Decorating everything with a subscript 0, the same is true for $X_0 \to T_0$. Last, recall [**SGA 7**, XIII 1.3.3 (ii)], which says that $Y \to \operatorname{Spec} k$ with separable closure $\overline{k}$ and a sheaf $\mathcal{F}$ on Y, the functor $\mathcal{F} \mapsto \overline{\mathcal{F}}$ induces an equivalence between the category of sheaves of sets on Y with that of sheaves of sets on $\overline{Y}$ equipped with a continuous action of $\operatorname{Gal}(\overline{k}/k)$ compatible with the action of $\operatorname{Gal}(\overline{k}/k)$ on $\overline{Y}$. □

(2.4.2.2) By the Berkovich result in the proof of the previous lemma, we may assume $T = \operatorname{Spec} k\{x\}$; now invoke not (2.2.2.1) but rather the discussion following (2.2.2.2). A choice of retraction for i_* (in the notation of (2.2.2)) corresponds to an extension of V to a lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on $\mathbf{G}_{m,k}$. Let $j : \mathbf{G}_{m,k} \hookrightarrow A$; for (i) put $K := j_!\mathcal{F}$ and for (ii) and (iii) put $\alpha_! j_! \mathcal{F}[1]$. In all three cases, K is a perverse sheaf [**BBD**, 4.1.3] and $\pi^*K = V_![1]$.

2.5. In order to complete the proof (4.4) of Deligne's main theorem in Weil II, it is only necessary to have (2.5.3.1) (i) (Kummer does not make an appearance). The proof of (2.5.3.1) (i) is an elegant synthesis of (2.3.2)–(2.4.2), and the only necessary auxiliary computation is that of $\mathscr{F}(\overline{\mathbf{Q}}_{\ell,A-\{0\}})|\eta_{\infty'}$. There is an exact sequence of perverse sheaves on A [**BBD**, 4.1.10]

$$0 \to \overline{\mathbf{Q}}_{\ell,\{0\}} \to \overline{\mathbf{Q}}_{\ell,A-\{0\}}[1] \to \overline{\mathbf{Q}}_{\ell,A}[1] \to 0.$$

Applying $\mathscr{F}$ gives the exact sequence of perverse sheaves on A′

$$0 \to \overline{\mathbf{Q}}_{\ell,A'}[1] \to \mathscr{F}(\overline{\mathbf{Q}}_{\ell,A-\{0\}})[1] \to \overline{\mathbf{Q}}_{\ell,\{0'\}}(-1) \to 0$$

which finds $\mathscr{F}(\overline{\mathbf{Q}}_{\ell,A-\{0\}})|\eta_{\infty'} = \overline{\mathbf{Q}}_{\ell,\eta_{\infty'}}$ and completes the proof of (i).

4.2.2. (4.2.2.2) Write $\mathscr{F} := F$ to harmonize with Deligne's notation. The condition that 'the action of I_s be unipotent of echelon 2.' corresponds, in the vocabulary of [**Weil II**, 1.7.2], to the property that the filtration of local monodromy on $\mathscr{F}_{\overline{\eta}}$ has at most

3 nonzero graded pieces; i.e. if I does not act trivially on $\mathscr{F}_{\overline{\eta}}$, then the nilpotent operator N arising from the logarithm of the unipotent part of the local monodromy has $\mathrm{N} \neq 0$ but $\mathrm{N}^2 = 0$. In this case, as described in [**Weil II**, 1.6.1], the filtration has $\mathrm{M}_1 = \mathscr{F}_{\overline{\eta}}$, $\mathrm{M}_0 = \mathrm{F}_{\overline{\eta}}^{\mathrm{I}}$, and $\mathrm{M}_{-1} = \operatorname{im}\mathrm{N}$, and $\mathrm{Gr}_1^{\mathrm{M}} = \mathrm{F}_{\overline{\eta}}/\mathrm{F}_{\overline{\eta}}^{\mathrm{I}}$, which [**Weil II**, 1.8.4] says is ι-pure of weight $w+1$.

(4.2.2.3) Recall the decomposition (1.6.14.3) of Weil II and the fact that there is a typo in it (c.f. note to 1.6.14). Recall that, as in the proof of [**Weil II**, 1.8.4], if $s \in \mathrm{S}$, the fiber $(j_*\mathscr{F})_{\overline{s}}$ coincides with $\ker \mathrm{N} = \mathscr{F}_{\overline{\eta}}^{\mathrm{I}}$, where N is the logarithm of the local monodromy (we can assume that all of I acts unipotently). Recall that N is compatible with the filtration of local monodromy, that $\mathrm{Gr}_i^{\mathrm{M}}(\ker \mathrm{N}) \xrightarrow{\sim} \mathrm{P}_i$ [**Weil II**, 1.6.6], and that $\mathrm{P}_i = 0$ for $i > 0$ [**Weil II**, 1.6.4]. The eigenvalues of F_s on $j_*(\mathscr{F})_{\overline{s}}$ are in bijection with the eigenvalues of F on the $\mathrm{P}_i(\mathscr{F}_{\overline{\eta}})$, where F is the conjugation class of liftings of Frobenius in the Weil group (or one such [**Weil II**, 1.7.4]). Now Laumon's proof goes through after you correct all the typos: $\mathrm{P}_{-i}(\mathscr{F}_{\overline{\eta}})$ is ι-pure of weight $w+i$ [**Weil II**, 1.8.4],

$$\mathrm{P}_{-i}(\check{\mathscr{F}}_{\overline{\eta}}) \simeq \mathrm{P}_{-i}(\mathscr{F}_{\overline{\eta}})^{\vee}(i),$$

and α is an eigenvalue of Frobenius on some $\mathrm{P}_{-i}(\mathscr{F}_{\overline{\eta}})$ (for this you need to know that the filtration of local monodromy is stable under the action of $\mathrm{W}(\overline{\eta}, \eta)$, which is true by [**Weil II**, 1.7.5, 1.8.5]).

4.3.1. (4.3.1.1) It is implicitly claimed that $\mathrm{H}^0(\mathrm{U}, \mathscr{F}) = 0$, which is a consequence of the fact that $\mathscr{F}$ is assumed irredudible and not geometrically constant. If $\overline{u}$ is a geometric point of $\overline{\mathrm{U}} = \mathrm{U} \times_{\mathbf{F}_q} \mathbf{F}$, there is an exact sequence

$$e \to \pi_1(\overline{\mathrm{U}}, \overline{u}) \to \pi_1(\mathrm{U}, \overline{u}) \to \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q) \to e$$

(SGA 1 6.1), and $\mathscr{F}$ is not geometrically constant if $\pi_1(\overline{\mathrm{U}}, \overline{u})$ does not act trivially on $\mathscr{F}_u$; i.e. the reciprocal image of $\mathscr{F}$ on $\overline{\mathrm{U}}$ is not constant. As $\mathscr{F}_{\overline{u}}^{\pi_1(\overline{\mathrm{U}}, \overline{u})}$ is stable under $\pi_1(\mathrm{U}, \overline{u})$, it must be 0, as $\mathscr{F}$ is assumed irreducible, so $\Gamma(\mathrm{U}, \mathscr{F}) = 0$.

4.3.2. Following Deligne, we decorate with a subscript 0 those objects over $\mathbf{F}_q$ and remove this subscript to indicate the extension of scalars to $\mathbf{F}$.

(4.3.2.1) Reduction to $X_0 = D_0$. We notate $j_0 : U_0 \hookrightarrow X_0$ and $\mathscr{F}_0$ lisse on U_0. The choice of a nonconstant meromorphic function on X_0 gives rise to a finite morphism $f_0 : X_0 \to D_0$. The only condition on the function is that it induce a morphism with nonempty étale locus. As X_0 is smooth, this is easy: just pick a point $x \in X_0$ and a generator of the local ring $\mathscr{O}_{X,x}$ (SGA 1 I 9.11). Leray gives

$$R\Gamma(D, f_{0*}j_*\mathscr{F}_0) \simeq R\Gamma(X, j_*\mathscr{F}_0).$$

It suffices to find some open $j_0' : U_0' \hookrightarrow D_0$ a lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}_0'$ on U_0', ι-pure of weight w, and an identification $f_{0*}j_{0*}\mathscr{F}_0 \simeq j'_{0*}\mathscr{F}_0'$ on D_0.

Lemma. — Let $\mathscr{F}$ *be a lisse sheaf on a normal connected curve* S *and* $j : U \hookrightarrow S$ *an open immersion. The unit of adjunction* $\mathscr{F} \to j_*j^*\mathscr{F}$ *is an isomorphism.*

Admitting the lemma, pick some nonempty U_0' such that $X_0 \times_{D_0} U_0' \to U_0'$ is étale, and let $u_0 : V_0 := U_0 \cap (X_0 \times_{D_0} U_0') \hookrightarrow U_0$. As the diagram

$$\begin{tikzcd}
V_0 \arrow[r, "u_0"] \arrow[d, "f_0|V_0"] & U_0 \arrow[r, "j_0"] & X_0 \arrow[d, "f_0"] \\
U_0' \arrow[rr, "j_0'"] & & D_0
\end{tikzcd}$$

commutes,

$$j'_{0*}(f_0|V_0)_*(u_0^*\mathscr{F}_0) = f_{0*}j_{0*}u_{0*}u_0^*\mathscr{F}_0 \simeq f_{0*}j_{0*}\mathscr{F}_0,$$

and the sheaf $(f_0|V_0)_*(u_0^*\mathscr{F}_0)$ is lisse. More precisely, if $\overline{\eta}$ denotes a geometric generic point of V_0 (and its image in U_0'), and $u_0^*\mathscr{F}_0$ is defined by the monodromy representation $\pi_1(V_0, \overline{\eta}) \to \operatorname{Aut}\mathscr{F}_{\overline{\eta}}$, the étale $f_0|V_0$ induces a morphism

$$\pi_1(V_0, \overline{\eta}) \to \pi_1(U_0', \overline{\eta})$$

which is an injection of the former group onto the open subgroup of the latter corresponding to the revêtement étale $f_0|V_0$. The sheaf $(f_0|V_0)_*(u_0^*\mathscr{F}_0)$ is defined by the induced representation

$$\mathscr{F}_{\overline{\eta}} \otimes_{\pi_1(V_0,\overline{\eta})} \pi_1(U_0', \overline{\eta}).$$

Therefore we take $\mathscr{F}_0' := (f_0|V_0)_*(u_0^*\mathscr{F}_0)$.

Proof of lemma. — The lemma can be recovered as a corollary of the *reprise* below, and, perhaps more cheekily, from [**BBD**, 4.3.2] in light of the note to 1.4.2 above. A simple direct proof goes as follows. Checking the statement fiberwise at a geometric point $\overline{s}$ centered on a point $s \in \mathrm{S} - \mathrm{U}$ reduces us to the setting of a henselian trait $(\mathrm{S}, \eta, s, \overline{\eta}, \overline{s})$ [**Weil II**, 0.6] and $\mathscr{F}$ a lisse sheaf on S. We have the usual exact sequence

$$e \to \mathrm{I} \to \mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{Gal}(\overline{s}/s) \to e.$$

where by SGA 1 V 8.2 & Arcata IV 2.2 the map $\mathrm{Gal}(\overline{\eta}/\eta) \twoheadrightarrow \mathrm{Gal}(\overline{s}, s)$ factors as

$$\mathrm{Gal}(\overline{\eta}/\eta) \twoheadrightarrow \pi_1(\mathrm{S}, \overline{\eta}) \xrightarrow{\sim} \pi_1(\mathrm{S}, \overline{s}) \xrightarrow{\sim} \pi_1(s, \overline{s}) \simeq \mathrm{Gal}(\overline{s}, s).$$

The stalk $(\eta_*\eta^*\mathscr{F})_{\overline{s}}$ can be identified with $\mathscr{F}_{\overline{\eta}}^{\mathrm{I}}$, where here $\mathscr{F}_{\overline{\eta}}$ is the $\mathrm{Gal}(\overline{\eta}, \eta)$-representation defined by $\mathrm{Gal}(\overline{\eta}, \eta) \to \pi_1(\mathrm{S}, \overline{\eta})$. The factorization above shows that I dies in this quotient and therefore acts trivially. □

Returning to the proof of (4.3.2.1), Poincaré duality on D as stated rests on

Theorem (*Deligne, SGA* $4\frac{1}{2}$ Dualité *1.3*). — *Let* $j : \mathrm{U} \hookrightarrow \mathrm{S}$ *be a dense open of a regular scheme* S *purely of dimension 1 and* $\mathscr{F}$ *a locally constant constructible sheaf of* $\mathbf{Z}/n$*-modules on* U. *One has* $\mathrm{D} j_*\mathscr{F} = j_*\mathrm{D}\mathscr{F}$, *i.e.* $\underline{\mathrm{Hom}}(j_*\mathscr{F}, \mathbf{Z}/n) = j_*\underline{\mathrm{Hom}}(\mathrm{F}, \mathbf{Z}/n_{\mathrm{U}})$ *and* $\underline{\mathrm{Ext}}^i(j_*\mathscr{F}, \mathbf{Z}/n) = 0$ *for* $i > 0$.

Let $\overline{\mathrm{A}} := \mathrm{A} \otimes_{\mathbf{F}_q} \overline{k}$ and $\overline{\mathrm{D}}$ likewise. To calculate $\mathrm{H}^i_c(\overline{\mathrm{A}}, j_*\mathscr{F})$, the perfect pairing

$$\mathrm{H}^i_c(\overline{\mathrm{A}}, j_*\mathscr{F}) \otimes \mathrm{H}^{2-i}(\overline{\mathrm{A}}, j_*\check{\mathscr{F}}) \to \overline{\mathbf{Q}}_\ell(-1)$$

and Artin's theorem give $\mathrm{H}^0_c(\overline{\mathrm{A}}, j_*\mathscr{F}) \simeq 0$ and $\mathrm{H}^2_c(\overline{\mathrm{A}}, j_*\mathscr{F}) \simeq \mathrm{M}(-1)$, as $\mathrm{H}^0(\overline{\mathrm{A}}, j_*\check{\mathscr{F}}) = \check{\mathrm{M}}$. The exact sequence

$$0 = \mathrm{H}^0_c(\overline{\mathrm{A}}, j_*\mathscr{F}) \to \mathrm{H}^0(\overline{\mathrm{D}}, j_*\mathscr{F}) \to \mathscr{F}_{\overline{\infty}} \xrightarrow{\partial} \mathrm{H}^1_c(\overline{\mathrm{A}}, j_*\mathscr{F}) \to \mathrm{H}^1(\overline{\mathrm{D}}, j_*\mathscr{F}) \to 0$$

reduces the calculation of $\mathrm{H}^1_c(\overline{\mathrm{A}}, j_*\mathscr{F})$ to that of $\mathrm{H}^1(\overline{\mathrm{D}}, j_*\mathscr{F})$, as $\partial = 0$. Relative purity (Arcata V 3.4) gives that $j_*\mathscr{F}$ is constant on $\overline{\mathrm{A}}$, and the standard computation $\mathrm{H}^1(\overline{\mathrm{X}}, \mathbf{Z}/\ell^n(1)) \simeq \underline{\mathrm{Pic}}(\overline{\mathrm{X}}/\overline{k})_{\ell^n}$ for a smooth connected curve $\overline{\mathrm{X}}$ over $\overline{k}$ shows that $\mathrm{H}^1(\overline{\mathrm{D}}, \mathbf{Z}/\ell^n) = 0$ and hence the same is true for $\mathrm{H}^1(\overline{\mathrm{D}}, j_*\mathscr{F})$ (Arcata III 3.1, IV 6.2).

Reprise: le sorite des faisceaux localement constants. The action of local inertia on a normal curve S determines the net locus (and therefore the étale locus by SGA 1 I 9.11) of the normalization of S in a finite separable extension of its function field in the following way. Let $\overline{\eta}$ be a geometric point centered on the generic point η of S, L a finite Galois extension of the function field of S, and S′ the normalization of S in L. Let $\overline{s}$ be a geometric point centered on a closed point of S, $\mathrm{S}_{(s)}$ and $\mathrm{S}_{(\overline{s})}$ the spectra of the henselization and strict henselization, respectively of the local ring of S at s. Both are henselian traits. Let $\mathrm{S}'_{(s)} := \mathrm{S}' \times_{\mathrm{S}} \mathrm{S}_{(s)}$; $\mathrm{S}'_{(s)} \to \mathrm{S}_{(s)}$ is generically étale. As $\mathrm{S}_{(s)}$ is a henselian local ring, $\mathrm{S}'_{(s)}$ splits as a disjoint union of henselian local rings $\mathrm{S}'_{(s'_i)}$ indexed by $\xi_i \in \operatorname{Spec} \mathrm{L} \otimes_{k(\eta)} k(\eta_s)$ (in bijection with the scheme-theoretic fiber of $\mathrm{S}' \to \mathrm{S}$ over s) Let s'_i denote the corresponding closed point of $\mathrm{S}'_{(s'_i)}$ (and by abuse of notation the corresponding point in the scheme-theoretic fiber of $\mathrm{S}' \to \mathrm{S}$ over s). $\mathrm{S}'_{(s'_i)}$ is pro-étale over S′ and henselian, with the same residue field as $\mathscr{O}_{\mathrm{S}',s'_i}$, therefore coincides with the henselization of $\mathscr{O}_{\mathrm{S}',s'_i}$, hence is also normal. $\mathrm{S}'_{(s'_i)}$ is finite and generically étale over $\mathrm{S}_{(s)}$, *a fortiori* integral. Hence $\mathrm{S}'_{(s'_i)}$ coincides with the normalization of $\mathrm{S}_{(s)}$ in $k(\xi_i)$. Considering s'_i as a point in the scheme-theoretic fiber over s, $\mathrm{S}' \to \mathrm{S}$ is étale at s'_i iff $\mathrm{S}_{(s'_i)} \to \mathrm{S}_{(s)}$ by faithfully flat descent (Stacks tag `02VN`). Fixing some i and letting $s' := s'_i$, $\xi := \xi_i$ we reduce to studying the normalization of the henselian trait $\mathrm{S}_{(s)}$ in a finite separable extension $k(\xi)$ of its function field. Let $\overline{\eta}_s$ be a geometric point centered on the generic point η_s of $\mathrm{S}_{(s)}$. The local inertia at s is defined by the exact sequence

$$e \to \mathrm{I}_s \to \mathrm{Gal}(\overline{\eta}/\eta_s) \to \mathrm{Gal}(\overline{s}/s) \to e.$$

Compare the following lemma to [**SGA1**, Exp. V §2].

Lemma. — $\mathrm{S}_{(s')} \to \mathrm{S}_{(s)}$ *is étale iff* I_s *acts trivially on* $\mathrm{S}_{(s')}$.

Proof. — '⇒' Arcata IV 2.2. '⇐' $k(\xi)$ is a finite separable extension of $k(\eta_s)$; let K denote the Galois closure of $k(\xi)$ in $k(\overline{\eta})$, $\mathrm{S}_{(s)} =: \operatorname{Spec} \mathrm{A}$, $\mathrm{S}_{(s')} =: \operatorname{Spec} \mathrm{A}'$, and B the normalization of A in K. $\mathrm{A} \subset (\mathrm{A}')^{\mathrm{I}_s} = \mathrm{A}' \subset \mathrm{B}^{\mathrm{I}_s}$ and $\mathrm{A} \subset \mathrm{B}^{\mathrm{I}_s}$ is étale (Stacks tag `09EH`), so $\mathrm{S}_{(s')} \to \mathrm{S}_{(s)}$ is étale. □

A question that should be easy is: describe the category of lisse sheaves on a normal connected curve in terms of the Galois group of the function field. More precisely, given a lisse sheaf on an open subscheme S of a smooth complete curve $\overline{S}$, describe the locus of $\overline{S}$ over which the sheaf can be extended to a lisse sheaf. In the case of finite coefficients, the answer goes like this. Let η be a geometric point centered on the generic point η of the curve S. A l.c.c. sheaf $\mathscr{F}$ on S is the same as a revêtement étale $X \to S$; i.e. an open subgroup of $\pi_1(S, \overline{\eta})$. S in turns corresponds to a closed subgroup Q of $\mathrm{Gal}(\overline{\eta}, \eta)$ via the exact sequence

$$e \to Q \to \mathrm{Gal}(\overline{\eta}, \eta) \to \pi_1(S, \overline{\eta}) \to e.$$

To each point $s \in S$ we attach an inertia subgroup I_s which is a subgroup of $\mathrm{Gal}(\overline{\eta}/\eta_s)$, where η_s is the generic point of $S_{(s)}$. As $S_{(s)}$ is a projective limit of revêtements étales of S, I_s embeds into $\mathrm{Gal}(\overline{\eta}/\eta)$ via

$$I_s \subset \mathrm{Gal}(\overline{\eta}/\eta_s) \subset \mathrm{Gal}(\overline{\eta}/\eta).$$

We must have that each I_s be contained in Q, since as we saw above, the map $\mathrm{Gal}(\overline{\eta}/\eta_s) \twoheadrightarrow \mathrm{Gal}(\overline{s}/s)$ factors through the projection $\mathrm{Gal}(\overline{\eta}/\eta_s) \twoheadrightarrow \pi_1(S, \overline{\eta})$. On the other hand, there may be other points in $\overline{S} - S$ with nontrivial monodromy; i.e. the corresponding inertia acts nontrivially. This means that the direct image along $S \hookrightarrow \overline{S}$ of the sheaf on S represented by X is not locally constant on a neighborhood of such a point. Geometrically, we can take the normalization X′ of $\overline{S}$ in the function field of X; $X' \times_{\overline{S}} S \simeq X$ (SGA 1 I 10.2). X′ is étale at a point if it is net there (SGA 1 I 9.11), so $\mathscr{F}$ extends to a lisse sheaf over $U \hookrightarrow \overline{S}$ iff $X' \times_{\overline{S}} U \to U$ is net. By the lemma above, this is true iff the inertia at each point $u \in U$ acts trivially on the $\mathrm{Gal}(\overline{\eta}/\eta_u)$-representation corresponding to the sheaf on $\overline{S}_{(u)}$ represented by $X' \times_{\overline{S}} \overline{S}_{(u)}$. Properly said, I_u acts on the fiber of this sheaf at $\overline{\eta}$. If I_u acts trivially, then X′ is net in the fiber over u and X extends (via X′) to a revêtement étale of $S \cup \{u\}$. (By the above lemma, is isomorphic to the direct image of $\mathscr{F}$ under $S \hookrightarrow S \cup \{u\}$.) Thinking about I_u as a subgroup of $\mathrm{Gal}(\overline{\eta}, \eta)$, the condition that I_u act trivially on $X' \times_{\overline{S}} \overline{S}_{(u)}$ is equivalent to that I_u act trivially on the $\mathrm{Gal}(\overline{\eta}/\eta)$-representation $\mathscr{F}_{\overline{\eta}}$. Therefore, the locus of $\overline{S}$ over which $\mathscr{F}$ can be extended to a lisse sheaf coincides with the union of the $s \in \overline{S}$ such that I_s acts trivially on $\mathscr{F}_{\overline{\eta}}$. This justifies the

Proposition. — Let S *be a normal connected curve with generic point* η *and* $\mathscr{C}$ *the category with objects pairs* $(\mathscr{F}, U)$ *with* U *a nonempty open of* S *and* $\mathscr{F}$ *a l.c.c. sheaf* *on* U*, modulo the equivalence relation* $(\mathscr{F}, \mathrm{U}) \sim (\mathscr{G}, \mathrm{V})$ *if* $\mathscr{F}|\mathrm{U}\cap\mathrm{V} \simeq \mathscr{G}|\mathrm{U}\cap\mathrm{V}$*, and morphisms*

$$\mathrm{Hom}_{\mathscr{C}}((\mathscr{F}, \mathrm{U}) \to (\mathscr{G}, \mathrm{V})) = \varinjlim_{\mathrm{W}\subset\mathrm{U}\cap\mathrm{V}} \mathrm{Hom}(\mathscr{F}|\mathrm{W}, \mathscr{G}|\mathrm{W}),$$

limit taken over nonempty opens W *contained in* $\mathrm{U}\cap\mathrm{V}$*.* $\mathscr{C}$ *is equivalent to the category of finite separable extensions of the function field* $k(\eta)$ *of* S *and* $k(\eta)$*-algebra morphisms. By Grothendieck's Galois theory, this category is in turn equivalent to the category of finite sets with continuous* $\mathrm{Gal}(\overline{\eta}/\eta)$ *action. Given such an extension* L *of* $k(\eta)$*, the normalization* X *of* S *in* L *is étale over a nonempty open* $\mathrm{U} \subset \mathrm{S}$ *and represents a l.c.c. sheaf on* U*. The maximal* $\mathrm{U} \subset \mathrm{S}$ *such that* $\mathrm{X} \times_{\mathrm{S}} \mathrm{U} \to \mathrm{U}$ *is étale coincides with the open subscheme of* S *with closed points*

$$|\mathrm{U}| = \{s \in \mathrm{S} : \mathrm{I}_s \textit{ acts trivially on } \mathrm{L}\}$$

where here I_s *acts via* $\mathrm{I}_s \subset \mathrm{Gal}(\overline{\eta}/\eta_s) \subset \mathrm{Gal}(\overline{\eta}/\eta)$*,* η_s *the generic point of* $\mathrm{S}_{(s)}$*. If* $\mathscr{F}$ *is the sheaf of local sections for* X *over this* U*, the pair* $(\mathscr{F}, \mathrm{U})$ *is distinguished in its class by the property that* U *is maximal for the filtered partial order given by inclusion. Given* $j : \mathrm{V} \hookrightarrow \mathrm{U}$ *and* $(\mathscr{G}, \mathrm{V}) \sim (\mathscr{F}, \mathrm{U})$*,* $\mathscr{F} \xrightarrow{\sim} j_*\mathscr{G}$*.*

Corollary. — Given a lisse ($\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathrm{R}, \mathrm{E}_\lambda, \overline{\mathbf{Q}}_\ell$)*-sheaf* $\mathscr{F}$ *on a nonempty open subscheme* U *of a normal connected curve* S *with geometric generic point* η*, the maximal locus* $\mathrm{U}' \subset \mathrm{S}$ *over which* $\mathscr{F}$ *extends to a lisse sheaf* $\mathscr{F}'$ *is defined by its set of closed points*

$$|\mathrm{U}'| = \{s \in \mathrm{S} : \mathrm{I}_s \textit{ acts trivially on } \mathscr{F}_{\overline{\eta}}\}.$$

If $j : \mathrm{U} \hookrightarrow \mathrm{U}'$ *denotes the open immersion,* $\mathscr{F}' \xrightarrow{\sim} j_*\mathscr{F}$*.*

Proof. — Disregrading the module structure, the sheaf $\mathscr{F}$ is represented by a projective system of revêtements étales, to each of which the proposition compatibly applies. □

Corollary. — Let $\mathscr{F}$ *be a lisse sheaf on a normal connected curve* S *and* $j : \mathrm{U} \hookrightarrow \mathrm{S}$ *an open immersion. The unit of adjunction* $\mathscr{F} \to j_*j^*\mathscr{F}$ *is an isomorphism.*

Corollary. — Let $\mathrm{A} \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathrm{R}, \mathrm{E}_\lambda, \overline{\mathbf{Q}}_\ell\}$*. The category* $\mathscr{C}$ *as above with l.c.c. sheaves replaced by lisse* A*-sheaves on a normal connected curve* S *is equivalent to* *the full subcategory of* A*-modules of finite type with continuous action of* $\mathrm{Gal}(\overline{\eta}/\eta)$ *generated by those* A*-modules* V *with the property that* I_s *acts trivially on* V *for all but finitely many* $s \in |\mathrm{S}|$.

Proof. — By SGA 5 Exp. VI (c.f. note to *Sommes trig.* 1.2), such a sheaf is equivalent the data of its monodromy representation, which is equivalent to a projective system of representations. Each representation in the projective system corresponds to a revêtement étale of some nonempty open $\mathrm{U} \subset \mathrm{S}$ together with the appropriate module structure on its fiber. By the proposition, each revêtement étale is unramified over an open set which can be calculated from the action of the local inertia. In order that there exist a nonempty open $\mathrm{U} \subset \mathrm{S}$ over which all the revêtements étales in the projective system are unramified (so that the corresponding projective system defines a lisse sheaf on U) it is necessary and sufficient that the local inertia at all but finitely many points act trivially on all the revêtements étales in the projective system. This is true for some I_s iff I_s acts trivially on the fiber of the lisse sheaf considered as A-module. ◻

Corollary. — *The kernel of the map*

$$\mathrm{Gal}(\overline{\eta}/\eta) \twoheadrightarrow \pi_1(\mathrm{U}, \overline{\eta})$$

is topologically generated by the subgroups $\{\mathrm{I}_s : s \in |\mathrm{U}|\}$.

Proof. — On one hand, the category of l.c.c. sheaves on U is a Galois category with group $\pi_1(\mathrm{U}, \overline{\eta})$. On the other, it is equivalent to the category of finite $\mathrm{Gal}(\overline{\eta}/\eta)$-sets on which I_s acts trivially for all $s \in \mathrm{U}$. ◻

Remark. The same argument can be used to extend a lisse sheaf on a normal scheme over the generic point of a divisor. If the scheme is moreover smooth, the sheaf can be extended over the points of codimension > 1 by the purity theorem of Zariski-Nagata (SGAA XVI 3.3).

Remark. It is a simple exercise to see that if $g : \eta \to \mathrm{X}$ is the inclusion of the generic point to a normal integral scheme X, and M_η is a constant sheaf on η with constant value M, then $g_*\mathrm{M}$ is likewise constant. Let M_X denote the constant sheaf on X with constant value M. We need to show

$$\mathrm{M_X} \to g_* g^* \mathrm{M_X} = g_* \mathrm{M}_\eta. \tag{†}$$

is an isomorphism. Fix a geometric point x of X and let $\mathrm{X}_{(x)}$ denote the (spectrum of the) strict henselization of X at x; $\mathrm{X}_{(x)}$ is a normal domain as X is normal. The map on stalks is

$$\mathrm{M} \to \Gamma(\eta \times_{\mathrm{X}} \mathrm{X}_{(x)}, \mathrm{M}_\eta).$$

$\eta_x := \eta \times_{\mathrm{X}} \mathrm{X}_{(x)}$ is the spectrum of the field of fractions of the strict henselization of X at x, which is separable algebraic over $k(\eta)$. Letting $\mathrm{I}_x := \mathrm{Gal}(\overline{\eta}/\eta_x)$,

$$\Gamma(\eta \times_{\mathrm{X}} \mathrm{X}_{(x)}, \mathrm{M}_\eta) = \mathrm{M}^{\mathrm{I}_x},$$

where here we have identified M with the stalk of M_η at a geometric point centered on η. So the condition that the map (†) be an isomorphism at the geometric point x is the same as I_x acting trivially on M, considered as $\mathrm{Gal}(\overline{\eta}/\eta)$- module. In particular, this condition is satisfied by the constant sheaf, which corresponds to M with trivial action of Galois.

(We implicitly use here the description of étale morphism with normal integral target in EGA IV_4 18.10.7.)

4.4. The short exact sequence in the second paragraph is the exact sequence of the first paragraph of (2.3.2) in light of the fact that $j'_*\mathrm{F}' = j'_{!*}\mathrm{F}'$ is concentrated in degree -1. The computation

$$\mathscr{H}^{-1}(\mathscr{F}(j_*\mathrm{F}[1])_{\overline{0}'}) = \mathrm{H}^1_c(\mathrm{A} \otimes_{\mathbf{F}_q} \overline{k}, j_*\mathrm{F})$$

follows from proper base change along the closed immersion $\{0'\} \hookrightarrow \mathrm{A}'$ applied to either (1.4.1.1), yielding

$$\mathscr{H}^{-1}(\mathscr{F}(j_*\mathrm{F}[1])_{\overline{0}'}) = \mathrm{H}^1(\alpha_! j_*\mathrm{F} \otimes \overline{\mathscr{L}}(x.0)) = \mathrm{H}^1_c(\mathrm{A} \times_{\mathbf{F}_q} \overline{0}, j_*\mathrm{F})$$

or simply from (1.3.1.1) directly, yielding

$$\begin{aligned}\mathscr{H}^{-1}(\mathscr{F}(j_*\mathrm{F}[1])_{\overline{0}'}) &= \mathscr{H}^{-1}(\mathrm{R}\,\mathrm{pr}'_!(\mathrm{pr}^*\, j_*\mathrm{F}[1] \otimes \mathscr{L}(x.x'))_{\overline{0}'}[1]) \\ &= \mathrm{H}^1_c(\mathrm{A} \times_{\mathbf{F}_q} \overline{0}, j_*\mathrm{F} \otimes \mathscr{L}(x.0)) = \mathrm{H}^1_c(\mathrm{A} \times_{\mathbf{F}_q} \overline{k}, j_*\mathrm{F}).\end{aligned}$$

In any event, this $G_{0'}$-module coincides with the stalk of j'_*F' at $\overline{0}'$, which is what is expressed by the equation

$$(F'_{\overline{\eta}_{0'}})^{I_{0'}} = H^1_c(A \times_{\mathbf{F}_q} \overline{k}, j_*F).$$

For the rightmost term of the short exact sequence, $I_{0'}$ acts trivially on $F_{\overline{\infty}}$ as F is by assumption unramified at infinity. let $\pi : \operatorname{Spec} k\{\pi\} := T \to D$ and $\pi' : \operatorname{Spec} k\{\pi'\} := T' \to A_{(0')}$ be defined by $\pi \mapsto 1/x$ and $\pi' \mapsto x'$. We have

$$\begin{aligned} R^{-1}\Phi_{\overline{\eta}_{0'}}(j'_*F') &= R^{-1}\Phi_{\overline{\eta}_{0'}}(\overline{\mathrm{pr}}^*(\alpha_! j_*F) \otimes \overline{\mathscr{L}}(x.x')[1])_{(\overline{\infty},\overline{0}')} && (2.3.2.1)\ (iii)\\ &= (\pi \times \pi')^* R^{-1}\Phi_{\overline{\eta}_{0'}}(\overline{\mathrm{pr}}^*(\alpha_! j_*F) \otimes \overline{\mathscr{L}}(x.x')[1]) \\ &= R^1\Phi_{\overline{\eta}_{0'}}(\mathrm{pr}^*(F_{\overline{\infty}!}) \otimes \mathscr{L}(\pi'/\pi))_{(\bar{t},\bar{t}')} && (2.4.2.1)\ (ii)\\ &= \mathscr{F}^{(\infty,0')}(F_{\overline{\infty}}) && (2.4.2.3)\\ &= F_{\overline{\infty}} \otimes \mathscr{F}^{(\infty,0')}(\overline{\mathbf{Q}}_\ell) = F_{\overline{\infty}}(-1). && (2.5.3.1)\ (i) \end{aligned}$$

The second-to-last equality is true because F is assumed unramified at infinity, so that the action of $G_\infty := \mathrm{Gal}(\overline{\eta}_\infty/\eta_\infty)$ on $F_{\overline{\infty}}$ factors through $G_\infty \twoheadrightarrow G_\infty/I_\infty = \mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$, which is procyclic generated by Frobenius, and by Schur's lemma $F_{\overline{\infty}}$ splits as a direct sum of 1-dimensional torsion sheaves (in the sense of [**Weil II**, 1.2.7]) so that we may assume $F_{\overline{\infty}} \simeq \overline{\mathbf{Q}}_\ell^{(b)}$ for $b \in \overline{\mathbf{Q}}_\ell^\times$ an ℓ-adic unit, which is the reciprocal image of a sheaf on $\operatorname{Spec} \mathbf{F}_p$. As twisting by such a sheaf induces an exact autoequivalence of the category (or derived category) of $\overline{\mathbf{Q}}_\ell$-sheaves, we find that $\overline{\mathbf{Q}}_\ell^{(b)} \otimes \mathscr{F}^{(\infty,0')}(V) = \mathscr{F}^{(\infty,0')}(\overline{\mathbf{Q}}_\ell^{(b)} \otimes V)$ for any G_∞-module V.

A few words on the application of the Deligne lemmata: it is used here that F is not ramified at infinity, so that if α denotes simultaneously $A \hookrightarrow D$ and $U \hookrightarrow U \cup \{\infty\}$ (depending on the context), α_*F is a lisse sheaf on $U \cup \{\infty\}$. This is not necessary to conclude by (4.2.2.1) that for each eigenvalue of Frobenius β on $F_{\overline{\infty}}$,

$$w_q(\beta) \leq w,$$

but it is necessary to connect the corresponding statement for $F^\vee$ to that for F in the following way. Namely, we find that for each eigenvalue of Frobenius γ on $(F^\vee)_{\overline{\infty}}$,

$$w_q(\gamma) \leq -w,$$

and as $\alpha_* F$ and $\alpha_* F^\vee$ are both lisse, $(\alpha_* F)^\vee = \alpha_*(F^\vee)$ and

$$(F^\vee)_{\overline{\infty}} := (\alpha_* F^\vee)_{\overline{\infty}} = (\alpha_* \underline{\mathrm{Hom}}(F, \overline{\mathbf{Q}}_\ell))_{\overline{\infty}} = \underline{\mathrm{Hom}}(\alpha_* F, \overline{\mathbf{Q}}_\ell)_{\overline{\infty}} = \mathrm{Hom}(F_{\overline{\infty}}, \overline{\mathbf{Q}}_\ell) = (F_{\overline{\infty}})^\vee,$$

which allows us to conclude from $w_{N(\infty)}(\gamma) \leq -w$ that

$$w_q(\beta) \geq w.$$

This shows in our application that F' is ι-pure of ι-weight w and (4.2.2.1) gives that for each eigenvalue ν of $F_{0'}$ on $(j'_* F')_{\overline{0}'} = H^1_c(A \times_{\mathbf{F}_q} \overline{k}, j_* F)$,

$$w_q(\nu) \leq w + 1.$$

In the effort to show that F' is ι-pure, since F is unramified at infinity, the P_∞-module $(F \oplus \check{F}(-w))_{\overline{\eta}_\infty}$ is trivial; i.e. purely of slope 0, so that $\mathscr{F}_\psi(j_*(F \oplus \check{F}(-w))[1])$ is of the form $j'_* G'_1[1]$ and $\mathscr{F}_{\psi^{-1}}(j_*(F \oplus \check{F}(-w))[1])$ is of the form $j'_* G'_2[1]$ and their direct sum is of the form $j'_* G'[1]$ for the lisse $\mathbf{Q}_\ell$-sheaf $G' := G'_1 \oplus G'_2$ on $A' - \{0\}$.

For the *coup de grâce*, it would be better to refer to (1.1.1), especially (1.1.1.3), and (1.2.1.2) than to (3.1). The point is that we can follow *Rapport sur la formula des traces* (1.6) and (3.1) (with t^f replaced by t so that degree is measured over $\mathbf{F}_q$, not $\mathbf{F}_p$, and shifting by $[-1]$ to cancel the shift of $[1]$ on G and in the definition of $\mathscr{F}$) to find that

$$\begin{aligned} &\det(1 - t\, F^*_{x'}, G') \\ &= \det(1 - t\, F^*_{x'}, H^0_c(A \times_{\mathbf{F}_q} \overline{x}', j_*(F \oplus \check{F}(-w)) \otimes \mathscr{L}_\psi(x.x')[1])) \\ &\quad \times \det(1 - t\, F^*_{x'}, H^0_c(A \times_{\mathbf{F}_q} \overline{x}', j_*(F \oplus \check{F}(-w)) \otimes \mathscr{L}_{\psi^{-1}}(x.x')[1])) \end{aligned}$$

Now the point is that as G' is concentrated in degree zero, letting $\mathscr{L} = \mathscr{L}_\psi$ or $\mathscr{L}_{\psi^{-1}}$,

$$R\Gamma(A \times_{\mathbf{F}_q} \overline{x}', j_*(F \oplus \check{F}(-w)) \otimes \mathscr{L}(x.x')) = H^1_c(A \times_{\mathbf{F}_q} \overline{x}', j_*(F \oplus \check{F}(-w)) \otimes \mathscr{L}(x.x'))[-1].$$

Grothendieck's trace formula gives therefore that $\iota \det(1 - t\, F^*_{x'}, G')$ can be computed as stated in terms of the polynomials $Q_{x,x'}$ in light of [**Sommes trig.**, 1.7.6 & 1.7.7] (compare (1.1.1.2) and (1.1.3.3)), which allow us to write for each $x \in |A \times_k x'|$

$$F^*_x | (j_*(F \oplus \check{F}(-w)) \otimes \mathscr{L}_\psi(x.x'))_{\overline{x}} = \psi(\mathrm{Tr}_{k(x)/\mathbf{F}_p}(x.x'))\ F^*_x | (j_*(F \oplus \check{F}(-w)))_{\overline{x}}.$$

If $P(t) \in \mathbf{R}[t]$ and $\alpha = e^{i\theta}$, then $P(\alpha t) P(\alpha^{-1} t) \in \mathbf{R}[t]$ as well, since $P(\alpha t) P(\overline{\alpha} t)$ is fixed by complex conjugation. Q.E.D.

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7. *Théorèmes de finitude en cohomologie ℓ-adique*

1.3. Let's see why, under the hypotheses of 1.1, Rf_* is of finite cohomological dimension. Localizing, we are in the situation of $f : X \to Y$ a morphism of schemes of finite type over S the spectrum of a field or a dvr. Let $y \to Y$ be a geometric point (which we may assume maps to the closed point of S) we wish to show that $R\Gamma(X \times_Y Y_{(y)}, -)$ is of finite ℓ-cohomological dimension, where $Y_{(y)}$ denotes the strict localization of Y at y. We use the criterion of [**SGAA**, X 4.2] (as $X \times_Y Y_{(y)} \to Y_{(y)}$ is of finite type, the former is noetherian). To show that cd_ℓ of the residue fields of the points of $X \times_Y Y_{(y)}$ is bounded, it suffices by [**SGAA**, X 2.1] to show the same for the residue fields of points of $Y_{(y)}$. Let $a \in Y_y$ with image $b \in Y$, and let Z denote the closure of the image of b in Y; y is a geometric point of Z and $Y_{(y)} \times_Y Z$ is the strict localization of Z at y (`05WR` & `04GH`). As $Y_{(y)} \times_Y b$ is the spectrum of the ring of rational functions R on $Y_{(y)} \times_Y Z$, which is the product of the residue fields of the generic points of the finitely many irreducible components of $Y_{(y)} \times_Y Z$ (`07QQ`), $\mathrm{cd}_\ell(R) = \dim Y_{(y)} \times_Y Z \leq \dim Y$ [**SGAA**, X 3.2] (`06LK`). In particular, a is the generic point of an irreducible component of $Y_{(y)} \times_Y Z$, so $\mathrm{cd}_\ell(k(a)) \leq \dim Y$.

To establish (ii) of the criterion, we need that for every ring A which is the strict localization of an irreducible closed subscheme W of $X \times_Y Y_{(y)}$ at a geometric point $w \to W$, $\mathrm{cd}_\ell R(A) \leq \dim A$. Let η denote the generic point of W and replace X and Y with the respective closures of the images of η. As the strict localization of W at a geometric point $w \to W$ can be obtained as a localization of $W \times_Y Y_{(w)}$, we may assume w maps via $W \to Y$ to y. The strict localization of $X \times_Y Y_{(y)}$ at w therefore coincides with $X_{(w)}$. Applying `0AH1` to $X_{(w)} \to X \times_Y Y_{(y)}$ and $X \times_Y Y_{(y)} \to X$, as both are cofiltered limits of étale maps, we find that both maps have discrete, reduced fibers, and η is therefore isolated in the generic fiber of the latter map. It follows that if $X_{(w)}$ is equidimensional, $\dim X_{(w)} = \dim W_{(w)}$, as $W_{(w)}$ is obtained from $X_{(w)}$ by throwing away some irreducible components. By the same logic, if $X_{(w)} = \mathrm{Spec}(C)$, $\mathrm{cd}_\ell R(A) \leq \mathrm{cd}_\ell R(C)$, and one concludes as $\mathrm{cd}_\ell R(C) = \dim X_{(w)}$ by [**SGAA**, X 3.2].

It remains to show that $X_{(w)}$ is equidimensional. Let $(B, \mathfrak{m})$ denote the Zariski local ring of X at the point on which w is centered. Then $B \to C$ is faithfully flat (`07QM`), so satisfies going down. Therefore all the irreducible components of $X_{(w)}$ dominate X, and it will suffice to show that given one of them, $X' = \operatorname{Spec} C'$, $\dim X' = \dim B$. If $\mathfrak{p} \subset C$ is the minimal prime so that $C' = C/\mathfrak{p}$, $\mathfrak{p}$ is finitely generated as C is noetherian, so that there is an étale local ring map $(B, \mathfrak{m}) \to (B', \mathfrak{m}')$ so that the generators of $\mathfrak{p}$ belong to B'. Let $\mathfrak{p}'$ be the restriction of $\mathfrak{p}$ via $B' \to C$; then by construction $\mathfrak{p}'C = \mathfrak{p}$ and $C/\mathfrak{p} = C'$ is the strict henselization of $B'/\mathfrak{p}'$ (`05WS`), so $\dim C' = \dim B'/\mathfrak{p}'$ (`06LK`). As $\mathfrak{p}$ restricts to $(0) \subset B$, so does $\mathfrak{p}'$. As X is of finite type over S, which is the spectrum of a field or of a dvr, X is universally catenary; as any localization of a universally catenary ring is universally catenary, B is universally catenary, so we have the formula (`02IJ`)

$$\dim B'/\mathfrak{p}' = \dim B + \operatorname{trdeg}_B(B'/\mathfrak{p}') - \operatorname{trdeg}_{k(\mathfrak{m})} k(\mathfrak{m}').$$

(Recall that an étale local ring map of noetherian local rings is by definition essentially of finite type (`0258`)). As $B \to B'$ is étale, both residual extensions are finite, and we have shown $\dim C' = \dim B'/\mathfrak{p}' = \dim B$, as promised.

1.6. Given a morphism of schemes $f : X \to Y$ and sheaves $\mathscr{F}, \mathscr{G}$ on Y, there is always a morphism

$$f^*\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) \longrightarrow \underline{\mathrm{Hom}}(f^*\mathscr{F}, f^*\mathscr{G}).$$

When $\mathscr{F}$ is locally constant constructible, this map is a bijection.

If $\mathscr{I}$ is an injective abelian sheaf on X and $j : U \to X$ belongs to the topology on X, then $\mathscr{I}(U)$ is an injective object, since if $\underline{A}$ is the constant sheaf on U with value A,

$$\mathrm{Hom}(A, \mathscr{I}(U)) = \mathrm{Hom}(j_!\underline{A}, \mathscr{I}),$$

and $j_!$ is exact. The restriction $\mathscr{I}_U$ is also an injective sheaf, since for $\mathscr{F}$ on U

$$\mathrm{Hom}(\mathscr{F}, \mathscr{I}_U) = \mathrm{Hom}(j_!\mathscr{F}, \mathscr{I}).$$

Likewise, for any (geometric) point $x \in X$, $\mathscr{I}_x$ is an injective object, from the general fact that a inductive limit of injective modules over a noetherian ring is injective: this follows from Baer's criterion in an obvious way and holds not only for injective abelian sheaves but also for injective sheaves of Λ-modules for Λ a noetherian ring.

All this to say that $\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})|_{\mathrm{U}} = \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}|_{\mathrm{U}},\mathcal{G}|_{\mathrm{U}})$, and when $\mathcal{F}$ is l.c.c., $\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})|_x = \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}|_x,\mathcal{G}|_x)$, and $\mathrm{R}\,\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})$ is constructible (resp. l.c.c.) when $\mathcal{G}$ is. This also shows that for $\mathcal{F}$ constant, one can compute $\mathrm{R}\,\underline{\mathrm{Hom}}$ by a projective resolution of the constant value of $\mathcal{F}$.

1.11. $\mathrm{R}\,\mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathrm{K} = b^*\mathrm{R}\Gamma(\mathrm{X},\mathrm{K})$ is base change for $\mathrm{R}a_*$ along b. In the string of equalities, the first one is just Leray, and the second and last are again base change morphisms, discussed at length in SGAA XVII §4. Given

$$\begin{array}{ccc} \mathrm{X}' & \xrightarrow{g'} & \mathrm{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{S}' & \xrightarrow{g} & \mathrm{S}, \end{array}$$

at the level of complexes, the map $\mathrm{L}\otimes g^*f_*\mathrm{K} \to f'_*(f'^*\mathrm{L}\otimes g'^*\mathrm{K})$ factors as

$$\mathrm{L}\otimes g^*f_*\mathrm{K} \to \mathrm{L}\otimes f'_*g'^*\mathrm{K} \to f'_*f'^*\mathrm{L}\otimes f'_*g'^*\mathrm{K} \to f'_*(f'^*\mathrm{L}\otimes g'^*\mathrm{K}),$$

where the last arrow comes from the fact that $\mathscr{F}(\mathrm{U})\otimes\mathscr{G}(\mathrm{U})$ are among the sections of $\mathscr{F}\otimes\mathscr{G}$ over U.

2.1. It is claimed that $\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F},\mathbf{Z}/m) \xleftarrow{\sim} \underline{\mathrm{Hom}}(\mathcal{F},\mathbf{Z}/m)$ when $\mathcal{F}$ is a l.c.c. sheaf (c.f. SGAA XVIII 3.2.6). There is evidently an arrow, and by the note to 1.6 (and since $\mathbf{Z}/m$ is an injective $\mathbf{Z}/m$-module), it is an isomorphism. It holds more generally with $\mathbf{Z}/m$ replaced by any l.c.c. sheaf locally isomorphic to $\mathbf{Z}/m$, such as one obtained by twisting.

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8. *La classe de cohomologie associée à un cycle*

1.1.1. The reference is Giraud, *Cohomologie non abélienne* III §3, in particular §3.5 for the abelian case.

1.1.4. '$H^1_D(X, F)$ classifies the F-torsors on X, trivialized on U' & 'Pour toute section $f \in H^0(U, F)$, $\partial f \in H^1_D(X, F)$ est la classe du torseur trivial F, trivialisé sur U par la section f' $\rightsquigarrow$ implicitly, two trivializations over U of a torsor on X are considered the same if their difference extends to a trivialization of the torsor on all of X.

In concrete terms, therefore, two sections $f, g \in H^0_D(X, H)$ give the same class in $H^1_D(X, F)$ if their difference $f - g \in H^0_D(X, H)$ has the property that the trivialization of the associated torsor $\beta^{-1}(f - g)$ provided by the zero section (which $f - g$ restricts to over U) extends to a global trivialization of $\beta^{-1}(f - g)$; this is true iff $f - g$ is in the image of some section of $H^0_D(X, G)$.

In the second construction, the trivial torsor F with a trivialization f over U represents the class $0 \in H^1_D(X, F)$ iff f extends to a global section in $H^0(X, F)$.

1.1.5. Let α, β denote the nonzero arrows of the exact sequence and $f \in H^0(U, F)$. The long exact sequence in $R\Gamma$ shows that the image of $\partial\beta(f)$ by the map $H^1_D(X, F) \to H^1(X, F)$ is zero, so $\partial\beta(f)$ indeed represents the trivial torsor $0 \in H^1(X, F)$. The map $\partial\beta : H^0(U, F) \to H^1_D(X, F)$ has kernel $H^0(X, F)$ since $H^1_D(X, j_*j^*F) = 0$. The class of $\partial\beta(f) \in H^1_D(X, F)$ is represented by $\beta^{-1}(\beta(f)) \subset j_*j^*F$ with the trivialization over U given by the zero section. Transform this torsor by $\beta^{-1}(\beta(f)) \mapsto \beta^{-1}(\beta(f)) - f$. The result is again a trivial torsor, this time with trivialization given by $-f$.

The distinction is that the torsor $\beta^{-1}(\beta(f))$ is trivialized over all of X by the global section $f \in H^0(X, \beta^{-1}(\beta(f))) \subset H^0(X, j_*j^*F)$ but we pick the trivialization given by the zero section over U, while the torsor $\beta^{-1}(\beta(f)) - f$ is trivialized over all of X by the zero section, but we pick the trivialization given by $-f$ over U. They represent the same class in $H^1_D(X, F)$ and show that the map $\partial\beta$ of (1.1.5) is indeed the opposite of the map given in the second paragraph of (1.1.4).

1.2.2. (1.2.2.1) induces (1.2.2.2) in the way discussed in the section on cup-products in the notes to Astérisque 100. In short: `068G`.

Suppose we stay in D^-. To obtain (1.2.2.3), take truncated flasque resolutions with flat stalks A and B of K and L as discussed in the note to 1.2.3 below. Then $A \otimes B$ computes $K \otimes^{\mathbf{L}} L$, i^*L is still adapted to tensor product, and we get a map

$$Ri^!K \otimes^{\mathbf{L}} i^*L = i^!A \otimes i^*B \to i^!(A \otimes B).$$

Take a K-injective or truncated flasque resolution $A \otimes B \to I$ to get a map

$$i^!(A \otimes B) \to i^!(I) = Ri^!(A \otimes B) = Ri^!(K \otimes L),$$

and hence (1.2.2.3). Now to deduce (1.2.2.1) from (1.2.2.3) by applying $R\Gamma(Y, -)$, follow the recipe of the note to 1.2.4. It starts by replacing $Ri^!K$ and i^*L by complexes C and D simultaneously adapted to tensor product and global sections, and writing the map $\Gamma(Y, C) \otimes \Gamma(Y, D) \to \Gamma(Y, C \otimes D)$ coming from the map (1.2.2.3), since $\Gamma(Y, C)$ and $\Gamma(Y, D)$ compute $R\Gamma(Y, Ri^!K)$ and $R\Gamma(Y, i^*L)$, respectively, and $C \otimes D = Ri^!K \otimes^{\mathbf{L}} i^*L$. Now take an injective resolution $C \otimes D \to I$, flat resolutions $P_1 \to \Gamma(Y, C)$ and $P_2 \to \Gamma(Y, D)$, and write

$$\begin{aligned} R\Gamma(Y, Ri^!K) \otimes^{\mathbf{L}} R\Gamma(Y, i^*L) = P_1 \otimes P_2 \to \Gamma(Y, C) \otimes \Gamma(Y, D) \\ \to \Gamma(Y, C \otimes D) \to \Gamma(Y, I) = R\Gamma(Y, Ri^!K \otimes^{\mathbf{L}} i^*L). \end{aligned}$$

1.2.3. Indeed, if the right derived functors are of finite cohomological dimension so that one can stay within D^-, [**SGAA**, XVII 4.2.10] shows that there are enough complexes which are adapted to both tensor product and direct image.

It is worth mentioning the work of Spaltenstein and Serpé which extend $\otimes^{\mathbf{L}}$ to $D \times D$ via K-flat resolutions (`06Y7`) and extend any right derived functor with no hypothesis of cohomological finiteness to all of the unbounded derived category using K-injective resolutions (`079I`, `070G`). However, this still doesn't get around the issue of producing complexes simultaneously adapted for both tensor product and a left-exact functor addressed above.

1.2.4. Let's assume we stay within D^-. The isomorphism $j_*\mathrm{F}\otimes j_!\mathrm{G} \xleftarrow{\sim} j_!(\mathrm{F}\otimes\mathrm{G})$ derives into an isomorphism $\mathrm{R}j_*\mathrm{F}\otimes^{\mathbf{L}} j_!\mathrm{G} \xleftarrow{\sim} j_!(\mathrm{F}\otimes^{\mathbf{L}}\mathrm{G})$, e.g. by taking a flat resolution of G and K-injective or truncated flasque resolution of F (or injective resolution if F is also in D^+). Once this isomorphism is achieved, one can apply $\mathrm{R}\Gamma(\mathrm{X},-)$ to get $\mathrm{R}\Gamma(\mathrm{X},\mathrm{R}j_*\mathrm{F}\otimes^{\mathbf{L}} j_!\mathrm{G}) \to \mathrm{R}\Gamma(\mathrm{X}, j_!\mathrm{M})$. Let A and B be complexes simultaneously adapted for tensor product and global sections representing $\mathrm{R}j_*\mathrm{F}$ and $j_!\mathrm{G}$, respectively. Then $\mathrm{R}j_*\mathrm{F}\otimes^{\mathbf{L}} j_!\mathrm{G} = \mathrm{A}\otimes\mathrm{B}$, $\Gamma(\mathrm{X},\mathrm{A}) = \mathrm{R}\Gamma(\mathrm{X},\mathrm{R}j_*\mathrm{F})$, $\Gamma(\mathrm{X},\mathrm{B}) = \mathrm{R}\Gamma(\mathrm{X}, j_!\mathrm{G})$, and we have an evident map $\Gamma(\mathrm{X},\mathrm{A})\otimes\Gamma(\mathrm{X},\mathrm{B}) \to \Gamma(\mathrm{X},\mathrm{A}\otimes\mathrm{B})$. Take any K-injective or truncated flasque resolution $\mathrm{A}\otimes\mathrm{B}\to\mathrm{I}$ (or injective if $\mathrm{A}\otimes\mathrm{B}$ is in D^+) and we have a map

$$\Gamma(\mathrm{X},\mathrm{A}\otimes\mathrm{B}) \to \Gamma(\mathrm{X},\mathrm{I}) = \mathrm{R}\Gamma(\mathrm{X},\mathrm{R}j_*\mathrm{F}\otimes^{\mathbf{L}} j_!\mathrm{G}) \to \mathrm{R}\Gamma(\mathrm{X}, j_!\mathrm{M}).$$

Take flat resolutions of $\mathrm{P}_1\to\Gamma(\mathrm{X},\mathrm{A})$ and $\mathrm{P}_2\to\Gamma(\mathrm{X},\mathrm{B})$ to get a map

$$\mathrm{R}\Gamma(\mathrm{X},\mathrm{R}j_*\mathrm{F})\otimes^{\mathbf{L}}\mathrm{R}\Gamma(\mathrm{X}, j_!\mathrm{G}) = \Gamma(\mathrm{X},\mathrm{A})\otimes^{\mathbf{L}}\Gamma(\mathrm{X},\mathrm{B}) = \mathrm{P}_1\otimes\mathrm{P}_2 \to \Gamma(\mathrm{X},\mathrm{A})\otimes\Gamma(\mathrm{X},\mathrm{B}).$$

Putting these maps together, we find the promised map

$$\mathrm{R}\Gamma(\mathrm{X},\mathrm{R}j_*\mathrm{F})\otimes^{\mathbf{L}}\mathrm{R}\Gamma(\mathrm{X}, j_!\mathrm{G}) \to \Gamma(\mathrm{X},\mathrm{A})\otimes\Gamma(\mathrm{X},\mathrm{B}) \to \Gamma(\mathrm{X},\mathrm{A}\otimes\mathrm{B}) \to \mathrm{R}\Gamma(\mathrm{X}, j_!\mathrm{M}).$$

1.2.5. The local product on Y referred to at the end is of course

$$\mathrm{R}i^!\mathrm{K}\otimes^{\mathbf{L}} i^*\mathrm{L} \to \mathrm{R}i^!(\mathrm{K}\otimes^{\mathbf{L}}\mathrm{L}) \to \mathrm{R}i^!\mathrm{M}.$$

1.3.6. A morphism f is homogeneous of degree d if f sends the homogeneous piece of degree n to the homogeneous piece of degree $n+d$. Suppose f and g homogeneous of degrees d and e respectively, so that we can represent fg as a linear combination of elements of the form $w_f\otimes v_f^\vee\otimes v_g\otimes w_g^\vee$ where the degrees of w_f, v_f, w_g, v_g are $n+d, n, m+e, m$ (so that $v_f^\vee$ and $w_g^\vee$ have degree $-n$ and $-m-e$, respectively, but the grading is $\mathbf{Z}/2\mathbf{Z}$ so it doesn't matter). To move the above element into the form $\mathrm{V}^\vee\otimes\mathrm{V}\otimes\mathrm{W}^\vee\otimes\mathrm{W}$, the corresponding $\mathrm{N} = (n+d)n+(n+d)(m+e)+(n+d)m \mod 2$.

Now gf can be represented as a linear combination of elements of the form $v_g\otimes w_g^\vee\otimes w_f\otimes v_f^\vee$ with the same degrees as above. To move into the form $\mathrm{V}^\vee\otimes\mathrm{V}\otimes\mathrm{W}^\vee\otimes\mathrm{W}$ the corresponding $\mathrm{N} = (n+d)n+n(m+e)+nm \mod 2$. The sum of these two N is $de = \deg f \deg g \mod 2$.

2.1.3. The anticommutativity is a consequence of [**BBD**, 1.1.11] in light of the commutativity of the diagram below where each row and column is distinguished.

$$
\begin{tikzcd}
{} & {} & {} \\
\Gamma(\mathrm{X}-\mathrm{D}, \mu_n) \arrow[r] \arrow[u] & \Gamma(\mathrm{X}-\mathrm{D}, \mathbf{G}_m) \arrow[r] \arrow[u] & \Gamma(\mathrm{X}-\mathrm{D}, \mathbf{G}_m) \arrow[r] \arrow[u] & {} \\
\Gamma(\mathrm{X}, \mu_n) \arrow[r] \arrow[u] & \Gamma(\mathrm{X}, \mathbf{G}_m) \arrow[r] \arrow[u] & \Gamma(\mathrm{X}, \mathbf{G}_m) \arrow[r] \arrow[u] & {} \\
\Gamma_{\mathrm{D}}(\mathrm{X}, \mu_n) \arrow[r] \arrow[u] & \Gamma_{\mathrm{D}}(\mathrm{X}, \mathbf{G}_m) \arrow[r] \arrow[u] & \Gamma_{\mathrm{D}}(\mathrm{X}, \mathbf{G}_m) \arrow[r] \arrow[u] & {}
\end{tikzcd}
$$

2.1.4. By (2.1.3) it suffices to see that the μ_n-torsor of n^{th} roots of unity generates $\mathrm{H}^1(\mathrm{X}-\mathrm{D}, \mu_n)$. The argument is contained in [**SGAA**, XVI 3.6]. The fact that t is a regular parameter and D is regular implies that the Kummer covering $\mathrm{U}' := \operatorname{Spec} \mathcal{O}_{\mathrm{X}-\mathrm{D}}[t^{1/n}] \to \mathrm{X}-\mathrm{D}$ is a principal Galois covering of group μ_n. As X is strictly local, $\mathrm{X}' := \operatorname{Spec} \mathcal{O}_{\mathrm{X}}[t^{1/n}]$ is too (it is regular by [**SGA1**, XIII 5.1]), so any étale covering of U′ that extends to an étale covering of X′ is trivial. Abhyankar's lemma [**SGA1**, XIII 5.2] implies that every principal Galois covering of $\mathrm{U} := \mathrm{X}-\mathrm{D}$ with group μ_n has the property that its reciprocal image on U′ extends to an étale covering of X′, hence is trivial. We know that $\mathrm{H}^1(\mathrm{U}, \mu_n)$ classifies μ_n-torsors on U, i.e. principal Galois covers of U of group μ_n, and that these are classified by $\operatorname{Hom}(\pi_1(\mathrm{U}, u), \mu_n)$ for a geometric point u of U. As Abhyankar tells us any such homomorphism must factor through the quotient $\pi_1(\mathrm{U}, u) \twoheadrightarrow \mu_n$, the class of the μ_n-torsor U′ generates $\mathrm{H}^1(\mathrm{U}, \mu_n) = \operatorname{Hom}(\mu_n, \mu_n) = \mathbf{Z}/n\mathbf{Z}$, as U′ corresponds to $\mathrm{id} : \mu_n \to \mu_n$.

2.1.5. Recall that the trace map for a curve over an algebraically closed field of characteristic exponent p not dividing n and with projective completion $\overline{\mathrm{X}}$ is defined [**SGAA**, XVIII 1.1.3] via the isomorphism

$$\mathrm{H}^2_c(\mathrm{X}, \mu_n) \xrightarrow{\sim} \mathrm{H}^2(\overline{\mathrm{X}}, \mu_n) = \operatorname{Pic}(\overline{\mathrm{X}})/n = (\mathbf{Z}/n)^c$$

where c is the number of connected components of $\overline{\mathrm{X}}$. The trace map (when X is reduced) is the composition of these isomorphisms with the sum map $(\mathbf{Z}/n)^c \to \mathbf{Z}/n$.

As the image of $c\ell(\mathrm{P})$ in $\mathrm{H}^2(\overline{\mathrm{X}}, \mu_n)$ is the same as that of the line bundle $\mathscr{O}(\mathrm{P})$, as element of $\mathrm{Pic}(\overline{\mathrm{X}})$, and this line bundle has degree 1, one concludes.

(The morphism $\mathrm{R}\Gamma_{\mathrm{P}} \to \mathrm{R}\Gamma_!$: if $j : \mathrm{X} \hookrightarrow \overline{\mathrm{X}}$ and $i : \mathrm{P} \hookrightarrow \mathrm{X}$ or $\overline{\mathrm{X}}$, and $\mathscr{F}$ is a sheaf on X, then $i^!\mathscr{F} = i^! j_!\mathscr{F}$, so

$$\Gamma_{\mathrm{P}}(\mathrm{X}, \mu_n) := \Gamma(\mathrm{P}, i^!\mathscr{F}) = \Gamma(\mathrm{P}, i^! j_!\mathscr{F}) =: \Gamma_{\mathrm{P}}(\overline{\mathrm{X}}, j_!\mathscr{F}) \to \Gamma(\overline{\mathrm{X}}, j_!\mathscr{F}) =: \Gamma_!(\mathrm{X}, \mathscr{F}).$$

2.2.1. Recall that the Koszul complex of a regular sequence of length c generating the ideal $\mathfrak{a}$ provides a projective resolution of length c of $\mathrm{A}/\mathfrak{a}$. It should read $i > c$.

$\dim(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) = c \rightsquigarrow$ by Nakayama's lemma, this amounts to showing that the c equations generating $\mathfrak{a}$ constitute a minimal generating set. As these c equations $x_1, \dots, x_c$ form a regular sequence, and any regular sequence in A can be extended to a sequence of parameters for A [**S**, IV-16 Prop. 7], so for $1 \leq k \leq c$, $\dim \mathrm{A}/(x_1, \dots, x_k) = \dim \mathrm{A} - k$ [**S**, III-11 Prop. 6], $x_1, \dots, x_c$ form a minimal generating set for $\mathfrak{a}$.

The last statement is that any sequence of elements $y_1, \dots, y_c$ of $\mathfrak{a}$ whose image in $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$ is a basis, is a regular sequence generating $\mathfrak{a}$. The 'generating' part is Nakayama's lemma, which also tells us that $y_1, \dots, y_c$ is a minimal generating set for $\mathfrak{a}$. That $y_1, \dots, y_c$ is a regular sequence in A follows, e.g. from Kaplansky, *Commutative Rings* Theorem 129, taking $\mathrm{I} = \mathfrak{a} = (y_1, \dots, y_c)$, $\mathrm{R} = \mathrm{A}$, and $\mathrm{A} = \mathrm{A}$ there. (In his notation, $\mathrm{G}(\mathrm{I}, \mathrm{A}) = c$, the common length of all maximal A-regular sequences contained in the ideal $\mathrm{I} = \mathfrak{a}$.) In light of the above, Kaplansky's theorem implies the easier-to-remember

Proposition. — An ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence.

For, as we have seen, any minimal set of generators for the ideal $\mathfrak{a}$ has cardinality equal to the dimension of $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$ as an $\mathrm{A}/\mathfrak{m}$-vector space.

2.2.2. c) Digression on restricting cohomology classes and $c\ell$ in particular, let

$$\begin{tikzcd} \mathrm{D}_1 \cap \mathrm{D}_2 \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"] & \mathrm{D}_2 \arrow[d, hook, "i_2"] \\ \mathrm{D}_1 \arrow[r, hook, "i_1"] & \mathrm{X} \end{tikzcd} \qquad \begin{tikzcd} u^{-1}\mathrm{D} \arrow[r, hook, "i"] \arrow[d, "u"] & \mathrm{X}' \arrow[d, "u"] \\ \mathrm{D} \arrow[r, hook, "i"] & \mathrm{X} \end{tikzcd}$$

set notation. There is a morphism of restriction $\Gamma_{D_1}(X, F) \to \Gamma_{D_1 \cap D_2}(D_2, F)$ obtained as

$$\Gamma_{D_1}(X, F) = \Gamma(X, i_{1*}i_1^! F) \to \Gamma(X, i_{1*}i_1^! i_{2*}i_2^* F) = \Gamma(X, i_{2*}i_{1*}i_1^! i_2^* F) = \Gamma_{D_1 \cap D_2}(D_2, F).$$

As for $c\ell$, well given $u : X' \to X$ as in (2.1.1), we can pull back the torsor and choice of trivialization, which corresponds to the following on H^1_D:

$$\begin{aligned}\Gamma_D(X, \mathbf{G}_m) \to \Gamma_D(X, u_*u^*\mathbf{G}_m) \to \Gamma_D(X, u_*\mathbf{G}_m) &= \Gamma(D, i^! u_*\mathbf{G}_m) \\ = \Gamma(D, u_*i^!\mathbf{G}_m) = \Gamma(u^{-1}D, i^!\mathbf{G}_m) &= \Gamma_{u^{-1}D}(X', \mathbf{G}_m).\end{aligned}$$

Of course we can do the same for $\mathbf{G}_m$ replaced by μ_n and the square

$$\begin{tikzcd} H^i_D(X, \mathbf{G}_m) \arrow[r] \arrow[d, "u^*"] & H^{i+1}_D(X, \mu_n) \arrow[d, "u^*"] \\ H^i_{u^{-1}D}(X', \mathbf{G}_m) \arrow[r] & H^{i+1}_{u^{-1}D}(X', \mu_n) \end{tikzcd}$$

commutes, showing that provided u^*D is again a Cartier divisor, $c\ell_n(u^*D) = u^*c\ell_n(D)$.

One would like to define a product of the type

$$H^*_{D_1}(X) \otimes H^*_{D_1 \cap D_2}(D_1) \otimes \ldots \otimes H^*_Y(D_1 \cap \ldots \cap D_{c-1}) \longrightarrow H^*_Y(X).$$

naïvely following the prescription of (1.2.1), and everything starts out all right: it is easy to define a map

$$\Gamma_{D_1}(X, F) \otimes \Gamma_{D_2}(D_1, G) \to \Gamma_{D_2}(X, F \otimes G)$$

as in (1.2.1), but I have no idea how to derive this map in general.

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9. *The derived category of perverse sheaves*

1.1. The condition that k be algebraically closed when $\mathrm{D}(\mathrm{X}) = \mathrm{D}^b_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$ is at first to secure [**BBD**, 2.2.14], and later to secure the existence of good neighborhoods.

2.1. The category $\mathrm{D}(\eta)$ coincides with the Verdier quotient of $\mathrm{D}(\mathrm{X})$ by the thick subcategory generated by the complexes acyclic at η.

2.1.1. In $\dim \mathrm{X} = 0$, the statement is clear as the abelian category of $\overline{\mathbf{Q}}_\ell$-sheaves on such an X is semisimple.

To find the neighborhood U, we proceed as follows: first shrink U so that $\mathrm{M}_\mathrm{U}, \mathrm{N}_\mathrm{U}$ are lisse, and U is integral and contained in the smooth locus (X is reduced by assumption). By Artin's good neighborhood theorem (k is algebraically closed), we find a Z smooth over k and a k-morphism $f : \mathrm{U} \to \mathrm{Z}$ with geometrically irreducible fibers of dimension 1 (f is smooth of relative dimension 1 in the sense of Hartshorne, recalling EGA IV_4 17.8.2 or SGA 1 II 2.1). In particular, Z is regular and, as $f(\mathrm{U}) = \mathrm{Z}$, irreducible hence integral. Replace Z by an affine open Spec A contained in it and U by an affine open Spec B contained in the preimage; the only thing that changes is that some fibers of f may now be empty; if a fiber is nonempty, it is still geometrically irreducible of dimension 1. As f is still smooth, hence in particular open, there is some $a \in \mathrm{A}$ with the property that $\mathrm{D}(a) \subset f(\mathrm{U}) \subset \mathrm{Z}$ and that the restriction of $\mathrm{R}^q f_* \underline{\mathrm{Hom}}(\mathrm{M}_u, \mathrm{N}_\mathrm{U})$ to $\mathrm{D}(a)$ is lisse for all q. As f is dominant, the induced ring homomorphism $\mathrm{A} \to \mathrm{B}$ is injective, the map of spectra ¶ : $\operatorname{Spec} \mathrm{B}_a \to \operatorname{Spec} \mathrm{A}_a$ is surjective, and the R^q¶$_*\underline{\mathrm{Hom}}(\mathrm{M}_u, \mathrm{N}_\mathrm{U})$ are lisse. Set $\mathrm{U} = \operatorname{Spec} \mathrm{B}_a$, $\mathrm{Z} = \operatorname{Spec} \mathrm{A}_a$: ¶ : $\mathrm{U} \to \mathrm{Z}$ now has all the desired properties; it is smooth, affine, with 1-dimensional fibers (which are moreover geometrically irreducible), Z is smooth over k hence regular, and the L^q are lisse sheaves on Z.

Now to see that $\mathrm{L}^q = 0$ unless $q = 0$ or 1. The argument below works for finite coefficients; we let R be the ring of integers of a finite extension E_λ of $\mathbf{Q}_\ell$ with maximal ideal m and note that R/m^n is injective as a module over itself by Baer's criterion. Let $\Lambda := \mathrm{R}/m^n$ and we consider constructible sheaves of Λ-modules. Let $\mathcal{J}$ be any l.c.c.

sheaf of Λ-modules on U; we wish to show that $R^q\P_*\mathcal{J} = 0$ unless $q = 0, 1$. En effet,

$$R\P_*\mathcal{J}[2] \simeq R\P_*R\underline{\mathrm{Hom}}(\mathcal{J}^\vee(1), \Lambda(1)[2]) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\P_!(\mathcal{J}^\vee(1)), \Lambda)$$
$$= \underline{\mathrm{Hom}}(R\P_!(\mathcal{J}^\vee(1)), \Lambda)$$

and it reduces to showing that $R\P_!(\mathcal{J}^\vee(1))$ is concentrated in degrees 1 and 2, which will follow from proper base change from the corresponding fiberwise statement. In other words, we must show that given a l.c.c. sheaf $\mathcal{F}$ on a smooth affine curve Y over an algebraically closed field k, $H^i_c(Y, \mathcal{F})$ unless $i = 1, 2$. This follows immediately from Poincaré duality on Y

$$H^{2-i}(Y, \mathcal{F}^\vee(1)) \xrightarrow{\sim} H^i_c(Y, \mathcal{F})^\vee$$

and Artin's theorem.

The abutment of the Leray spectral sequence

$$H^p(Z, L^q) \Rightarrow H^{p+q}(U, \underline{\mathrm{Hom}}(M_U, N_U)) \simeq H^{p+q}R\,\mathrm{Hom}(\overline{\mathbf{Q}}_\ell, \underline{\mathrm{Hom}}(M_U, N_U))$$

is naturally isomorphic to

$$\mathrm{Ext}^{p+q}_{D(U)}(M_U, N_U) \simeq H^{p+q}R\,\mathrm{Hom}(\overline{\mathbf{Q}}_\ell, R\underline{\mathrm{Hom}}(M_U, N_U))$$

because as M_U is a lisse $\overline{\mathbf{Q}}_\ell$-sheaf, $\underline{\mathrm{Hom}}(M_U, -)$ is an exact functor. To see that $\underline{\mathrm{Ext}}^i(M_U, N_U) = 0$ for $i > 0$, note that since M_U is lisse, for any geometric point x of U, $\underline{\mathrm{Hom}}(M_U, N_U)_x = \underline{\mathrm{Hom}}((M_U)_x, (N_U)_x)$ and likewise for $R\underline{\mathrm{Hom}}$ (c.f. note to Th. fin. 1.6), so that $R\underline{\mathrm{Hom}}(M_U, N_U)$ is connective iff $\underline{\mathrm{Hom}}((M_U)_x, (N_U)_x)$ is, and we can compute the latter via a projective resolution of $(M_U)_x$. But M_U is a lisse $\overline{\mathbf{Q}}_\ell$-sheaf, so that we can represent it on the level of a finite coefficient ring Λ by a torsion-free sheaf, in which case $(M_U)_x$ is a projective Λ-module.

2.1.2. On the definition of the arrows

$$H^p(Z, R^0\P_*\underline{\mathrm{Hom}}(M_U, N_U)) \to H^p(Y', R^0\P_{Y'*}(\underline{\mathrm{Hom}}(M_{U_{Y'}}, Q_{U_{Y'}}))) :$$

let $j : \mathrm{Y}' \hookrightarrow \mathrm{Z}$ denote the immersion and its base extensions; then

$$\begin{aligned}
\mathrm{R}\Gamma(\mathrm{Z}, \mathrm{R}^0\P_*\underline{\mathrm{Hom}}(\mathrm{M_U}, \mathrm{N_U})) &\xrightarrow{\sim} \mathrm{R}\Gamma(\mathrm{Z}, j_*j^*\mathrm{R}^0\P_*\underline{\mathrm{Hom}}(\mathrm{M_U}, \mathrm{N_U}))\\
&\xrightarrow{\sim} \mathrm{R}\Gamma(\mathrm{Z}, j_*\mathrm{R}^0\P_{\mathrm{Y}'*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U}_{\mathrm{Y}'}}, \mathrm{N}_{\mathrm{U}_{\mathrm{Y}'}}))\\
&\to \mathrm{R}\Gamma(\mathrm{Z}, j_*\mathrm{R}^0\P_{\mathrm{Y}'*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U}_{\mathrm{Y}'}}, \mathrm{Q}_{\mathrm{U}_{\mathrm{Y}'}}))\\
&\to \mathrm{R}\Gamma(\mathrm{Z}, \mathrm{R}j_*\mathrm{R}^0\P_{\mathrm{Y}'*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U}_{\mathrm{Y}'}}, \mathrm{Q}_{\mathrm{U}_{\mathrm{Y}'}}))\\
&= \mathrm{R}\Gamma(\mathrm{Y}', \mathrm{R}^0\P_{\mathrm{Y}'*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U}_{\mathrm{Y}'}}, \mathrm{Q}_{\mathrm{U}_{\mathrm{Y}'}})),
\end{aligned}$$

where the first arrow is (inconsequentially) an isomorphism since L^0 is lisse. (Of course $j^*\mathrm{R}^0\P_* = \mathrm{H}^0 j^*\mathrm{R}\P_* \xrightarrow{\sim} \mathrm{H}^0\mathrm{R}\P_{\mathrm{Y}'*}j^* = \mathrm{R}^0\P_{\mathrm{Y}'*}j^*$.)

Proof of (2.1.2) $\Rightarrow$ (2.1.1). In the proof, (2.1.2 b) is invoked with Z and replaced by Y and sheaves $\mathrm{M}_{\mathrm{U_Y}}$ and $\mathrm{P}_{\mathrm{U_Y}}$, where $\mathrm{P}_{\mathrm{U_Y}}$ is lisse on $\mathrm{U_Y}$ but the condition that the $\mathrm{R}^q\P_{\mathrm{Y}*}(\mathrm{M}^*_{\mathrm{U_Y}} \otimes \mathrm{P}_{\mathrm{U_Y}})$ be lisse on Y is not secured by (2.1.2 a). The condition that the L^q be lisse is used in the proof of (2.1.2 b). Therefore (2.1.2 a) should say

(2.1.2 a) There exists an open $\mathrm{Y} \subset \mathrm{Z}$, a lisse $\mathrm{P}_{\mathrm{U_Y}}$ in $\mathrm{M}(\mathrm{U_Y})$ and an injection $\mathrm{N}_{\mathrm{U_Y}} \hookrightarrow \mathrm{P}_{\mathrm{U_Y}}$ such that the sheaves $\mathrm{R}^q\P_{\mathrm{Y}*}(\mathrm{M}^*_{\mathrm{U_Y}} \otimes \mathrm{P}_{\mathrm{U_Y}})$ are lisse and the corresponding arrow $\mathrm{R}^1\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{N}_{\mathrm{U_Y}}) \to \mathrm{R}^1\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{P}_{\mathrm{U_Y}})$ is null.

Of course, the original (2.1.2 a) implies this statement.

As for the proof of (2.1.2) $\Rightarrow$ (2.1.1), one must observe that restriction from U to $\mathrm{U_Y}$ and the arrow $\mathrm{N}_{\mathrm{U_Y}} \hookrightarrow \mathrm{P}_{\mathrm{U_Y}}$ together induce a morphism of spectral sequences which is the zero map on $\mathrm{H}^{i-1}(\mathrm{Z}, \mathrm{R}^1\P_*\underline{\mathrm{Hom}}(\mathrm{M_U}, \mathrm{N_U})) \to \mathrm{H}^{i-1}(\mathrm{Y}, \mathrm{R}^1\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{P}_{\mathrm{U_Y}}))$ (this arrow is defined as the one in (2.1.2 b)), implying the first statement as the image of the E_2 page has

$$\mathrm{im}(\mathrm{H}^i(\mathrm{U}, \mathrm{R}^0\P_*\underline{\mathrm{Hom}}(\mathrm{M_U}, \mathrm{N_U})) \to \mathrm{H}^i(\mathrm{Y}, \mathrm{R}^0\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{P}_{\mathrm{U_Y}})))$$

as the only nonzero term of the ith diagonal, so that its image in the abutment, which coincides with its image on the E_3 page, which coincides with its image in $\mathrm{H}^i(\mathrm{Y}, \mathrm{R}^0\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{P}_{\mathrm{U_Y}}))/d_2(\mathrm{H}^{i-2}(\mathrm{Y}, \mathrm{R}^1\P_{\mathrm{Y}*}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{P}_{\mathrm{U_Y}})))$, coincides with the image of $\mathrm{Ext}^i_{\mathrm{D(U)}}(\mathrm{M_U}, \mathrm{N_U})$ in $\mathrm{Ext}^i_{\mathrm{D}(\mathrm{U_Y})}(\mathrm{M}_{\mathrm{U_Y}}, \mathrm{P}_{\mathrm{U_Y}})$. This is in turn killed by (2.1.2 b) for $i > 0$.

Proof of 2.1.2 a.

$$\begin{aligned} L^{1*} \otimes L^1 &= \underline{Hom}(L^1, L^1) = \underline{Hom}(L^1, R^1¶_*\underline{Hom}(M, N)) \\ &\simeq H^1\underline{Hom}(L^1, R¶_*\underline{Hom}(M, N)) \simeq R^1¶_*\underline{Hom}(¶^*L^1, \underline{Hom}(M, N)) \\ &\simeq R^1¶_*\underline{Hom}(¶^*L^1 \otimes M, N) \end{aligned}$$

where the first isomorphism on the second line holds because L^1 is a lisse $\overline{\mathbf{Q}}_\ell$-sheaf so that $\underline{Hom}(L^1, -)$ is an exact functor, as discussed above.

The injection $N \hookrightarrow P_{U_Y}$ induces the following map on $L^{1*} \otimes L^1$:

$$L^{1*} \otimes L^1 \xrightarrow{\sim} R^1¶_*\underline{Hom}(¶^*L^1 \otimes M, N) \to R^1¶_*\underline{Hom}(¶^*L^1 \otimes M, P_{U_Y}); \quad \textit{i.e.}$$
$$\underline{Hom}(L^1, L^1) = \underline{Hom}(L^1, R^1¶_*\underline{Hom}(M, N)) \to \underline{Hom}(L^1, R^1¶_*\underline{Hom}(M, P_{U_Y})).$$

If α is the image of the global extension $\tilde{\alpha}$, then the injection $N \hookrightarrow P_{U_Y}$ defined by the extension of class α will annihilate $\tilde{\alpha}$ (c.f. note to BBD 3.1.17). As α is the image of $\tilde{\alpha}$ under a natural map (i.e. $N \hookrightarrow P_{U_Y}$ induces a morphism of Leray spectral sequences), this injection likewise kills α. But $L^{1*} \otimes L^1 = \underline{Hom}(L^1, L^1)$, and under this isomorphism $\alpha \mapsto \mathrm{id}$, so that the global section $\alpha \in H^0(Z, L^{1*} \otimes L^1)$ goes to zero precisely when the section $\mathrm{id} \in \mathrm{Hom}(L^1, L^1)$ does; i.e. precisely when L^1 is annihilated.

$$\mathrm{Hom}(L^1,\ L^1) \ni \mathrm{id} \longmapsto 0$$
$$L^1 = \underline{Hom}(L^1, R^1¶_*\underline{Hom}(M, N)) \longrightarrow \underline{Hom}(L^1, R^1¶_*\underline{Hom}(M, P_{U_Y})$$

The induction step is used to kill the obstruction in the following way: $\dim Z = -1 + \dim X$ so the induction hypothesis gives $Y \subset Z$ and $\varphi : L^{1*}_Y \hookrightarrow K_Y$ inducing zero on $\mathrm{Ext}^2_{D(Z)}(L^{0*}, L^{1*}) \to \mathrm{Ext}^2_{D(Y)}(L^{0*}_Y, L^{1*}_Y)$. But

$$\begin{aligned} \mathrm{Ext}^2_{D(Z)}(L^{0*}, L^{1*}) &= H^2 R\,\mathrm{Hom}(L^{0*}, L^{1*}) = H^2 R\,\mathrm{Hom}(\overline{\mathbf{Q}}_\ell, R\underline{Hom}(L^{0*}, L^{1*})) \\ &= H^2 R\,\mathrm{Hom}(\overline{\mathbf{Q}}_\ell, \underline{Hom}(L^{0*}, L^{1*})) = H^2(Z, L^0 \otimes L^{1*}), \end{aligned}$$

and likewise on Y.

Typos: it is clear from the above that

$$\mathrm{K_Y} \otimes \mathrm{L^1_Y} \simeq \mathrm{R^1}\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\P^*\mathrm{K^*_Y} \otimes \mathrm{M_{U_Y}}, \mathrm{N_{U_Y}}) \qquad \textit{and}$$
$$\widetilde{\varphi(\alpha)} \in \mathrm{Ext^1}(\P^*\mathrm{K^*_Y} \otimes \mathrm{M_{U_Y}}, \mathrm{N_{U_Y}})$$

with $\P^*\mathrm{K^*_Y}$ instead of $\P^*\mathrm{K_Y}$ and $\mathrm{M_{U_Y}}, \mathrm{N_{U_Y}}$ in lieu of $\mathrm{M_U}, \mathrm{N_U}$. Finally, $\mathrm{P_{U_Y}}$ is defined by an extension of $\P^*\mathrm{K^*_Y} \otimes \mathrm{M_{U_Y}}$, not $\P_*\mathrm{K_Y} \otimes \mathrm{M_{U_Y}}$, by $\mathrm{N_{U_Y}}$.

The point is that, localizing about any point of the base Y, the sheaves $\mathrm{L^{1*}}$ and $\mathrm{K_Y}$ become constant so that along any fiber, $\mathrm{L^{1*}} \hookrightarrow \mathrm{K_Y}$ admits a complement splitting $\mathrm{K_Y}$. Therefore the monomorphism $\mathrm{N_{U_Y}} \hookrightarrow \mathrm{P_{U_Y}}$ killing $\varphi(\alpha)$ also kills α since it does so along every fiber, and it does so along every fiber because along any fiber there is a retraction to $\alpha \mapsto \varphi(\alpha)$.

Explanation of proof: our objective is to find an injection $\mathrm{N_{U_Y}} \hookrightarrow \mathrm{P_{U_Y}}$ such that the corresponding arrow $\mathrm{R^1}\P_{\mathrm{Y}*}(\underline{\mathrm{Hom}}(\mathrm{M_{U_Y}}, \mathrm{N_{U_Y}}) \to \mathrm{R^1}\P_{\mathrm{Y}*}(\mathrm{M_{U_Y}}, \mathrm{P_{U_Y}})$ is null. The first part of the proof is enough to construct such a $\mathrm{P_{U_Y}}$ if we look over the strict henselization of Y at a closed point $y \in \mathrm{Y}$, because of course any lisse sheaf on Y_y is constant so that if we set $\mathrm{Z} = \mathrm{Y}_y$, $\mathrm{Ext^1}(\P^*\mathrm{L^1} \otimes \mathrm{M}, \mathrm{N}) = \mathrm{H^0}(\mathrm{Y}_y, \mathrm{L^{1*}} \otimes \mathrm{L^1})$ since $\mathrm{R\Gamma}(\mathrm{Y}_y, -) = \mathrm{H^0}(\mathrm{Y}_y, -)$ is an exact functor so that

$$\begin{aligned}
&\mathrm{Ext^1}(\P^*\mathrm{L^1} \otimes \mathrm{M}, \mathrm{N}) = \mathrm{H^1R\Gamma\ R}\P_*\mathrm{R}\underline{\mathrm{Hom}}(\P^*\mathrm{L^1} \otimes \mathrm{M}, \mathrm{N}) \\
&= \mathrm{R\Gamma\ H^1R}\P_*\mathrm{R}\underline{\mathrm{Hom}}(\P^*\mathrm{L^1} \otimes \mathrm{M}, \mathrm{N}) = \mathrm{H^0}(\mathrm{Y}_y, \mathrm{R^1}\P_*\underline{\mathrm{Hom}}(\P^*\mathrm{L^1} \otimes \mathrm{M}, \mathrm{N})) \\
&\qquad = \mathrm{H^0}(\mathrm{Y}_y, \mathrm{L^{1*}} \otimes \mathrm{L^1}).
\end{aligned}$$

Now the point is that even though it is simple to find a $\mathrm{P_{U_Y}}$ looking over each closed fiber, the existence of a global $\mathrm{P_{U_Y}}$ is obstructed by the possibly nonzero class $\partial(\alpha)$. Therefore the trick is to take an injection $\varphi : \mathrm{L^{1*}_Y} \hookrightarrow \mathrm{K_Y}$ into a lisse $\mathrm{K_Y}$ on $\mathrm{Y} \subset \mathrm{Z}$ which kills $\partial(\alpha)$. Now, $\varphi(\alpha) \in \mathrm{H^0}(\mathrm{Y}, \mathrm{K_Y} \otimes \mathrm{L^1_Y})$ is annihilated by $\mathrm{L^1_Y} \to \mathrm{R^1}\P_{\mathrm{Y}*}\underline{\mathrm{Hom}}(\mathrm{M_{U_Y}}, \mathrm{P_{U_Y}})$ for the same reasons as before, but now the existence of $\mathrm{P_{U_Y}}$ is guaranteed by design. The injection φ induces an injection of global sections $H^0(Y, L_Y^{1*} \otimes L_Y^1) \hookrightarrow H^0(Y, K_Y \otimes L_Y^1)$.

$$\begin{array}{ccc} H^0(Y, L_Y^{1*} \otimes L_Y^1) & \xhookrightarrow{\varphi \otimes \mathrm{id}} & H^0(Y, K_Y \otimes L_Y^1) \\ \downarrow & & \downarrow \\ H^0(Y, L_Y^{1*} \otimes R^1 ¶_{Y*}\underline{Hom}(M_{U_Y}, P_{U_Y})) & \hookrightarrow & H^0(Y, K_Y \otimes R^1 ¶_{Y*}\underline{Hom}(M_{U_Y}, P_{U_Y})) \end{array}$$

As the rightmost vertical arrow kills $\varphi(\alpha)$, the leftmost vertical arrow must kill α. As α corresponds to $\mathrm{id}_{L^1} \in \mathrm{Hom}(L_Y^1, L_Y^1)$, the fact that α goes to zero under

$$\mathrm{Hom}(L_Y^1, L_Y^1) \to \mathrm{Hom}(L_Y^1, R^1 ¶_{Y*}\underline{Hom}(M_{U_Y}, P_{U_Y}))$$

implies that the map $L_Y^1 \to R^1 ¶_{Y*}\underline{Hom}(M_{U_Y}, P_{U_Y})$ is null.

Proof of 2.1.2. b. The $Q_{U'_Y}$ of the statement of (2.1.2 b) is denoted $O_{U'_Y}$ in the proof. About the cocartesian square: indeed pushouts exist in any topos, and if you like fancy words, any topos is an adhesive category, meaning that it has pullbacks and pushouts of monomorphisms, and pushout squares of monomorphisms are also pullback squares and are stable under pullback. In particular, $O_{U_{Y'}}$ defined by a pushout, has $N_{U_{Y'}} \hookrightarrow O_{U_{Y'}}$ injective since $¶_{Y'}^* L_{Y'}^0 \otimes M_{U_{Y'}} \hookrightarrow ¶_{Y'}^* Q_{Y'} \otimes M_{U_{Y'}}$ is. The reason any Grothendieck topos is adhesive is as simple as you think it is, namely that the category of sets is adhesive and adhesivity is a condition on colimits and finite limits, hence preserved by functor categories and left-exact localizations (c.f. nLab).

The canonical arrow is given by composition

$$¶_{Y'}^* L_{Y'}^0 \otimes M_{U_{Y'}} = ¶_{Y'}^* ¶_{Y'*}(M_{U_{Y'}}^* \otimes N_{U_{Y'}}) \otimes M_{U_{Y'}} \to (M_{U_{Y'}}^* \otimes N_{U_{Y'}}) \otimes M_{U_{Y'}} \to N_{U_{Y'}}.$$

The pushout

$$\begin{array}{ccc} ¶_{Y'}^* L_{Y'}^0 \otimes M_{U_{Y'}} & \hookrightarrow & ¶_{Y'}^* Q_{Y'}^0 \otimes M_{U_{Y'}} \\ \downarrow & & \downarrow \\ N_{U_{Y'}} & \hookrightarrow & O_{U_{Y'}} \end{array}$$

defines by adjunction the commutative diagram

$$\begin{tikzcd}
\P^*_{Y'}L^0_{Y'} \arrow[r, hook] \arrow[d] & \P^*_{Y'}Q^0_{Y'} \arrow[d] \\
\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}}) \arrow[r] & \underline{\mathrm{Hom}}(M_{U_{Y'}}, O_{U_{Y'}})
\end{tikzcd}$$

and again by adjunction the commutative diagram

$$\begin{tikzcd}
L^0_{Y'} \arrow[r, hook] \arrow[d] & Q^0_{Y'} \arrow[d] \\
\P_{Y'*}\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}}) \arrow[r] & \P_{Y'*}\underline{\mathrm{Hom}}(M_{U_{Y'}}, O_{U_{Y'}})
\end{tikzcd}$$

Now to check that the leftmost vertical arrow in the diagram above is the identity it suffices to show that the identity is taken by adjunction (from the bottom diagram to the top) to the given arrow in the pushout diagram that defines $O_{U_{Y'}}$. The image of $\mathrm{id} : L^0_{Y'} \to \P_{Y'*}\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}})$ under adjunction is given by the counit of the adjunction at the object $\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}})$

$$\varepsilon(\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}})) : \P^*_{Y'}\P_{Y'*}\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}}) \to \underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}}).$$

Now in general a morphism $f : A \to \mathrm{Hom}(B, C)$ is taken under tensor-hom adjunction to the morphism $A \otimes B \to C$ which is the composition of $f \otimes \mathrm{id}$ followed by evaluation. This is exactly the definition of the canonical arrow above: $\varepsilon(\underline{\mathrm{Hom}}(M_{U_{Y'}}, N_{U_{Y'}})) \otimes \mathrm{id}$ followed by evaluation. For each $i > 0$, applying the functor $H^i(Y', -)$ to the last commutative diagram and precomposing with the arrow $H^i(Z, L^0) \to H^i(Y', L^0_{Y'})$ coming from the unit $\eta(L^0) : L^0 \to \P_{Y'*}\P^*_{Y'}L^0 = \P_{Y'*}L^0_{Y'}$ effaces $H^i(Z, L^0)$.

Perversity. All of the arguments above hold for any perversity.

2.2. By general considerations [**BBD**, 1.2], the morphism $N \to N'$ is a monomorphism if it completes to an exact triangle (N, N', C) with C in $M(X)$, or equivalently to an exact triangle (K, N, N') with $K[1]$ in $M(X)$, so it suffices to show that ${}^pH^0K = 0$. As j_i^* is t-exact, it suffices to show that ${}^pH^0 j_i^* K = 0$ for all i. By hypothesis $N_{U_i} \hookrightarrow N'_{U_i}$ so that $(N_{U_i}, j_i^*N', j_i^*K[1])$ has $j_i^*K[1]$ in $M(U_i)$.

To see that the injection $\mathrm{N} \hookrightarrow \mathrm{N}'$ kills all $\mathrm{Ext}^j_{\mathrm{D(X)}}(\mathrm{M}, \mathrm{N})$, $j > 0$, write

$$\mathrm{Ext}^j_{\mathrm{D(X)}}(\mathrm{M}, \mathrm{N}) = \oplus_i \mathrm{Ext}^j_{\mathrm{D(X)}}(\mathrm{M}, j_{i*}\mathrm{N}'_{\mathrm{U}_i}) = \oplus_i \mathrm{Ext}^j_{\mathrm{D(U}_i)}(j_i^*\mathrm{M}, \mathrm{N}'_{\mathrm{U}_i});$$

this map sends $\varphi \in \mathrm{Ext}^j_{\mathrm{D(X)}}(\mathrm{M}, \mathrm{N})$ to $\oplus_i \varphi_i$, where $\varphi_i \in \mathrm{Ext}^j_{\mathrm{D(U}_i)}(j_i^*\mathrm{M}, \mathrm{N}'_{\mathrm{U}_i})$ is given by

$$\varphi_i : \mathrm{M}_{\mathrm{U}_i}[-j] \xrightarrow{j_i^*\varphi} \mathrm{N}_{\mathrm{U}_i} \to \mathrm{N}'_{\mathrm{U}_i}.$$

This composition is zero by assumption.

Perversity. j_{i*} no longer t-exact and the same argument doesn't work if you replace j_{i*} with ${}^p j_{i*}$; j^* is still t-exact so that $j_i^{*p} j_{i*} = j_i^{*p}\mathrm{H}^0 j_{i*} = {}^p\mathrm{H}^0 j_i^* j_{i_*} = {}^p\mathrm{H}^0 = \mathrm{id}$ on $\mathrm{M(U}_i)$ and $(j_i^*, {}^p j_{i*})$ form an adjoint pair $\mathrm{M(U}_i) \leftrightarrows \mathrm{M(X)}$ but not $\mathrm{D(U}_i) \leftrightarrows \mathrm{D(X)}$.

2.2.1. We have

$$\mathrm{Ext}^i_{\mathrm{D(X)}}(\mathrm{M}, \mathrm{N}) := \mathrm{Hom}_{\mathrm{D(X)}}(\mathrm{M}, \mathrm{N}[i]) = \mathrm{Hom}_{\mathrm{D(X)}}(\mathrm{M}, i_*i^*\mathrm{N}[i]) = \mathrm{Hom}_{\mathrm{D(Y)}}(i^*\mathrm{M}, i^*\mathrm{N}[i]).$$

Note that, although the embedding $\mathrm{M(Y)} \hookrightarrow \mathrm{M(X)}$ is via the t-exact functor i_*, in order to show I is an isomorphism we cannot naïvely make use of an adjunction between i^* and i_* as i^* is no longer t-exact. (It is true however that $({}^p i^*, {}^p i_* = i_*)$ form an adjoint pair [**BBD**, 1.4.16].)

The part about $\mathrm{Ext}^i_{\mathrm{M(X)}}(\mathrm{M}, \mathrm{N})$ coinciding with the set of connected components of $\mathrm{E}^i_{\mathrm{M(X)}}(\mathrm{M}, \mathrm{N})$ is (3.2.2) in Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239**. The sequence of morphisms in $\mathrm{E}^i_{\mathrm{M(X)}}(\mathrm{M}, \mathrm{N})$ connecting $\mathrm{C}^\cdot$ with $\Phi_f(\mathrm{C}^\cdot)$ is as written

$$\mathrm{C}^\cdot \to \mathrm{C}^\cdot \oplus \Xi_f(\mathrm{C}^\cdot_{\mathrm{U}}) \to (\mathrm{C}^\cdot \oplus \Xi_f(\mathrm{C}^\cdot_{\mathrm{U}}))/j_!(\mathrm{C}^\cdot_{\mathrm{U}}) \leftarrow \Phi_f(\mathrm{C}^\cdot). \qquad (\dagger)$$

Recall that the unipotent vanishing cycles functor $\Phi_f : \mathrm{M(X)} \to \mathrm{M(Y)}$ is defined as the H^0 of the complex of objects in $\mathrm{M(X)}$

$$j_!j^*\mathrm{K} \to \Xi_f j^*\mathrm{K} \oplus \mathrm{K} \to j_*j^*\mathrm{K}$$

where K is a perverse sheaf on X (see §6 of Morel's notes and the accompanying note). The point is that the ends of $\mathrm{C}^\cdot$ are supported on Y, and for a perverse sheaf K in $\mathrm{M(X)}$

supported on Y, the maps in (†) degenerate to

$$\mathrm{K} \to \mathrm{K} \oplus 0 \to \mathrm{K} \oplus 0 \leftarrow \mathrm{K} = \Phi_f(\mathrm{K})$$

with every map the identity.

2.2.2. Let's show that given an adjunction of exact functors $\mathrm{F} \dashv \mathrm{G}$ between abelian categories $\mathscr{A} \underset{\mathrm{F}}{\overset{\mathrm{G}}{\rightleftarrows}} \mathscr{B}$, we have

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(\mathrm{FB}, \mathrm{A}) = \mathrm{Hom}_{\mathrm{D}(\mathscr{B})}(\mathrm{B}, \mathrm{GA}).$$

The extension of the adjunction $\mathrm{F} \dashv \mathrm{G}$ to complexes of objects of $\mathscr{A}$ and $\mathscr{B}$ is simple, and it is easy to verify that the adjunction sends homotopic morphisms to homotopic morphisms, hence descends to an adjunction $\mathrm{K}(\mathscr{A}) \leftrightarrows \mathrm{K}(\mathscr{B})$. As the functors are both exact, they preserve quasi-isomorphisms, so to conclude on derived categories we write

$$\begin{tikzcd} & \mathrm{C} \arrow[dl, "\sim"'] \arrow[dr, "f"] & \\ \mathrm{B} & & \mathrm{GA} \end{tikzcd} \rightsquigarrow \begin{tikzcd} & \mathrm{FC} \arrow[dl, "\sim"'] \arrow[dr, "\mathrm{F}f"] & & \\ \mathrm{FB} & & \mathrm{FGA} \arrow[r, "\varepsilon \mathrm{A}"] & \mathrm{A} \end{tikzcd} \rightsquigarrow \begin{tikzcd} & \mathrm{C} \arrow[dl, "\sim"'] \arrow[r, "\eta \mathrm{C}"'] \arrow[rrrd, dashed, bend left, "f"] & \mathrm{GFC} \arrow[dl, "\sim"] \arrow[dr, "\mathrm{GF}f"] & & \\ \mathrm{B} \arrow[r, "\eta \mathrm{B}"'] & \mathrm{GFB} & & \mathrm{GFGA} \arrow[r, "\varepsilon \mathrm{A}"] & \mathrm{GA} \end{tikzcd}$$

(the dashed arrow coincides with f since it coincides with applying the adjunction isomorphism backwards and forwards to f), and

$$\begin{tikzcd} & \mathrm{D} \arrow[dl, "\sim"'] \arrow[dr, "g"] & \\ \mathrm{FB} & & \mathrm{A} \end{tikzcd} \rightsquigarrow \begin{tikzcd} & & \mathrm{GD} \arrow[dl, "\sim"'] \arrow[dr, "\mathrm{G}g"] & \\ \mathrm{B} \arrow[r, "\eta \mathrm{B}"] & \mathrm{GFB} & & \mathrm{GA} \end{tikzcd} \rightsquigarrow \begin{tikzcd} & & & \mathrm{FGA} \arrow[r, "\varepsilon \mathrm{A}"] & \mathrm{A} \\ & & \mathrm{FGD} \arrow[ur, "\mathrm{FG}g"] \arrow[r, "\varepsilon \mathrm{D}"] \arrow[dl, "\sim"'] & \mathrm{D} \arrow[ur, "g"'] \arrow[dl, "\sim"] & \\ \mathrm{FB} \arrow[r, "\mathrm{F}\eta \mathrm{B}"] \arrow[rr, dashed, bend right, "\mathrm{id}"'] & \mathrm{FGFB} \arrow[r, "\varepsilon \mathrm{FA}"] & \mathrm{FA} & & \end{tikzcd}$$

Together these diagrams show that the maps on Hom induced by the first $\rightsquigarrow$ of each row are mutually inverse and induce the isomorphism $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(\mathrm{FB}, \mathrm{A}) = \mathrm{Hom}_{\mathrm{D}(\mathscr{B})}(\mathrm{B}, \mathrm{GA})$.

2.2.3. Of course $j_! j^* \mathrm{M} \to \mathrm{M}$ has nonzero image since this arrow restricts to the identity over U and j^* is exact functor so that j^* of the image is M. As M is irreducible, $j_! j^* \mathrm{M} \to \mathrm{M}$ is therefore an epimorphism. Then $(\mathrm{K}, j_! j^* \mathrm{M}, \mathrm{M})$ is a distinguished triangle of perverse sheaves and applying j^* finds $j^* \mathrm{K} = 0$.

X needn't be irreducible, so needn't have a unique generic point. However, we reduce to the case X irreducible in the next paragraph, allowing us to assume X irreducible with generic point η in the rest of this paragraph. I see no reason why

$$
\begin{aligned}
&(\mathrm{co})\ker(\mathrm{Ext}^{\bullet}_{\mathrm{M(X)}}(\mathrm{M},\mathrm{N}) \to \mathrm{Ext}^{\bullet}_{\mathrm{D(X)}}(\mathrm{M},\mathrm{N}))\\
&= (\mathrm{co})\ker(\mathrm{Ext}^{\bullet}_{\mathrm{M(X)}}(\mathrm{M}, j_*\mathrm{N_U}) \to \mathrm{Ext}^{\bullet}_{\mathrm{D(X)}}(\mathrm{M}, j_*\mathrm{N_U}));
\end{aligned}
$$

take some stupid example like

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{C_3} \arrow[r] \arrow[d] & \mathrm{C_1 \oplus C_2 \oplus C_3} \arrow[r] \arrow[d, equal] & \mathrm{C_1 \oplus C_2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{C_2 \oplus C_3} \arrow[r] & \mathrm{C_1 \oplus C_2 \oplus C_3} \arrow[r] & \mathrm{C_1} \arrow[r] & 0.
\end{tikzcd}
$$

We instead make a slightly more subtle analysis. For each $\mathrm{U} \ni \eta$, we have the following commutative diagram with exact rows & columns.

$$
\begin{tikzcd}
\mathrm{Ext}^{i-1}_{\mathrm{M(U)}}(\mathrm{M_U},\mathrm{N_U}) \arrow[r] \arrow[d, "\beta^{i-1}"] & \mathrm{Ext}^{i-1}_{\mathrm{M(X)}}(\mathrm{M},\mathrm{L^U}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^{i}_{\mathrm{M(X)}}(\mathrm{M},\mathrm{N}) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Ext}^{i}_{\mathrm{M(U)}}(\mathrm{M_U},\mathrm{N_U}) \arrow[r] \arrow[d, "\beta^{i}"] & \mathrm{Ext}^{i}_{\mathrm{M(X)}}(\mathrm{M},\mathrm{L^U}) \arrow[d, "\wr"] \\
\mathrm{Ext}^{i-1}_{\mathrm{D(U)}}(\mathrm{M_U},\mathrm{N_U}) \arrow[r] & \mathrm{Ext}^{i-1}_{\mathrm{D(X)}}(\mathrm{M},\mathrm{L^U}) \arrow[r] & \mathrm{Ext}^{i}_{\mathrm{D(X)}}(\mathrm{M},\mathrm{N}) \arrow[r] & \mathrm{Ext}^{i}_{\mathrm{D(U)}}(\mathrm{M_U},\mathrm{N_U}) \arrow[r] & \mathrm{Ext}^{i}_{\mathrm{M(X)}}(\mathrm{M},\mathrm{L^U})
\end{tikzcd}
$$

These diagrams form an inductive system with respect to étale neighborhoods of η, and in the limit β^{i-1} and β are isomorphisms. By the five lemma, α is an isomorphism. (In more words, say we have an element $x \in \ker\alpha$. Shrinking U, we can make its image in $\mathrm{Ext}^{i}_{\mathrm{M(U)}}(\mathrm{M_U},\mathrm{N_U})$ zero. Chasing the diagram, we produce an element $y \in \mathrm{Ext}^{i-1}_{\mathrm{D(U)}}(\mathrm{M_U},\mathrm{N_U})$; shrinking U further we produce a $z \in \mathrm{Ext}^{i-1}_{\mathrm{M(U)}}(\mathrm{M_U},\mathrm{N_U})$ with $\beta^{i-1}(z) = y$ whose image in $\mathrm{Ext}^{i}_{\mathrm{M(X)}}(\mathrm{M},\mathrm{N})$ equals x. Hence x is zero. Given $w \in \operatorname{coker}\alpha$, $\operatorname{coker}\alpha \hookrightarrow \operatorname{coker}\beta^i$ injects by the strong four lemma. Shrinking U, we can make the image of w in $\operatorname{coker}\beta^i$ zero so that $w = 0$, etc.)

Reduction to the case X *irreducible.* Let X_i denote the irreducible components of X with generic points η_i. Since (2.2.3) covers the case when $\dim\operatorname{supp}\mathrm{N} < \dim\mathrm{X}$, (2.2.4) should more properly say that we consider an irreducible N supported at the generic point η of an irreducible component X_1 of X with $\dim\mathrm{X}_1 = \dim\mathrm{X}$. This doesn't preclude the possibility that N is supported at the generic points η_i of two (or more) such irreducible components X_i with $\dim\mathrm{X}_i = \dim\mathrm{X}$ and possibly nonempty pairwise intersection. If so, we should pick open affine $U \subset X$ with $U = \coprod_i U_i$, $\eta_i \in U_i \subset X_i$ so that the U_i are irreducible and have $U_i \cap U_j = \emptyset$. We have to compute $\mathrm{Ext}^{\bullet}(M_U, N_U)$ for this U. Ordinary sheaves on U decompose as a direct sum of sheaves each supported on one U_i, and morphisms between two such sheaves $\mathcal{F}, \mathcal{G}$ respect this decomposition so that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \oplus_i \mathrm{Hom}(\mathcal{F}_{U_i}, \mathcal{G}_{U_i})$. Objects of $D(U)$ are complexes of sheaves on U and therefore decompose similarly into a direct sum; as $D = D^b$, in particular bounded below, $D(U)$ is equivalent to the homotopy category of bounded below complexes of injective sheaves, which likewise decompose, inducing a decomposition of the category $D(U) \simeq \oplus_i D(U_i)$ and therefore a decomposition $D(U) \supset M(U) \simeq \oplus_i M(U_i)$. Therefore

$$\mathrm{Ext}^{\bullet}_{D(U)}(M_U, N_U) = \oplus_i \mathrm{Ext}^{\bullet}_{D(U_i)}(M_{U_i}, N_{U_i}),$$

and we will have the same decomposition for $\mathrm{Ext}^{\bullet}_{M(U)}$ in light of $M(U) \simeq \oplus_i M(U_i)$, provided we can show that $\mathrm{Ext}^{\bullet}_{M(U_i \coprod U_j)}(M_{U_i}, N_{U_j}) = 0$ for $i \neq j$. Since $M(U_i \coprod U_j) \simeq M(U_i) \oplus M(U_j)$, this follows from Yoneda's description of Ext^p: any acyclic complex

$$M_{U_i} \to L^{-p+1} \to \cdots \to L^0 \to N_{U_j}$$

will decompose as the direct sum of acyclic complexes

$$\begin{aligned} &M_{U_i} \to L^{-p+1}_{U_i} \to \cdots \to L^0_{U_i} \to 0 \qquad \text{and} \\ &0 \to L^{-p+1}_{U_j} \to \cdots \to L^0_{U_j} \to N_{U_j}, \end{aligned}$$

demonstrating a decomposition

$$\mathrm{Ext}^p_{M(U_i \coprod U_j)}(M_{U_i}, N_{U_j}) = \mathrm{Ext}^p_{M(U_i)}(M_{U_i}, 0) \oplus \mathrm{Ext}^p_{M(U_j)}(0, N_{U_j}) = 0.$$

Epilogue: Madhav's *Constructible Sheaves*. Theorem 1 uses a transcendental input. The content of §3 does not, although the proof of Proposition 3.10 must be modified in order to work in the setting of constructible étale sheaves on a k-variety with k of possibly nonzero characteristic.

Proof of Theorem 1. Arranging the projection π: follow the directions of the proof of the basic lemma (second form). The linear change of coordinates required to make f monic in the last variable x_n amounts to the substitutions $x_i \mapsto x_i + x_n$ for $i < n$ (which ensures that the coefficient of the highest power of x_n appearing in f is a nonzero scalar $\alpha \in k^\times$), followed by $x_n \mapsto \alpha^{-1/m} x_n$. If we work over a field which is not algebraically closed, $y^m - 1/\alpha$ may not have any roots. However, we may replace k by its perfect closure (a universally integral, radicial, surjective extension), and then assume we need an m^{th} root of a nonzero scalar with $p \nmid m$; the minimal polynomial over k for such an element defines a finite separable extension $k' \supset k$; denote by f the corresponding map on spectra. If we can prove Theorem 1 for $\mathbf{A}^n_{k'}$, then we can deduce the result that any constructible sheaf $\mathcal{F}$ admits a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ which induces the null map $\mathrm{H}^i(\mathbf{A}^n_k, \mathcal{F}) \to \mathrm{H}^i(\mathbf{A}^n_k, \mathcal{G})$ by finding a monomorphism $\alpha : f^*\mathcal{F} \hookrightarrow \mathcal{G}'$ with $\mathrm{H}^i(\mathbf{A}^n_{k'}, \mathcal{G}') = 0$ for $i > 0$ and then taking the monomorphism $\mathcal{F} \xhookrightarrow{\text{res}} f_* f^* \mathcal{F} \xhookrightarrow{f_*\alpha} f_* \mathcal{G}' =: \mathcal{G}$.

In any event, the projection π obtained by making these linear changes of coordinates and then forgetting x_n enjoys all the properties of the usual coordinate projection: π is smooth with fibers which are copies of $\mathbf{A}^1_k$, hence π is acyclic. If $\mathcal{F}$ is a constructible sheaf on $\mathbf{A}^n_k$, we know that after a linear change of coordinates inducing an automorphism $f : \mathbf{A}^n_k \to \mathbf{A}^n_k$, $f^*\mathcal{F}$ satisfies the conditions of (2.2) & (1.3A) for $\pi : \mathbf{A}^n \to \mathbf{A}^{n-1}$ the usual coordinate projection, so we get a monomorphism $f^*\mathcal{F} \hookrightarrow \mathcal{G}$ with $\mathrm{H}^i(\mathbf{A}^n_k, \mathcal{G}) = 0$ for $i > 0$. Now apply f_*. In this way we reduce to π the usual coordinate projection when applying (2.2) & (1.3A).

In the proof itself, both morphisms of short exact sequences are constructed by forming H and G by pushout, and then forming cokernels. To see that the third vertical maps constructed from the universal property of cokernels are isomorphisms, consider by way of example the second diagram. Notate by α, β the nontrivial monomorphisms of the upper and lower rows, respectively. Then coker α = coker β since to give a map from G that kills H is the same as giving a map from π^*J that kills $\pi^*\pi_*$H.

2.1. As $\mathcal{A} = \Delta^* \operatorname{pr}_1^* \mathcal{A}$, the map $p_1^*\mathcal{A} \twoheadrightarrow \Delta_*\mathcal{A}$ is of course the unit of adjunction.

2.2. As remarked in the note to the proof of Theorem 1 above, we may assume that π is the canonical projection. The relevant maps are defined by the cartesian square

$$\begin{tikzcd} \mathbf{A}^2_X \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, "\Pi"] & \mathbf{A}^1_X \arrow[d, "\pi"] \\ \mathbf{A}^1_X \arrow[r, "\pi"] & X. \end{tikzcd}$$

Proof of (2) To see that $p_2|_Y$ is finite, note that it suffices to show that p_2 is finite when restricted to $\Delta(\mathbf{A}^1_X)$ and $p_1^{-1}V$ separately, as if $X \times \mathbf{A}^1 = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with ideals $I, J \subset B$ defining the closed loci of which Y is the union (i.e. $I \cap J = 0$), the square

$$\begin{tikzcd} B = B/I \cap J \arrow[r] \arrow[d] & B/I \arrow[d] \\ B/J \arrow[r] & B/I \cup J \end{tikzcd}$$

is bicartesian, so that if B/I and B/J are finite as modules over A, so too is B. To check that $p_2|_{p_1^{-1}V}$ is finite, pull back the square above along $V \hookrightarrow \mathbf{A}^1_X$ to produce another cartesian square

$$\begin{tikzcd} V \times \mathbf{A}^1 \arrow[r] \arrow[d] \arrow[rr, bend left, "p_2|_{p_1^{-1}V}"] & \mathbf{A}^2_X \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbf{A}^1_X \arrow[d, "\pi"] \\ V \arrow[r] & \mathbf{A}^1_X \arrow[r, "\pi"] & X. \end{tikzcd}$$

Now we need to show the vanishing of $p_{2*}\mathscr{B}$, which by (1.3A) is equivalent to the vanishing of global sections of $p_1^*\mathscr{A}$ along every geometric fiber of p_2. Fix a geometric point $x \to \mathbf{A}^1_X$ defining also the geometric point $x \to X$, and let $p_2^{-1}x \to \mathbf{A}^2_X$ be the geometric fiber of p_2 above x; $p_2^{-1}x$ is isomorphic to the geometric fiber $F_x : \pi^{-1}x \to \mathbf{A}^1_X$ of π, and the restriction of $p_1^*\mathscr{A}$ to p_2^{-1} is therefore the same as the restriction of $\mathscr{A}$ to F_x (which is just a copy of $\mathbf{A}^1_{\bar{k}}$). Since $\pi|_V$ is surjective and $\mathscr{A}|_V = 0$, $F_x^{-1}V =: V_x$ is a nonempty closed locus. Denoting by $j_x : U_x \hookrightarrow F_x$ the complement of V_x in the fiber, we have that $j_x^*(\mathscr{A}|_{F_x})$ is locally constant and $\mathscr{A}|_{F_x} = j_{x!}j_x^*(\mathscr{A}|_{F_x})$; such a sheaf on $\mathbf{A}^1_{\bar{k}}$ has no nonzero global section.

Proof of (1) In the notation of (1.3B), $\mathrm{M} = \mathrm{X}, \mathrm{M}' = \mathbf{A}^1_{\mathrm{X}}, \mathrm{L} = \mathbf{A}^1_{\mathrm{X}}, \mathrm{L}' = \mathbf{A}^2_{\mathrm{X}}, g = \pi = f$, so that both $\mathrm{L} \xrightarrow{f} \mathrm{M}$ and $\mathrm{M}' \xrightarrow{g} \mathrm{M}$ coincide with $\mathbf{A}^1_{\mathrm{X}} \xrightarrow{\pi} \mathrm{X}$. The business about $\pi_* \mathscr{A} = 0$ reduces, after proper base change, to the same situation just discussed, namely to the fact that if $j : \mathrm{U} \subset \mathbf{A}^1_{\bar{k}}$ is a nonempty open and $\mathscr{E}$ a lisse sheaf on U, $j_! \mathscr{E}$ has no nonzero global section. Again, it is crucial that V intersect every fiber of π nontrivially.

2.3. (As $\mathrm{R}p_{2*}\Delta_* = \mathrm{id}$, $\mathrm{R}^q p_{2*}\Delta_* \mathscr{A} = 0$ for $q > 0$.)

3.8. Here are the diagrams necessary to obtain the conclusion.

$$\begin{tikzcd}
\mathrm{FGB} \arrow[r, hook, "\mathrm{F}u"] \arrow[d, "\varepsilon(\mathrm{B})"] & \mathrm{FA} \arrow[d, "t"] \\
\mathrm{B} \arrow[r, hook, "v"] & \mathrm{B}'
\end{tikzcd}$$

$$u : \mathrm{GB} \hookrightarrow \mathrm{A}$$

$$w : \mathrm{A} \xrightarrow{\eta(\mathrm{A})} \mathrm{GFA} \xrightarrow{\mathrm{G}t} \mathrm{GB}'.$$

We want to show that the composition

$$\begin{tikzcd}
\mathrm{GB} \arrow[r, "u"] & \mathrm{A} \arrow[r, "\eta(\mathrm{A})"] \arrow[rr, bend right, "w"'] & \mathrm{GFA} \arrow[r, "\mathrm{G}t"] & \mathrm{GB}'
\end{tikzcd}$$

coincides with $\mathrm{G}v$. This follows from the below commutative diagram; that the dashed arrow coincides with the identity is a basic fact about adjunctions; c.f. Mac Lane IV.1 Th. 1.

$$\begin{tikzcd}
\mathrm{GB} \arrow[rr, hook, "u"] \arrow[dr, "\eta(\mathrm{GB})"] \arrow[ddr, dashed, bend right, "\mathrm{id}"] & & \mathrm{A} \arrow[dr, "\eta(\mathrm{A})"] & \\
& \mathrm{GFGB} \arrow[rr, hook, "\mathrm{GF}u"] \arrow[d, "\mathrm{G}\varepsilon(\mathrm{B})"] & & \mathrm{GFA} \arrow[d, "\mathrm{G}t"] \\
& \mathrm{GB} \arrow[rr, hook, "\mathrm{G}v"] & & \mathrm{GB}'.
\end{tikzcd}$$

From this general fact we deduce the admissibility of $j_!\mathrm{Q}$ on X from that of Q on U in the following way.

$(j_!, j^*)$: if $\mathrm{Ext}^q_{\mathrm{Sh(U)}}(\mathcal{F}, -)$ is effaceable on $\mathrm{C(U)}$, then $\mathrm{Ext}^q_{\mathrm{Sh(U)}}(\mathcal{F}, j^*(-))$ is effaceable on $\mathrm{C(X)}$. Let $\mathcal{G}$ be a constructible sheaf on X and $\mathcal{I}^\bullet$ be a complex of injectives on $\mathrm{Sh(X)}$ resolving $\mathcal{G}$; $j^*\mathcal{I}^\bullet$ is a complex of injectives on $\mathrm{Sh(U)}$ so that

$$\mathrm{R\,Hom}^\bullet_{\mathrm{Sh(U)}}(\mathcal{F}, j^*(\mathcal{G})) = \mathrm{Hom}^\bullet_{\mathrm{Sh(U)}}(\mathcal{F}, j^*\mathcal{I}^\bullet) = \mathrm{Hom}^\bullet_{\mathrm{Sh(X)}}(j_!\mathcal{F}, \mathcal{I}^\bullet) = \mathrm{R\,Hom}^\bullet_{\mathrm{Sh(X)}}(j_!\mathcal{F}, \mathcal{G}).$$

Taking H^q finds

$$\mathrm{Ext}^q_{\mathrm{Sh(U)}}(\mathcal{F}, j^*\mathcal{G}) = \mathrm{Ext}^q_{\mathrm{Sh(X)}}(j_!\mathcal{F}, \mathcal{G}),$$

so that if the former is effaceable on $\mathrm{C(X)}$, the latter is.

3.10. Here are the necessary modifications: we may replace k by its perfect closure, as the extension is radicial and doesn't affect the étale topos. We may also take the reduced scheme structure on X for the same reason so that X becomes geometrically reduced (`035U`). Then the map $\mathrm{U}' \cap \mathrm{Z}_1 \to \mathbf{A}^d$ can be chosen to be generically finite étale. There is a nonempty Zariksi open of $\mathbf{A}^d$ over which the map is finite étale (Hartshorne II.3 Ex. 3.7).

Nori's theorem for finite coefficients Λ. We wish to show that any constructible sheaf $\mathcal{F}$ is admissible. By dévissage one reduces to $\mathcal{F} = j_!\mathcal{F}'$ for j the immersion of a locally closed stratum and $\mathcal{F}'$ locally constant. Factoring j as a composition of a closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{U}$ followed by an open immersion, we see from the exact sequence

$$0 \to \Lambda_{\mathrm{U-Z}} \to \Lambda_{\mathrm{U}} \to \Lambda_{\mathrm{Z}} \to 0$$

that $\mathcal{H}om(i_*\Lambda, -) = i^!(-)$ commutes with filtered colimits since the other two do. Also $\mathrm{Hom}(\mathcal{F}', -)$ commutes with filtered colimits by [**SGAA**, VI 5.8], as $\mathcal{F}'$ is locally constant. Therefore filtered colimits of injective objects are acyclic for both functors, and $j^!$ sends a filtered colimit of injectives to a filtered colimit of injectives. Let $\mathcal{G}_i$ be an filtered system of sheaves giving rise to a filtered system $\mathcal{I}^\bullet_i$ of complexes of injective sheaves. Recalling that filtered colimits are exact in the abelian category of étale sheaves,

$$\begin{aligned}\mathrm{Ext}^q(\mathcal{F}, \varinjlim \mathcal{G}_i) &= \mathrm{Ext}^q(j_!\mathcal{F}', \varinjlim \mathcal{G}_i) = \mathrm{H}^q\mathrm{R}\,\mathrm{Hom}(\mathcal{F}', \mathrm{R}j^!(\varinjlim \mathcal{G}_i))\\ &= \mathrm{H}^q\,\mathrm{Hom}(\mathcal{F}', j^!(\varinjlim \mathcal{J}_i^\bullet)) = \mathrm{H}^q\,\mathrm{Hom}(\mathcal{F}', \varinjlim j^!(\mathcal{J}_i^\bullet))\\ &= \mathrm{H}^q \varinjlim \mathrm{Hom}(\mathcal{F}', j^!(\mathcal{J}_i^\bullet)) = \varinjlim \mathrm{H}^q\,\mathrm{Hom}(\mathcal{F}', j^!(\mathcal{J}_i^\bullet))\\ &= \varinjlim \mathrm{H}^q\,\mathrm{Hom}(\mathcal{F}, \mathcal{J}_i^\bullet) = \varinjlim \mathrm{Ext}^q(\mathcal{F}, \mathcal{G}_i).\end{aligned}$$

Embedding $\mathcal{G}$ as a subsheaf of an injective sheaf $\mathcal{J}$ and passing to the limit along constructible subsheaves of $\mathcal{J}$ containing $\mathcal{G}$ [**SGAA**, IX 2.9 (iii)], one finds a constructible $\mathcal{G} \hookrightarrow \mathcal{G}'$ inducing the null map $\mathrm{Ext}^q(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^q(\mathcal{F}, \mathcal{G}')$ for $q > 0$.

Bibliography

[B1] *On the derived category of perverse sheaves* by Beilinson

[B2] *How to glue perverse sheaves* by Beilinson

[BBD] *Faisceaux Pervers* par Beilinson, Bernstein, Deligne & Gabber

[SGAA] SGA 4

10. *How to glue perverse sheaves*

Sasha's exposé. Sasha gave (12/5/19) a lecture outlining how to glue perverse sheaves based on his article, which he says was written in a language somewhat better suited to $\mathscr{D}$-modules, so he gave an exposé tailored to perverse sheaves. This section is an attempt to write down the contents of his exposé.

$$\begin{tikzcd} \mathrm{Y} \arrow[r, hook, "i"] & \mathrm{X} \arrow[d, "f"] & \mathrm{U} \arrow[l, hook', "j"'] \\ & \mathbf{A}^1 & \end{tikzcd}$$

The theorem is that the functor

$$\begin{array}{cccccc} \mathscr{F} \longmapsto (& \mathscr{F}_{\mathrm{Y}}, & \mathscr{F}_{\mathrm{U}}, & \text{gluing datum}) \\ & \| & \| & \\ & \Psi_f^{\mathrm{un}}\mathscr{F} & \mathscr{F}_{\mathrm{U}} & \mathscr{F}_{\mathrm{Y}} \rightleftarrows \Psi_f^{\mathrm{un}}\mathscr{F}_{\mathrm{U}} \end{array}$$

is an equivalence of categories between perverse sheaves on X and perverse sheaves on U and Y with a gluing datum which consists of morphisms arising (up to a twist) from composing arrows coming from

$$\Psi^{\mathrm{un}} \xrightarrow{\mathrm{can}} \Phi^{\mathrm{un}} \xrightarrow{\mathrm{var}} \Psi^{\mathrm{un}}(-1)$$

where var denotes variation. Choosing a generator for $\mathbf{Z}_\ell(1)$, the composition of these arrows gives $1-t$ (c.f. [**SGA 7**, Exp. XIII 1.4]). Now for the definition of the unipotent nearby and vanishing cycles.

$$\begin{tikzcd} & & \overline{\mathrm{U}} \arrow[d, "\pi"] \arrow[r] & \tilde{\mathrm{U}} \arrow[dl, "\tilde{\pi}"] \\ \mathrm{Y} \arrow[r, hook, "i"] & \mathrm{X} & \mathrm{U} \arrow[l, hook', "j"'] & \end{tikzcd}$$

Here, $\overline{\mathrm{U}}$ is coming from the normalization of U in the separable closure of its generic point, while $\tilde{\mathrm{U}}$ is the normalization of U in the extension of its generic point corresponding to the pro-ℓ part, isomorphic to $\mathbf{Z}_\ell(1)$. (In fact, this is all done on $\mathbf{A}^1 - \{0\}$ and pulled back to U via f. The Galois group at the generic point of $\mathbf{A}^1$ of course has the structure of an extension of $\mathbf{Z}_\ell(1)$ by a pro-p group, and the maximal pro-ℓ quotient corresponds to an closed subgroup and hence to a Galois extension of the generic point. The normalization of $\mathbf{A}^1 - \{0\}$ in this extension gives $\tilde{\mathbf{A}}^1$: it is the $\mathbf{Z}_\ell(1)$-torsor corresponding to the Kummer sheaves, and its pullback to U is also a $\mathbf{Z}_\ell(1)$-torsor, i.e. 'logarithmic sheaf' since in the complex situation a choice of generator corresponds to a branch of the logarithm – see the section on Poincaré duality in Weil I.) The usual nearby cycles (we work with finite coefficients $\mathbf{F}_\ell = \mathbf{Z}/\ell$; this changes nothing) is defined by

$$\Psi(\mathcal{F}_\mathrm{U}) = i^* j_* \pi_* \pi^* \mathcal{F}_\mathrm{U} = \mathcal{F}_\mathrm{U} \otimes \pi_* \pi^* \mathbf{F}_\ell,$$

and we define the unramified nearby cycles

$$\Psi^{\mathrm{un}}(\mathcal{F}_\mathrm{U}) = i^* j_* (\mathcal{F}_\mathrm{U} \otimes \tilde{\pi}_* \tilde{\pi}^* \mathbf{F}_\ell).$$

Next define the Iwasawa algebra

$$\mathrm{R} := \mathbf{F}_\ell[[\mathbf{Z}_\ell(1)]] = \varprojlim \mathbf{F}_\ell[\mathbf{Z}_\ell/\ell^n(1)].$$

Choosing a generator t of $\mathbf{Z}_\ell(1)$ allows us to write an isomorphism

$$\mathrm{R} \simeq \mathbf{F}_\ell[[t-1]];$$

for the details see the note to 1.1. below. Inside of R is the maximal ideal $\mathfrak{m}$ corresponding after a choice of generator to $(1-t)$. It is an invertible module (zeros $\leftrightarrow$ poles). The Iwasawa twist of a sheaf is the functor

$$\mathbf{Z}_\ell(1)' : \mathcal{F} \longmapsto \mathcal{F} \otimes \mathfrak{m} =: \mathcal{F}(1)'$$

After a choice of generator, Iwasawa twist becomes Tate twist. Now recognize $\Psi^{\mathrm{un}}(\mathcal{F}_\mathrm{U})$ as a cone on the morphism $(j_! \to j_*)(\mathcal{F}_\mathrm{U} \otimes \tilde{\pi}_* \tilde{\pi}^* \mathcal{F}_\mathrm{U})$. As $\Psi^{\mathrm{un}}(\mathcal{F}_\mathrm{U})[-1]$ is a perverse sheaf, it is actually the kernel of the surjection $j_! \twoheadrightarrow j_*$. Note that $\mathrm{H}^0(\mathrm{U}, \tilde{\pi}_* \tilde{\pi}^* \mathbf{F}_\ell)$ can be described as continuous functions from U to $\mathbf{F}_\ell$. Among them are obviously the constant functions. Therefore inside $\tilde{\pi}_* \tilde{\pi}^* \mathbf{F}_\ell$ sits a copy of $\mathbf{F}_\ell$ and of course $\mathcal{F}_\mathrm{U} \otimes \mathbf{F}_\ell \simeq \mathcal{F}_\mathrm{U}$. Define the maximal extension Ξ as the kernel of the composition

$$j_!(\mathcal{F}_\mathrm{U} \otimes \tilde{\pi}_* \tilde{\pi}^* \mathbf{F}_\ell) \twoheadrightarrow j_*(\mathcal{F}_\mathrm{U} \otimes \tilde{\pi}_* \tilde{\pi}^* \mathbf{F}_\ell) \twoheadrightarrow j_*((\mathcal{F}_\mathrm{U} \otimes \tilde{\pi}_* \tilde{\pi}^* \mathbf{F}_\ell)/\mathcal{F}_\mathrm{U}),$$

where the quotient by $\mathcal{F}_{\mathrm{U}}$ refers to the constant functions described above. The second arrow remains an epimorphism as j_* is exact (middle perversity). Note that the restriction of Ξ to U is $\mathcal{F}_{\mathrm{U}}$. We have exact sequences

$$0 \to \Psi \to \Xi \to j_* \to 0$$

$$0 \to j_!\mathcal{F}_{\mathrm{U}} \to \Xi \to \Psi(-1)' \to 0.$$

Now we can state a more refined version of the original theorem. We have complexes of perverse sheaves

$$j_!\mathcal{F}_{\mathrm{U}} \longrightarrow \Xi\mathcal{F}_{\mathrm{U}} \oplus \mathcal{F} \longrightarrow j_*\mathcal{F}$$

$$\Psi \longrightarrow \Xi \oplus \Phi \longrightarrow \Psi(-1)'$$

(probably the Ψ and Φ should be unramified), both concentrated in degree zero. The H^0 of the first is Φ^{un} and of the second is $\mathcal{F}$.

Remark. About the lisse sheaves $\mathrm{I}^{a,b}$: Sasha said that they are not really used in the glueing construction and that they have something to do with a unipotent nearby cycles construction that is nice with respect to Verdier duality.

'Glueing' the glueing construction. 1/10/20 Discussed with Sasha what happens if you try to extend this construction to the global situation of the complement of a Cartier divisor D; if you could do this, you could build up the category of perverse sheaves combinatorially. If D is described by the vanishing of f_1 on U_1 and f_2 on U_2, the construction would glue on $\mathrm{U}_1 \cap \mathrm{U}_2$ if to the invertible function f_1/f_2 one could associate a canonical isomorphism of the construction for f_1 and the one for f_2 on $\mathrm{U}_1 \cap \mathrm{U}_2$; however, Sasha goes on to describe that isomorphisms between the two constructions are in bijection with choices of all ℓ-power roots of f_1/f_2, and evidently $\mathbf{Z}_\ell(1)$ acts on these choices. So $\mathbf{Z}_\ell(1)$ acts on the set of isomorphisms between the constructions on $\mathrm{U}_1 \cap \mathrm{U}_2$. Closely related is Verdier's notion of monodromic sheaf, which, if I'm remembering correctly, has the property that the sheaf and its translation by an element of $\mathbf{G}_{m,k}$ are (abstractly) isomorphic, but does not provide a canonical isomorphism.

It seems to me that the choice of these ℓ-power roots is in a way of trying to 'glue' the unipotent nearby cycles Ψ_f^{un}, which are by nature local. Because they are unipotent, they only depend on the tame nearby cycles Ψ^t_f, and glueing these should amount to something like extending the isomorphism of $U_1 \cap U_2$ onto itself to one between its universal tame covers. Therefore the crux of the issue seems to be the 'globalization of nearby cycles.'

Now we go into a more detailed analysis of the paper itself section-by-section, trying to adapt it to the étale language of Sasha's exposé.

1.1. Of course, $\mathbf{A}^1_{\mathbf{C}}$ is replaced by $\mathbf{A}^1_k$ with $\operatorname{char} k \neq \ell$ and the isomorphism $\mathbf{Z}(1) \simeq \pi_1((\mathbf{A}^1 - \{0\})(\mathbf{C}), 1)$ is replaced by the isomorphism $\hat{\mathbf{Z}}(1)(\overline{k}) \simeq \pi_1(\mathbf{G}_{m,\overline{k}}, 1)^{\mathrm{mod}}$, where the latter group is defined in [**Laumon**, 2.2.2] and corresponds to the étale coverings tamely ramified at 0 and ∞.

Likewise, we will only consider the completed Iwasawa algebra (A° in the article and R in the exposé) at the prime ℓ

$$A^\circ = \mathbf{F}_\ell(\mathbf{Z}_\ell(1)) := \varprojlim \mathbf{F}_\ell[\mathbf{Z}/\ell^n(1)]$$

where as always $\mathbf{Z}/\ell^n(1)$ denotes the group of ℓ^n roots of unity (of the field $\overline{k}$) equipped with action of $\operatorname{Gal}(\overline{k}/k)$, and $\mathbf{F}_\ell[\mathbf{Z}/\ell^n(1)]$ denotes the group algebra, where here $\mathbf{F}_\ell$ can be taken to be $\mathbf{Z}_\ell$ or $\mathbf{Z}/\ell^m$ for any m and we will still have the isomorphism

$$\mathbf{F}_\ell[[\tilde{t} - 1]] \xrightarrow{\sim} A^\circ$$

where t is a topological generator of $\mathbf{Z}_\ell(1)$. The proof that this is an isomorphism is the same of course as for the Iwasawa algebra of the ℓ-adic integers, and the best source for this is Serre, *Classes des Corps Cyclotomiques* §6 (it is simply a matter of noting that the polynomial $(1+T)^{\ell^n} - 1$ lies in $(\ell, T)^n \subset \mathbf{F}_\ell[[T]]$).

Let $\gamma := \tilde{t} - 1$. In order to see $\operatorname{Gr} A^\circ = \oplus_{i \geq 0} \mathbf{F}_\ell(i)$, we ask what is the action of Galois on A^i/A^{i+1}, which is free of rank 1 as $\mathbf{F}_\ell$-module. Galois acts via the cyclotomic character $\chi : \operatorname{Gal}(\overline{k}/k) \to \mathbf{Z}_\ell^\times$, where after fixing an isomorphism of $\mathbf{F}_\ell$-modules $\mathbf{F}_\ell(1) \simeq \mathbf{F}_\ell$ (which amounts to a choice of generator $t \mapsto 1$), the action $t \mapsto gt$ of an element

$g \in \mathrm{Gal}(\overline{k}/k)$ corresponds to the map $1 \mapsto \chi(g)$. In particular, $g\tilde{t} = \tilde{t}^{\chi(g)}$ and

$$\begin{aligned} g\gamma^n \mod \gamma^{n+1} &= g((\gamma+1)-1)^n \mod \gamma^{n+1} \\ &= ((\gamma+1)^{\chi(g)}-1)^n \mod \gamma^{n+1} = \chi(g)^n\gamma^n \mod \gamma^{n+1}. \end{aligned}$$

This shows that $\mathrm{A}^i/\mathrm{A}^{i+1} \simeq \mathbf{F}_\ell(i)$. As $\mathrm{A}^\circ \simeq \mathbf{F}_\ell[[\gamma]]$, $\mathrm{A} = \mathrm{A}^\circ_{(\tilde{t}-1)} = \mathbf{F}_\ell((\gamma))$ and we find

$$\begin{aligned} g\gamma^{-n} \mod \gamma^{1-n} &= g((\gamma+1)-1)^{-n} \mod \gamma^{1-n} = ((\gamma+1)^{\chi(g)}-1)^{-n} \mod \gamma^{1-n} \\ &= \left((\chi(g)\gamma)\left(1+\binom{\chi(g)}{2}\gamma+\dots\right)\right)^{-n} \mod \gamma^{1-n} = \chi(g)^{-n}\gamma^{-n} \mod \gamma^{1-n}. \end{aligned}$$

This shows that $\mathrm{Gr}\,\mathrm{A} = \oplus_{i\in\mathbf{Z}}\mathbf{F}_\ell(i)$. These statements have been made in terms of a particular generator t but they are independent of the choice of generator; i.e. these isomorphisms are canonical.

Now we come to the pairing $\langle\ ,\ \rangle : \mathrm{A}\times\mathrm{A} \to \mathbf{F}_\ell(-1)$. Skew-symmetry: put

$$a := fg^- = \frac{a_n}{\gamma^n} + \cdots + \frac{a_1}{\gamma} + \cdots .$$

As $gf^- = (fg^-)^-$, it amounts to showing that

$$\mathrm{res}_{\gamma=0}\left(a^-\frac{d\gamma}{\tilde{t}}\right) = -\,\mathrm{res}_{\gamma=0}\left(a\frac{d\gamma}{\tilde{t}}\right).$$

Note that

$$\gamma^- = -\frac{\gamma}{\gamma+1} = -\frac{\gamma}{\tilde{t}} \quad \text{and} \quad \frac{1}{\tilde{t}} = 1-\gamma+\gamma^2-\gamma^3+\cdots$$

so that

$$\begin{aligned} \mathrm{res}_{\gamma=0}\left(a\frac{d\gamma}{\tilde{t}}\right) &= \mathrm{res}_{\gamma=0}\left(\left(\frac{a_n}{\gamma^n}+\cdots+\frac{a_1}{\gamma}+\cdots\right)\left(1-\gamma+\gamma^2-\gamma^3+\cdots\right)d\gamma\right) \\ &= a_1+\cdots+(-1)^{n-1}a_n, \\ \mathrm{res}_{\gamma=0}\left(a^-\frac{d\gamma}{\tilde{t}}\right) &= \mathrm{res}_{\gamma=0}\left(\left(\frac{(-1)^n a_n\tilde{t}^n}{\gamma^n}+\cdots+\frac{(-a_1\tilde{t})}{\gamma}+\cdots\right)\frac{d\gamma}{\tilde{t}}\right) \\ &= \frac{(-1)^n a_n\tilde{t}^{n-1}}{\gamma^n}+\cdots+\frac{(-a_1)}{\gamma}+\cdots \\ &= -a_1+\cdots+(-1)^n a_n. \end{aligned}$$

$\mathbf{Z}_\ell(1)$-invariance is simply the statement that $\langle \tilde{t}^i f, \tilde{t}^i g\rangle = \langle f, g\rangle$, $i \in \mathbf{Z}_\ell$, and is obvious. Likewise $(\mathrm{A}^i)^\perp = \mathrm{A}^{-i}$ is obvious. The induced pairing on Gr A should read

$$\langle \mathrm{S}_i, \mathrm{S}_{-i-1}\rangle = (-1)^{i+1}\mathrm{S}_i \cdot \mathrm{S}_{-i-1}, \qquad \mathrm{S}_j \in \mathbf{F}_\ell(j)$$

because as a pairing $\mathrm{Gr}\,\mathrm{A} \times \mathrm{Gr}\,\mathrm{A} \to \mathbf{F}_\ell(-1)$ it can be described simply as the piece of

$$(\overline{f}, \overline{g}) \mapsto (\overline{f}, \mathrm{Gr}(-)(\overline{g})) \mapsto \overline{f}\,\mathrm{Gr}(-)(\overline{g}) \mapsto \mathrm{Gr}\left(\times\frac{1}{\tilde{t}}\right)(\overline{f}\,\mathrm{Gr}(-)(\overline{g})) = \overline{f}\,\mathrm{Gr}(-)(\overline{g})$$

in $\mathrm{Gr}_{-1}\mathrm{A}$. Some explanation: the involution map $a \mapsto a^-$ respects the filtration, as $\gamma^- = -\gamma/\tilde{t} = -\gamma(1-\gamma+\cdots)$. Therefore it descends to a morphism $\mathrm{Gr}(-) : \mathrm{Gr}\,\mathrm{A} \to \mathrm{Gr}\,\mathrm{A}$ determined by $\gamma \mapsto -\gamma$. Similarly, multiplication by $1/\tilde{t}$ respects the filtration on A and descends to the identity map $\mathrm{Gr}\,\mathrm{A} \to \mathrm{Gr}\,\mathrm{A}$.

To obtain the isomorphism

$$\mathrm{A}^a/\mathrm{A}^b \xrightarrow{\sim} \mathrm{Hom}(\mathrm{A}^{-b}/\mathrm{A}^{-a}, \mathbf{F}_\ell(-1)),$$

simply induct on $b-a$, the case $b-a=1$ just obtained, using the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{A}^{b-1}/\mathrm{A}^b & \longrightarrow & \mathrm{A}^a/\mathrm{A}^b & \longrightarrow & \mathrm{A}^a/\mathrm{A}^{b-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathrm{A}^{-b}/\mathrm{A}^{-b+1}, \mathbf{F}_\ell(-1)) & \longrightarrow & \mathrm{Hom}(\mathrm{A}^{-b}/\mathrm{A}^{-a}, \mathbf{F}_\ell(-1)) & \longrightarrow & \mathrm{Hom}(\mathrm{A}^{-b+1}/\mathrm{A}^{-a}, \mathbf{F}_\ell(-1)) & \longrightarrow & 0
\end{array}$$

whose rows are exact as $\mathrm{A}^a/\mathrm{A}^b$ is free as $\mathbf{F}_\ell$-module.

Finally, we should verify that the pairing is $\mathrm{Gal}(\overline{k}, k)$-equivariant; i.e. that for $\sigma \in \mathrm{Gal}(\overline{k}, k)$ we have $\langle \sigma f, \sigma g\rangle = \sigma\langle f, g\rangle = \chi(\sigma)^{-1}\langle f, g\rangle$. The point is that as $\chi(g) \in \mathbf{Z}_\ell^\times$, t generates $\mathbf{Z}_\ell(1)$ iff $\chi(g)t$ does, and

$$d\log(\widetilde{\sigma^{-1}t}) = d\log(\tilde{t}^{1/\chi(\sigma)}) = \chi(\sigma)^{-1} d\log\tilde{t}, \qquad \text{so that indeed}$$
$$(\langle \sigma f, \sigma g\rangle, t) = (\langle f, g\rangle, \sigma^{-1}t) = (\langle f, g\rangle, t/\chi(\sigma)) = (\chi(\sigma)^{-1}\langle f, g\rangle, t).$$

We proceed to define the sheaf I (and sheaves $\mathrm{I}^{a,b}$) of A-modules on $\mathbf{G}_{m,k}$ by endowing the Iwasawa algebra A with an action of $\pi_1(\mathbf{G}_{m,k}, 1)$, which acts via the quotient by the kernel of

$$\Gamma : \pi_1(\mathbf{G}_{m,\overline{k}}, 1) \twoheadrightarrow \pi_1(\mathbf{G}_{m,\overline{k}}, 1)^{\mathrm{mod}} \simeq \hat{\mathbf{Z}}(1)(\overline{k}) \twoheadrightarrow \mathbf{Z}_\ell(1),$$

so that $\pi_1(\mathbf{G}_{m,k}, 1)$ acts on I via $\mathrm{G} := \pi_1(\mathbf{G}_{m,k}, 1)/\ker \Gamma$; as $\pi_1(\mathbf{G}_{m,k}, 1)$ can be described as the canonically split extension

$$1 \to \pi_1(\mathbf{G}_{m,\bar{k}}, 1) \to \pi_1(\mathbf{G}_{m,k}, 1) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

G can be described as the canonically split extension [**SGA1**, Exp. IX 6.1]

$$0 \to \mathbf{Z}_\ell(1) \to \mathrm{G} \to \mathrm{Gal}(\bar{k}/k) \to 1. \tag{†}$$

The action of $l \in \mathbf{Z}_\ell(1)$ on $\mathrm{A} \simeq \mathbf{F}_\ell[\mathbf{Z}_\ell(1)]$ is by multiplication by $\tilde{l} \in \mathrm{A}$; the action of $\mathrm{Gal}(\bar{k}, k)$ is obvious.

'Coincides on I_1 with the above $\langle\ ,\ \rangle$': $\mathrm{I}_1 = \mathrm{A}$ the stalk of I at 1.

A3. Generalities on lim**.** In the below we use the words 'exact pair,' 'short exact sequence,' and 'conflation' interchangeably. Let $(\mathscr{A}, \mathscr{E})$ be an exact category with exact structure specified by class $\mathscr{E}$ of exact pairs. $\mathbf{Z} \times \mathbf{Z}$ has the product partial order $(a, b) \leq (c, d) \Leftrightarrow a \leq c$ and $b \leq d$. Let's verify the axioms of exact category (following Keller's notes *Derived categories and their uses*) for $\mathscr{A}^\Pi$. We notate an object of $\mathscr{A}^\Pi$ by $\underline{\mathrm{X}}_{ij}$. First note that the class $\mathscr{E}^\Pi$ of exact pairs of $\mathscr{A}^\Pi$ are the pairs $\underline{\mathrm{X}}_{ij} \to \underline{\mathrm{Y}}_{ij} \to \underline{\mathrm{Z}}_{ij}$ such that the pair in $\mathscr{A}$ corresponding to (i, j) is in $\mathscr{E}$ for all $(i, j) \in \Pi$. We verify that the composition of inflations is an inflation; this is true if given $\underline{\mathrm{X}}_{ij}, \underline{\mathrm{Y}}_{ij}$ in $\mathscr{A}^\Pi$ and inflation $\underline{\mathrm{X}}_{ij} \to \underline{\mathrm{Y}}_{ij}$, we can find $\underline{\mathrm{Z}}_{ij}$ in $\mathscr{A}^\Pi$ making an exact pair $\underline{\mathrm{X}}_{ij} \to \underline{\mathrm{Y}}_{ij} \to \underline{\mathrm{Z}}_{ij}$. We can find a Z_{ij} 'pointwise' for each $(i, j) \in \Pi$ from the exact structure $\mathscr{E}$, and we need to join these 'points' to a natural transformation. This is done in a unique way from the universal property of cokernel and achieves $\mathrm{Ex1}^{\mathrm{op}}$. Dually we can complete a diagram

$$\begin{tikzcd} & \mathrm{C}'_{ij} \arrow[d, "c_{ij}"] \\ \mathrm{B}_{ij} \arrow[r, "p_{ij}"] & \mathrm{C}_{ij} \end{tikzcd}$$

where p_{ij} comes from a deflation $p : \underline{\mathrm{B}}_{ij} \to \underline{\mathrm{C}}_{ij}$ in $\mathscr{E}^\Pi$ to a square

$$\begin{array}{ccccc} \mathrm{A}'_{ij} & \xrightarrow{i'_{ij}} & \mathrm{B}'_{ij} & \xrightarrow{p'_{ij}} & \mathrm{C}'_{ij} \\ & & \downarrow{\scriptstyle b_{ij}} & & \downarrow{\scriptstyle c_{ij}} \\ \mathrm{A}_{ij} & \xrightarrow{i_{ij}} & \mathrm{B}_{ij} & \xrightarrow{p_{ij}} & \mathrm{C}_{ij} \end{array}$$

cartesian in $\mathscr{A}$ where (i'_{ij}, p'_{ij}) is exact in $\mathscr{E}$ and (i_{ij}, p_{ij}) is exact in $\mathscr{E}$ coming from an exact pair (i, p) in $\mathscr{E}^\Pi$. There is a unique way to make the B'_{ij} into an object $\underline{\mathrm{B}}'_{ij}$ of $\mathscr{A}^\Pi$ compatible with the status of B'_{ij} as a limit of the first diagram for all $(i, j) \in \Pi$; once this is done there is a unique way to make the A'_{ij} into an object $\underline{\mathrm{A}}'_{ij}$ of $\mathscr{A}^\Pi$ from the universal property that i'_{ij} is a kernel of p'_{ij} for all $(i, j) \in \Pi$. Finally we verify that $\underline{\mathrm{B}}'_{ij}$ is a limit of the first diagram in $\mathscr{A}^\Pi$ since its morphisms for $(i, j) \leq (i', j')$ were obtained from the universal property of limit of $\mathrm{B}'_{i'j'}$. This secures Ex2.

Now we pass to the definition and properties of admissible objects. Typo: admissible objects are obviously objects of $\mathscr{A}^\Pi$, not $\mathscr{A}$. The full subcategory $\mathscr{A}^\Pi_a \subset \mathscr{A}^\Pi$ would inherit an exact structure from the $\mathscr{E}^\Pi$ on $\mathscr{A}^\Pi$ if it were closed under extensions in the sense that the existence of a short exact sequence $\underline{\mathrm{X}}_{ij} \hookrightarrow \underline{\mathrm{Y}}_{ij} \twoheadrightarrow \underline{\mathrm{Z}}_{ij}$ in $\mathscr{E}^\Pi$ with $\underline{\mathrm{X}}_{ij}$ and $\underline{\mathrm{Z}}_{ij}$ admissible would imply $\underline{\mathrm{Y}}_{ij}$ admissible [**Bühler**, 10.20]. This would be secured if the statement 'in any short exact sequence in $\mathscr{A}^\Pi$ if any two objects are admissible then the third is.' The 3×3 lemma [**Bühler**, 3.6] gives precisely this, modulo the additional condition that to be able to conclude that $\underline{\mathrm{Y}}_{ij}$ is admissible from the admissibility of $\underline{\mathrm{X}}_{ij}$ and $\underline{\mathrm{Z}}_{ij}$, one must secure that for every $i \leq j \leq k$ the sequence $\mathrm{Y}_{ij} \to \mathrm{Y}_{ik} \to \mathrm{Y}_{jk}$ is a complex. Writing the composition as $\mathrm{Y}_{ij} \to \mathrm{Y}_{jj} \to \mathrm{Y}_{jk}$, we see that the composition is zero if Y_{jj} is a zero object of $\mathscr{A}$. We will show that X_{jj} and Z_{jj} are zero objects of $\mathscr{A}$. Suppose $\underline{\mathrm{X}}_{ij}$ is an admissible object of $\mathscr{A}^\Pi$. Let $i = j = k$; we find that $\mathrm{X}_{jj} \xrightarrow{\mathrm{id}} \mathrm{X}_{jj} \xrightarrow{\mathrm{id}} \mathrm{X}_{jj}$ is short exact; in particular is a complex. Therefore X_{jj} is a zero object for all $j \in \mathbf{Z}$. We therefore have an exact pair $\mathrm{X}_{jj} \to \mathrm{Y}_{jj} \to \mathrm{Z}_{jj}$ with X_{jj} and Z_{jj} zero objects of $\mathscr{A}$ so that both arrows in the exact pair are zero. In any additive category, of course the identity is a kernel for any zero morphism. As kernels are universal, $\mathrm{X}_{jj} \simeq \mathrm{Y}_{jj}$ and we have shown Y_{jj} is a zero object, and therefore that any extension of admissible objects is admissible. (Note that a consequence of the admissibility criterion is that all maps $X_{ij} \to X_{ik}$, $k \geq j$, are monomorphisms, and all maps $X_{ik} \to X_{jk}$, $j \geq i$, are epimorphisms.)

$\mathscr{A}$ embeds in $\mathscr{A}_a^\Pi$ via the functor which to $X \in \operatorname{ob} \mathscr{A}$ associates the Π-object $\underline{X}_{ij}$ with

$$X_{ij} = \begin{cases} X & i \leq -1, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and all arrows between $X \to X$ the identity, all other arrows 0. (There is a similar embedding for every $i < j$.) 'If $\varphi \leq \psi$ i.e. $\varphi(i) \leq \psi(i)$ for all i,' not 'for any i.' A Π-object of $\mathscr{A}$ is just a functor $F : \Pi \to \mathscr{A}$, and $\tilde{\varphi}(F)$ is the functor $\Pi \xrightarrow{\varphi\times\varphi} \Pi \xrightarrow{F} \mathscr{A}$.

The way $\underset{\leftrightarrow}{\lim}$ is defined doesn't work, but the simpler, obvious way to define it works flawlessly. The fewest conditions on the functions φ necessary to make the stated arrows into a multiplicative system are probably that (i) $\varphi : \mathbf{Z} \to \mathbf{Z}$ be surjective; and (ii) the function $|\varphi(i) - i|$ have an upper bound. But who cares! The only functions we need to consider in our localization are the shift functions $\varphi_N(i) = i + N$, $N \in \mathbf{Z}$.

Let's verify the axioms of multiplicative system for the system of morphisms $\tilde{\varphi}_N(X) \to \tilde{\varphi}_M(X)$, $X \in \operatorname{ob} \mathscr{A}_a^\Pi$, $N \leq M$ which arise from the obvious morphism of functors $\tilde{\varphi}_N \to \tilde{\varphi}_M$. (To be clear, this morphism of functors at an object X is defined by the arrows $X_{i+N,j+N} \to X_{i+M,j+M}$ which are part of the data of X as Π-object.) If N is a natural number and X in $\mathscr{A}_a^\Pi$, write $X_N := \tilde{\varphi}_N(X)$. Composition is trivial: $X_A \to X_B = Y_C \to Y_D$ implies $Y = X_{B-C}$ so the composition coincides with $X_A \to X_{B+D-C}$. The following diagrams verify the remaining conditions of a multiplicative system when $A \leq B$ are integers.

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & Y \arrow[d] \\ X_B \arrow[r] & Y_{B-A} \end{tikzcd} \qquad \begin{tikzcd} X_{A-B} \arrow[r] \arrow[d] & Y_A \arrow[d] \\ X \arrow[r] & Y_B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & Y \arrow[d] \\ X_{B-A} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y_{B-A} \end{tikzcd}$$

We localize $\mathscr{A}$ with respect to this multiplicative system to obtain the category $\underset{\leftrightarrow}{\lim} \mathscr{A}$. The functor $\underset{\leftrightarrow}{\lim} : \mathscr{A}_a^\Pi \to \underset{\leftrightarrow}{\lim} \mathscr{A}$ is surjective on (isomorphism classes) of objects. A morphism $\lim\limits_{\leftrightarrow} \underline{X} \to \lim\limits_{\leftrightarrow} \underline{Y}$ is represented by a diagram

$$\begin{tikzcd} & Y_A & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

where $A \geq 0$ and $Y \to Y_A$ is the morphism in the multiplicative system. Two such morphisms $f : X \to Y_A$ and $g : X \to Y_B$ agree if, supposing $A \leq B$, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y_A \arrow[r] \arrow[d] & Y_C \arrow[d, equal] \\ X \arrow[r, "g"] & Y_B \arrow[r] & Y_C \end{tikzcd}$$

commutes; this is true iff g coincides with the composition $X \xrightarrow{f} Y_A \to Y_B$.

The exact structure on $\mathscr{A}_a^\Pi$ induces one on $\lim\limits_{\leftrightarrow} \mathscr{A}$ as the localization functor $\lim\limits_{\leftrightarrow} : \mathscr{A}_a^\Pi \to \lim\limits_{\leftrightarrow} \mathscr{A}$ preserves finite limits & colimits, and if $\alpha : I \to \mathscr{A}_a^\Pi$ is an inductive or projective system in $\mathscr{A}_a^\Pi$ indexed by a finite category I such that $\varinjlim \alpha$ (resp. $\varprojlim \alpha$) exists in $\mathscr{A}_a^\Pi$, then $\varinjlim(\lim\limits_{\leftrightarrow} \circ \alpha)$ (resp. $\varprojlim(\lim\limits_{\leftrightarrow} \circ \alpha)$) exists in $\lim\limits_{\leftrightarrow} \mathscr{A}$ and is isomorphic to $\lim\limits_{\leftrightarrow}(\varinjlim \alpha)$ (resp. $\lim\limits_{\leftrightarrow}(\varprojlim \alpha)$; in particular, $\lim\limits_{\leftrightarrow}$ commutes with kernels, cokernels, finite products and finite coproducts (c.f. Kashiwara & Schapira, *Categories and Sheaves* (7.1.22)). In other words, $\lim\limits_{\leftrightarrow} : \mathscr{A}_a^\Pi \to \lim\limits_{\leftrightarrow} \mathscr{A}$ is exact and hence composes with the faithful exact embedding $\mathscr{A} \hookrightarrow \mathscr{A}_a^\Pi$ to give a faithful exact embedding $\lim\limits_{\leftrightarrow} : \mathscr{A} \hookrightarrow \lim\limits_{\leftrightarrow} \mathscr{A}$.

We will use the $\lim\limits_{\leftrightarrow}$ construction in a particular way in what follows. Let $\mathrm{Fil}(\mathscr{A})$ denote the filtered category of $\mathscr{A}$ with objects sequences

$$\cdots \to B^p \xrightarrow{j^p} B^{p-1} \to \cdots \qquad p \in \mathbf{Z}$$

of admissible monomorphisms in $\mathscr{A}$ such that $B^p = 0$ for $p \ll 0$ and $\operatorname{coker} j^p = 0$ for all $p \gg 0$. Morphisms in $\mathrm{Fil}(\mathscr{A})$ are those compatible with the filtration and correspond bijectively to sequences $f^p \in \mathrm{Hom}_{\mathscr{A}}(B_1^p, B_2^p)$ such that $f^{p-1} j_1^p = j_2^p f^p$ for all $p \in \mathbf{Z}$. The componentwise short exact sequences form an exact structure on $\mathrm{Fil}(\mathscr{A})$. Given a sequence

$$\cdots \to \mathrm{A}^p \to \mathrm{A}^{p-1} \to \cdots \qquad p \in \mathbf{Z}$$

of admissible monomorphisms in $\mathscr{A}$ (with no finiteness condition), let $\mathrm{A}^{a,b} := \operatorname{coker}(\mathrm{A}^b \to \mathrm{A}^a)$ whenever $b \geq a$, considered as object of $\mathrm{Fil}(\mathscr{A})$. Define the admissible Π-object $\underline{\mathrm{X}}$ in $\mathrm{Fil}(\mathscr{A})^{\Pi}_a$ by $\mathrm{X}_{ij} := \mathrm{A}^{-j,-i}$. When $(i_1, j_1) \leq (i_2, j_2)$ with both in Π we have for transition morphism $\mathrm{X}_{i_1 j_1} \to \mathrm{X}_{i_2 j_2}$ the canonical

$$\operatorname{coker}(\mathrm{A}^{-i_1} \to \mathrm{A}^{-j_1}) \to \operatorname{coker}(\mathrm{A}^{-i_2} \to \mathrm{A}^{-j_2}),$$

and $\underline{\mathrm{X}}$ is admissible in view of exact sequences ($i \leq j \leq k$)

$$\begin{array}{ccccc} \mathrm{X}_{ij} & \hookrightarrow & \mathrm{X}_{ik} & \twoheadrightarrow & \mathrm{X}_{jk} \\ \| & & \| & & \| \\ \mathrm{A}^{-j,-i} & \hookrightarrow & \mathrm{A}^{-k,-i} & \twoheadrightarrow & \mathrm{A}^{-k,-j}. \end{array}$$

Let $\mathrm{AM}(\mathscr{A})$ denote the subcategory of A with objects the objects of $\mathscr{A}$ and morphisms the admissible monomorphisms in $\mathscr{A}$. Let $\mathrm{Fil}_{\mathbf{Z}}(\mathscr{A})$ denote the category with objects functors $\mathbf{Z} \to \mathrm{AM}(\mathscr{A})$ and morphisms natural transformations of functors considered as functors $\mathbf{Z} \to \mathscr{A}$ ($\mathrm{Fil}_{\mathbf{Z}}(\mathscr{A})$ is just $\mathrm{Fil}(\mathrm{A})$ with no finiteness hypothesis). The above construction gives a functor

$$\mathrm{T} : \mathrm{Fil}_{\mathbf{Z}}(\mathscr{A}) \to \mathrm{Fil}(\mathscr{A})^{\Pi}_a.$$

1.2. Recall that $\mathrm{A}^{\circ} = \mathbf{F}_{\ell}[\mathbf{Z}_{\ell}(1)]$ with fraction field A; after choosing a generator t for $\mathbf{Z}_{\ell}(1)$, A° coincides with $\mathbf{F}_{\ell}[[\tilde{t} - 1]]$ while A coincides with $\mathbf{F}_{\ell}((\tilde{t} - 1))$. As before, $\mathbf{F}_{\ell}$ may coincide with $\mathbf{Z}/\ell^n$ or $\mathbf{Z}_{\ell}$, and we have $\mathrm{A}^i := (\tilde{t} - 1)^i \mathrm{A}^{\circ}$. It's time to 'see' the $\underset{\leftrightarrow}{\lim}$ construction in the construction of an étale avatar $\mathrm{A}_{\text{ét}}$ of A. The point is that A as an étale sheaf on $\operatorname{Spec} k$ is far from constructible. Just as we obtain ℓ-adic sheaves as projective systems of compatible constructible $\mathbf{Z}/\ell^n$-sheaves, we would like to obtain Laurent series as a suitable system of constructible (or ℓ-adic) sheaves. This is what the $\underset{\leftrightarrow}{\lim}$ construction does.

Let $\mathscr{P}_k$ denote the category of $\mathbf{F}_{\ell}$-sheaves on $\operatorname{Spec} k$. We would like to define an object $\underset{\leftrightarrow}{\lim}\,\mathrm{A}$ of $\underset{\leftrightarrow}{\lim}\,\mathscr{P}_k$ that behaves like the algebra A with action of Galois. Consider A with its $\mathbf{Z}$-filtration as an object of $\mathrm{Fil}_{\mathbf{Z}}(\mathscr{P}_k)$ and let $\mathrm{A}_{\text{ét}} := \mathrm{T}(\mathrm{A})$, where T is the functor defined at the end of the note to 1.1. To see why $\mathrm{A}_{\text{ét}}$ is a good avatar of the field A of Laurent series, observe that

$$\mathrm{Hom}_{\underset{\leftrightarrow}{\lim}\,\mathscr{P}_{\overline{k}}}(\mathbf{F}_\ell, \mathrm{A}_{\text{ét}}) = \mathrm{A}$$

with Galois acting on the left by transport of structure. Here the constant sheaf $\mathbf{F}_\ell$ in $\mathscr{P}_{\overline{k}}$ is considered embedded via the embedding discussed in the note to A3, which is the particular case $i = 0$ of the system of embeddings $\mathscr{P} \hookrightarrow \underset{\leftrightarrow}{\lim}\,\mathscr{P}$ pictured in Figure 1 (where (i, j) refers to the i^{th} row and j^{th} column). For, it is easily seen that

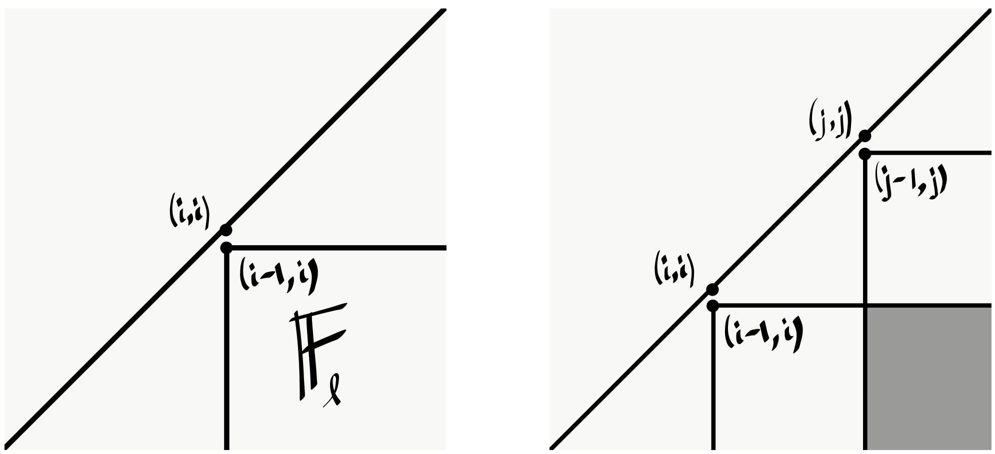

FIGURE 1.

$\mathrm{Hom}_{\underset{\leftrightarrow}{\lim}\,\mathscr{P}_{\overline{k}}}(\mathbf{F}_\ell, \mathrm{A}_{\text{ét}})$ is in bijection with the data of a choice of $i \in \mathbf{Z}$ and a morphism in $\mathrm{Fil}(\mathscr{P}_{\overline{k}})^{\Pi}_{a}$ from $\mathbf{F}_\ell$, embedded as in the left picture in Figure 1, to $\mathrm{A}_{\text{ét}}$, modulo the relation that two such morphisms are identified if they agree on their nontrivial overlap, as indicated in the right picture in Figure 1. If one associates to each nonzero Laurent series $p \in \mathrm{A}$ its initial form $\mathrm{in}(p) \in \mathrm{A}^{a,a+1}$, there is a unique morphism in $\mathrm{Hom}_{\underset{\leftrightarrow}{\lim}\,\mathscr{P}_{\overline{k}}}(\mathbf{F}_\ell, \mathrm{A}_{\text{ét}})$ with $\mathbf{F}_\ell$ embedded as in Figure 1 with $i = a$ corresponding to p. One recovers A^a via the morphisms that can be represented by a morphism in $\mathrm{Fil}(\mathscr{P}_{\overline{k}})^{\Pi}_{a}$ with $\mathbf{F}_\ell$ embedded in $i \leq -a$. Galois acts continuously on $\mathrm{Hom}_{\underset{\leftrightarrow}{\lim}\,\mathscr{P}_{\overline{k}}}(\mathbf{F}_\ell, \mathrm{A}_{\text{ét}}) = \mathrm{A}$ for the topology induced by the filtration.

I do not see a way to make $A_{\text{ét}}$ into a ring, or a ring object. The issue is of course that if we were dealing with an 'étalization' of A°, the multiplication would be easy to define as multiplication respects the filtration. Once we pass to fractions, however, the multiplication of two fractions no longer respects the topology. As I understand it, $A_{\text{ét}}$ should be understood in analogy with the adic formalism. The adic formalism, however, proceeds from the notion of A- linear category, where A is a commutative ring so that every object of the category has endomorphism ring an A-algebra. To define the category of $\mathbf{Q}_\ell$-sheaves, one formally inverts the endomorphism 'multiplication by ℓ' in the $\mathbf{Z}_\ell$-category. In our case, the algebra A (or, more precisely, A°) is not just a commutative ring, but a sheaf, a commutative ring with action of Galois. I have no doubt that A° can be 'étalized' in a very correct way, but it cannot be done directly from the formalism of SGA 5 for this reason. The matter is essentially this: the adic formalism in SGA 5 is enough to treat projective systems of étale sheaves which are modules for a *constant sheaf of rings* $\mathbf{Z}_\ell$, which does not really enter the picture directly as a sheaf of rings, but only via the commutative A in the notion of A-linear category. Of course, the notion of ringed topos is foundational, and so the way it is done in the adic formalism is not really satisfactory. I wonder if the pro-étale topos allows one to satisfactorily define an 'ℓ-adic sheaf of rings' which needn't be constant. Our 'étalization' of A° should be such an 'ℓ-adic sheaf of rings.'

I don't think the $\underset{\leftrightarrow}{\lim}$ formalism is up to the job of working 'ℓ-adically' for a 'nonconstant adic sheaf of rings' A, because I don't see how to make $A_{\text{ét}}$ into a ring object, and moreover I don't see how to have it act on $I_{\text{ét}}$. However, I'm not sure we actually need this. In the key lemma, forget the notion of π^{-1} and just notice that if ker and coker are annihilated by a power of π independent of a, b, say N, then both are killed by the map $\mathrm{id} \to \tilde{\varphi}_N$, which is an isomorphism in $\underset{\leftrightarrow}{\lim}$.

1.4. Given $\pi \in A^{1*}$, let $\tilde{t} := \exp \pi$ and $\gamma := \tilde{t} - 1$ so that $A \simeq \mathbf{F}_\ell[[\gamma]]$ and $\pi = \log \tilde{t}$. (As remarked, if t is a generator of the sheaf $\mathbf{Z}_\ell(1)$, $\pi = \log t$ is a logical choice.) The sheaf $I^{a,b}$ is defined by the action of the group G defined in the note to (1.1), and in view of the canonically split extension (†), it will suffice to show that if $l \in \mathbf{Z}_\ell(1), m \in \mathrm{Gal}(\overline{k}/k)$, $lm\sigma_\pi(x) = \sigma_\pi(lmx)$. Supposing $m = 1$, the statement is clear as l acts trivially on $\mathbf{Z}_\ell(-n)$ (which is the reciprocal image of a sheaf on Spec k) and of course $\tilde{l}$ commutes with π^n

as both are elements of the Iwasawa algebra A. In general, m acts by multiplication by $\chi(m)^{-n}$ on $\overline{\pi}^{-n}$ and sends $\tilde{t} \mapsto \tilde{t}^{\chi(m)}$ so $m\pi = \log(\tilde{t}^{\chi(m)}) = \chi(m)\pi$.

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11. Morel's notes

0. Introduction. We use the notation $\mathbf{G}_{m,k} \overset{j}{\hookrightarrow} \mathbf{A}^1_k \overset{i}{\hookleftarrow} \{0\}$. The nearby cycles functor Ψ_f is relative to the morphism $f : \mathrm{X} \to \mathbf{A}^1_k$; let $\mathrm{Y} := f^{-1}(0)$ and $\mathrm{U} := \mathrm{X}-\mathrm{Y}$. We use the setup of (2.2.2) in Laumon's Fourier transform article: write $\mathbf{A}^1_k = \mathrm{Spec}(k[u])$ with generic point $\eta = \mathrm{Spec}(k(u))$ and Zariski trait $(\mathbf{A}^1_k)_0 := \mathrm{Spec}(k[u]_{(u)})$ at 0; let $k\{u\}$ denote the henselization of the local ring $k[u]_{(u)}$, let $(\mathbf{A}^1_k)_{(0)} := \mathrm{Spec}(k\{u\})$ with generic point $\xi = \mathrm{Spec}(k\{u\}[u^{-1}])$, so that the inclusion $k[u, u^{-1}] \hookrightarrow k\{u\}[u^{-1}]$ induces a morphism of schemes $\iota : \xi \to \mathbf{G}_{m,k}$. We also use i, j to denote the inclusion of the closed and generic points to $(\mathbf{A}^1_k)_{(0)}$.

Fix a geometric point $\rho : \overline{\xi} \to \xi$, which via $\xi \to \eta$ we also consider to be centered on η. Given an object K of $\mathrm{D}^b_c(\mathrm{X}_\xi)$, $\mathrm{Gal}(\overline{\xi}, \xi)$ acts on the nearby cycles $\mathrm{R}\Psi_\xi(\mathrm{K})[-1]$ of [**SGA 7**, Exp. XIII] relative to the morphism f (abusively letting ι also denote the inclusion of ξ into $\mathbf{A}^1_k$). (See the note to BBD Appendix A for a detailed comparison of two ways of writing $\mathrm{R}\Psi_\xi$.)

Let S denote $(\mathbf{A}^1_k)_{(0)}$ with closed point 0. Let $\tilde{\xi}$ denote the maximal tamely ramified extension of ξ and let $\overline{\mathbf{A}}^1_k, \overline{\mathbf{G}}_{m,k}, \overline{\mathrm{S}}$ denote the normalisations of $\mathbf{A}^1_k, \mathbf{G}_{m,k}$, and S, respectively, in $\overline{\xi}$. Likewise for tildes. Note that $\tilde{0}$ is the spectrum of the separable closure of k and $\tilde{\mathrm{S}}$ is the spectrum of a valuation ring with value group $\mathbf{Z}_{(p)}$ (the localization of $\mathbf{Z}$ at the prime (p)) while $\overline{\mathrm{S}}$ is the spectrum of a valuation ring with value group $\mathbf{Q}$ and with residue field a purely inseparable extension of $k(\tilde{s})$. The commutative diagram on the right is obtained by localising the one on the left near 0.

$$\begin{tikzcd}
\bar{0} \arrow[r, "\bar{i}"] \arrow[d] & \overline{\mathbf{A}}^1_k \arrow[d] & \overline{\mathbf{G}}_{m,k} \arrow[l, "\bar{j}"'] \arrow[d] \\
\tilde{0} \arrow[r, "\tilde{i}"] \arrow[d] & \tilde{\mathbf{A}}^1_k \arrow[d, "\tilde{\pi}"] & \tilde{\mathbf{G}}_{m,k} \arrow[l, "\tilde{j}"'] \arrow[d, "\tilde{\pi}"] \\
0 \arrow[r, "i"] & \mathbf{A}^1_k & \mathbf{G}_{m,k}. \arrow[l, "j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{0} \arrow[r, "\bar{i}"] \arrow[d] & \overline{\mathrm{S}} \arrow[d] & \overline{\xi} \arrow[l, "\bar{j}"'] \arrow[d] \\
\tilde{0} \arrow[r, "\tilde{i}"] \arrow[d] & \tilde{\mathrm{S}} \arrow[d, "\tilde{\pi}"] & \tilde{\xi} \arrow[l, "\tilde{j}"'] \arrow[d] \\
0 \arrow[r, "i"] & \mathrm{S} & \xi. \arrow[l, "j"']
\end{tikzcd}$$

Let π denote the composition $\overline{\mathbf{A}}_k^1 \to \tilde{\mathbf{A}}_k^1 \xrightarrow{\tilde{\pi}} \mathbf{A}_k^1$ and its various base extensions to morphisms landing in $\mathbf{G}_{m,k}$ or S. To get $\pi_1(\mathbf{G}_{m,k}, 1)$ to act on nearby cycles, make the

Definition. For K in $\mathrm{D}_c^b(\mathrm{U})$, set $\Psi_f := \bar{i}^* \bar{j}_* \pi^* \mathrm{K}[-1]$.

(We notate also by i, j etc. the corresponding base extensions via the morphism f.)

As i factors as $\{0\} \to (\mathbf{A}_k^1)_{(0)} \to \mathbf{A}_k^1$, $\bar{i}$ factors as $\{\bar{0}\} \to \overline{\mathrm{S}} \to \overline{\mathbf{A}}_k^1$ so that $\Psi_f[1]$ coincides with the usual $\mathrm{R}\Psi_\xi$ for the henselian trait S. As i also factors as $\{0\} \to (\mathbf{A}_k^1)_0 \to \mathbf{A}_k^1$, Ψ_f also coincides with $\bar{i}^* \bar{j}_* \mathrm{K}_{\bar{\xi}}[-1]$ which carries an action of $\mathrm{Gal}(\bar{\xi}/\eta)$. ('Coincide,' means in this case 'coincide' in $\mathrm{D}_c^b(\mathrm{Y}_{\bar{k}})$, but these various ways of writing Ψ_f endow it with actions by different groups; in this case, one is a subgroup of the other.) When $\mathscr{L}$ is a lisse sheaf on $\mathbf{G}_{m,k}$, this action of factors through the quotient $\mathrm{Gal}(\bar{\xi}/\eta) \twoheadrightarrow \pi_1(\mathbf{G}_{m,k}, 1)$. For an arbitrary $\mathbf{Q}_\ell$-sheaf, the lisse locus may be a proper subscheme of $\mathbf{G}_{m,k}$ in which case the action of $\mathrm{Gal}(\bar{\xi}/\eta)$ may not factor through the stated quotient, but rather will factor through one with a smaller kernel. Regardless, the remark about a local system $\mathscr{L}$ is now obvious.

On the functorial exact triangle $\Psi_f \xrightarrow{\mathrm{T}-1} \Psi_f \to i^* j_* \to$: this is discussed with more detail in the proof of t-exactness of (shifted) nearby and vanishing cycles in [**BBD**, Appendix A]. The point is, with $\mathrm{X} \to \mathrm{S}$ of finite type over a strictly henselian trait (S, η, s) which is essentially of finite type over a field k of characteristic $\neq \ell$, starting with $\mathrm{K} \in \mathrm{D}_c^b(\mathrm{X}_\xi)$ and letting $\mathrm{G} := \mathrm{Gal}(\bar{\xi}/\xi)$, $\mathrm{K} \xrightarrow{\sim} \mathrm{R}\Gamma(\mathrm{G}, \rho_* \rho^* \mathrm{K})$ and therefore

$$i^* j_* \mathrm{K} = \mathrm{R}\Gamma(\mathrm{G}, \Psi_f \mathrm{K}).$$

Writing G as an extension of $\mathbf{Z}_\ell(1)$ by a pro-group Q of order prime to ℓ, since invariants by Q is an exact functor, $\mathrm{R}\Gamma(\mathrm{G}, -)$ is represented on $\mathrm{D}_c^b(\mathrm{X}_\xi \times \mathrm{BG})$ by

$$\mathrm{K} \mapsto \mathrm{Cone}(\mathrm{K}^{\mathrm{Q}} \xrightarrow{\mathrm{T}-1} \mathrm{K}^{\mathrm{Q}})[-1]$$

(an explication of why can be found in [**D**, 10.7]; in short, it is because a $\mathbf{Z}_\ell(1)$-module admits a 2-step acyclic resolution by coinduced modules), so, keeping in mind $\Psi_f := \mathrm{R}\Psi_\xi[-1]$, the triangle

$$\Psi_f(\mathrm{K})^{\mathrm{Q}} \xrightarrow{\mathrm{T}-1} \Psi_f(\mathrm{K})^{\mathrm{Q}} \to i^* j_*(\mathrm{K}) \to$$

is distinguished.

When S is no longer strictly henselian, the situation is the same, except of course $\mathrm{Gal}(\overline{k}/k)$ could be nontrivial, so $\mathrm{R}\Gamma(\pi_1(\mathbf{G}_{m,\overline{k}}, 1), \Psi_f(\mathrm{K})) = i^* j_*(\mathrm{K})$, 'packaged' as a sheaf on $\mathrm{Y}_{\overline{k}}$ with action of $\mathrm{Gal}(\overline{k}/k)$ (recall [**SGA 7**, XIII 1.1.3]: the functor $\mathscr{F} \mapsto \{\overline{\mathscr{F}}$ endowed with action of $\mathrm{Gal}(\overline{k}/k)\}$ is an equivalence of categories). This is why in the notes Morel writes 'the last term should be base changed from Y to $\mathrm{Y}_{\overline{k}}$,' but this is a distinction without a difference.

Morel takes T to be a topological generator of the prime-to-p quotient of $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)$, while in the above T is a topological generator of the maximal pro-ℓ quotient $\mathbf{Z}_\ell(1)$ of $\mathrm{Gal}(\overline{\xi}/\xi)$. This is insignificant: the invariants under Q functor is exact on a pro-ℓ module provided Q has order prime to ℓ so we could take T to be a topological generator of the prime-to-p quotient and take invariants under the maximal pro-p subgroup (which is a normal Sylow) or take T to be a topological generator of the maximal pro-ℓ quotient and take invariants under everything else (which is the approach taken above). Let's first analyse the maximal prime-to-p quotient of both groups; we will see that they coincide, which will imply that the maximal pro-ℓ quotients also coincide. The maximal prime-to-p quotient of $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)$ coincides with $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)^{\mathrm{mod}}$, and, after base changing to $\overline{k}$, the maximal prime-to-p quotient of $\mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k})$ coincides with $\mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k})^{\mathrm{mod}}$, the tame quotient of $\mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k})$. Both these groups are isomorphic to $\hat{\mathbf{Z}}(1)(\overline{k})$; c.f. [**Laumon**, 2.2.2.1] and the note to 2.2.2.2. To see that the prime-to-p quotient of $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)$ coincides with $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)^{\mathrm{mod}}$, simply recall that $\mathrm{Gal}(\overline{\xi}, \eta \otimes_k \overline{k}) \twoheadrightarrow \pi_1(\mathbf{G}_{m,\overline{k}}, 1)$ and the former group admits a dévissage

$$1 \to \mathrm{P}_0 \to \mathrm{Gal}(\overline{\xi}, \eta \otimes_k \overline{k}) \to \hat{\mathbf{Z}}(1)(\overline{k}) \to 1$$

where the wild inertia P_0 is a p-group so that the maximal prime-to-p quotient of $\mathrm{Gal}(\overline{\xi}, \eta \otimes_k \overline{k})$ is $\hat{\mathbf{Z}}(1)(\overline{k})$; moreover the same is true of the maximal prime-to-p quotient of the subgroup $\mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k}) \subset \mathrm{Gal}(\overline{\xi}, \eta \otimes_k \overline{k})$, and this subgroup surjects onto

$\mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k})^{\mathrm{mod}} \simeq \hat{\mathbf{Z}}(1)(\overline{k})$. We have a commutative diagram [**Laumon**, 2.2.2]

$$\begin{array}{ccccc} \mathrm{Gal}(\overline{\xi}, \eta \otimes_k \overline{k}) & \twoheadrightarrow & \pi_1(\mathbf{G}_{m,\overline{k}}, 1) & \twoheadrightarrow & \pi_1(\mathbf{G}_{m,\overline{k}}, 1)^{\mathrm{mod}} \\ & & \uparrow & & \wr \\ & & \mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k}) & \twoheadrightarrow & \mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k})^{\mathrm{mod}} \simeq \hat{\mathbf{Z}}(1)(\overline{k}) \end{array}$$

and from the description of $\mathrm{Gal}(\overline{\xi}, \eta \otimes_k \overline{k})$ it follows that $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)^{\mathrm{mod}} \simeq \hat{\mathbf{Z}}(1)(\overline{k})$ is the maximal prime-to-p quotient of $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)$. Moreover, a topological generator of either the maximal prime-to-p or pro-ℓ quotients of $\mathrm{Gal}(\overline{\xi}, \xi \otimes_k \overline{k})$ is carried onto the same in $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)$.

Regardless of whether one takes T to be a topological generator of the maximal pro-ℓ quotient or the maximal prime-to-p quotient (we will choose the latter to conform to Morel's notes), there is still the issue that after extending scalars $k \hookrightarrow \overline{k}$ the triangle

$$\Psi_f(\mathrm{K})^{\mathrm{Q}} \xrightarrow{\mathrm{T}-1} \Psi_f(\mathrm{K})^{\mathrm{Q}} \to i^* j_*(\mathrm{K}) \to$$

is distinguished, where we have taken Q-invariants, whereas Morel doesn't mention taking Q-invariants when she writes this triangle. This is most likely because she is using tamely-ramified nearby cycles $\tilde{\Psi}_f$ to begin with.

Definition. For $\mathrm{K} \in \mathrm{D}^b_c(\mathrm{U})$, let $\tilde{\Psi}_f(\mathrm{K}) := \tilde{i}^* \tilde{j}_* \tilde{\pi}^* \mathrm{K}[-1]$ denote tame nearby cycles.

(Compare [**SGA 7**, Exp. I 2.7].) Note that $\tilde{\Psi}_f(\mathrm{K}) = \mathrm{R}\Gamma(\mathrm{Q}, \Psi_f(\mathrm{K})) = \Psi_f(\mathrm{K})^{\mathrm{Q}}$ where Q is the wild inertia of $\mathrm{Gal}(\overline{\xi}, \xi)$. Therefore, when Morel writes Ψ_f, she is probably implicitly writing $\tilde{\Psi}_f$, and we will do the same in what follows.

1. Unipotent nearby cycles. We retain the notation from the previous section. We write $\mathbf{F}_\ell$ for Morel's F. Now $\mathbf{F}_\ell$ can be $\mathbf{Q}_\ell$, E_λ, $\overline{\mathbf{Q}}_\ell$, etc.

(1.1) We construct the unipotent nearby cycles. To find that the endomorphism ring is finite over $\mathbf{F}_\ell$, simply use the fact that $\mathrm{R}\underline{\mathrm{Hom}}$ preserves constructibility and write

$$\mathrm{End}_{\mathrm{D}^b_c}(\Psi_f \mathrm{K}) = \mathrm{H}^0 \mathrm{R}\,\mathrm{Hom}(\Psi_f \mathrm{K}, \Psi_f \mathrm{K}).$$

Let $\mathrm{P} := \Psi_f \mathrm{K}$ and $\mathrm{E} = \mathrm{End}\,\mathrm{P}$. We have $\mathbf{F}_\ell[\mathrm{T}] \to \mathrm{E}$ with kernel $a(\mathrm{T})$, a nonconstant polynomial as $\dim_{\mathbf{F}_\ell} \mathrm{E} < \infty$. Write $a = bc$ with $(\mathrm{T}-1) \nmid b$ and $c = (\mathrm{T}-1)^m$ for some

$m \in \mathbf{N}$. As $(b, c) = 1$, there exist $x, y \in \mathbf{F}_\ell[\mathrm{T}]$ s.t. $1 = xb + yc$ ($1 = \mathrm{id}_\mathrm{P}$). As $a = 0$ in E, we have $(xb)(yc) = xya = 0$,

$$xb = xb(xb + yc) = (xb)^2 + xya = (xb)^2,$$

and similarly $(yc)^2 = yc$, all in E. Let $e_1 = xb$ and $e_2 = yc$. We have found a pair of orthogonal idempotents e_1 and e_2 with $e_1 + e_2 = 1 := \mathrm{id}_\mathrm{P}$, and by the note to BBD 2.2.18, the triangulated category $\mathrm{D}^b_c(\mathrm{Y})$ is Karoubi complete: every idempotent splits. More precisely, given an object P of this category and an idempotent $e : \mathrm{P} \to \mathrm{P}$, there is an object P_e and a retraction $\mathrm{P}_e \xrightarrow{i} \mathrm{P} \xrightarrow{p} \mathrm{P}_e$ with $pi = \mathrm{id}_{\mathrm{P}_e}$ and $ip = e$. In particular, this means that i is a monomorphism, p an epimorphism, and P_e is simultaneously a limit and colimit of the diagram $\mathrm{P} \rightrightarrows \mathrm{P}$, where the parallel morphisms are id_P and $ip = e$, via the morphisms i and p, respectively. Moreover, P_e is an absolute (co)limit, i.e. preserved by every functor.

Therefore from the e_i we get P_i, i_i, p_i ($i = 1, 2$), and to verify that $\mathrm{P} = \mathrm{P}_1 \oplus \mathrm{P}_2$ it only remains (0103) to check that $p_2 \circ i_1 = 0 = p_1 \circ i_2$. As i_2 is a monomorphism and p_1 an epimorphism, in order to show $p_2 \circ i_1 = 0$ it suffices to write

$$i_2 \circ p_2 \circ i_1 \circ p_1 = e_2 \circ e_1 = 0.$$

Therefore $\mathrm{P} = \mathrm{P}_1 \oplus \mathrm{P}_2$. Given any polynomial $q(\mathrm{T})$, we consider $q(\mathrm{T})$ as endomorphism of P. As $i_2 \circ p_2 \circ q(\mathrm{T}) \circ i_1 \circ p_1 = e_2 q(\mathrm{T}) e_1 = q(\mathrm{T}) e_2 e_1 = 0$, in fact $p_2 \circ q(\mathrm{T}) \circ i_1 = 0$ and similarly $p_1 \circ q(\mathrm{T}) \circ i_2 = 0$. This means that $q(\mathrm{T})$ descends to endomorphisms of P_1 and P_2, and the decomposition $\mathrm{P} = \mathrm{P}_1 \oplus \mathrm{P}_2$ is $\mathbf{F}_\ell[\mathrm{T}]$-equivariant. Moreover, to calculate $q(\mathrm{T})$ on P_1, it suffices to know $p_1 \circ q(\mathrm{T}) \circ i_1$.

We see that as $(\mathrm{T} - 1)^m = c$ and $a = bc = 0$ as endomorphism of P, $(\mathrm{T} - 1)^m e_1 = cxb = xa = 0$. Therefore $p_1 \circ (\mathrm{T} - 1)^m \circ i_1 = 0$ (p_1 is an epimorphism) and $(\mathrm{T} - 1)^m = 0$ on P_1. On the other hand, $p_2 \circ (\mathrm{T} - 1)(y(\mathrm{T} - 1)^{m-1}) \circ i_2 = p_2 \circ e_2 \circ i_2 = \mathrm{id}_{\mathrm{P}_2}$ shows that $(\mathrm{T} - 1)$ is invertible as endomorphism of P_2. This shows that $\mathrm{T} - 1$ is nilpotent on P_1 and an automorphism of P_2.

We set $\Psi_f^{\mathrm{un}}\mathrm{K} := \mathrm{P}_1$ and $\Psi_f^{\mathrm{nu}}\mathrm{K} := \mathrm{P}_2$, call the former the unipotent nearby cycles, and the latter the non-unipotent nearby cycles. We have proved a $\mathbf{F}_\ell[\mathrm{T}]$-linear decomposition

$$\Psi_f\mathrm{K} = \Psi_f^{\mathrm{un}}\mathrm{K} \oplus \Psi_f^{\mathrm{nu}}\mathrm{K}.$$

Nothing in the above discussion changes if we replace Ψ_f by $\tilde{\Psi}_f$, so we have similarly

$$\tilde{\Psi}_f(\mathrm{K}) = \tilde{\Psi}_f^{\mathrm{un}}(\mathrm{K}) \oplus \tilde{\Psi}_f^{\mathrm{nu}}(\mathrm{K})$$

and a distinguished triangle

$$\tilde{\Psi}_f^{\mathrm{un}} \xrightarrow{\mathrm{T}-1} \tilde{\Psi}_f^{\mathrm{un}} \to i^*j_* \to .$$

(That $\tilde{\Psi}_f^{\mathrm{un}}$ preserves constructibility is deduced trivially from the fact that Ψ_f^{un} does, and as taking Q-invariants is an exact functor on ℓ-adic sheaves, $\mathrm{H}^i\tilde{\Psi}_f^{\mathrm{un}} = \mathrm{H}^i(\Psi_f^{\mathrm{un}})^{\mathrm{Q}} = (\mathrm{H}^i\Psi_f^{\mathrm{un}})^{\mathrm{Q}}$; now recall that every subsheaf of a constructible sheaf of modules over a noetherian ring on a noetherian scheme is constructible.)

2. Some local systems on $\mathbf{G}_{m,k}$. The local systems $\mathscr{L}_a$ appear to be stand-ins for Beilinson's $\mathrm{Gr}\,\mathrm{I}^{a,b}$, which, by the way, come with the pairing $\langle\ ,\ \rangle$. Let's look more closely at the analogy. The action of $\pi_1(\mathbf{G}_{m,k},1)$ on $\mathrm{I}^{a,b}$ is also via the projection onto the maximal pro-ℓ quotient $\mathbf{Z}_\ell(1)$, and $t \in \mathbf{Z}_\ell(1)$ acts on $\mathrm{I}^{a,b}$ by $\tilde{t} = \exp(\log\tilde{t})$. The point is that $\log\tilde{t} \in \mathrm{A}^{1*}$ in Beilinson's notation; i.e. $\mathrm{A}^\circ \simeq \mathbf{F}_\ell[[\log\tilde{t}]]$, and therefore $\log\tilde{t}$ acts on $\mathrm{Gr}\,\mathrm{I}^{-a,0}$ by Morel's N. Therefore indeed Morel's $\mathscr{L}_a \simeq \mathrm{Gr}\,\mathrm{I}^{-a,0}$. Morel claims an isomorphism $\mathscr{L}_a^\vee \simeq \mathrm{Gr}\,\mathrm{I}^{0,a}$, but if we want to upgrade the naïve isomorphism of sheaves on $\mathrm{Spec}\,k$ to one on $\mathbf{G}_{m,k}$ we run into the issue that the logarithm of geometric monodromy acts on $\mathscr{L}_a^\vee$ by $-\log\tilde{t}$, not $\log\tilde{t}$ and so t acts by $\exp(-\log\tilde{t})$. Therefore a $\pi_1(\mathbf{G}_{m,k},1)$-equivariant isomorphism is not provided by the identity morphism but rather by the isomorphism

$$\begin{array}{ccccccccc}
\mathrm{I}^{0,a} & = & \mathrm{F} & \oplus & \mathrm{F}(1) & \oplus & \cdots & \oplus & \mathrm{F}(a) \\
\downarrow\wr & & \downarrow\mathrm{id} & & \downarrow -\mathrm{id} & & & & \downarrow (-1)^a\,\mathrm{id} \\
\mathscr{L}_a^\vee & = & \mathrm{F} & \oplus & \mathrm{F}(1) & \oplus & \cdots & \oplus & \mathrm{F}(a).
\end{array}$$

In Beilinson's notation, $(\mathrm{Gr}\,\mathrm{I}^{-a,0})^\vee \simeq \mathrm{Gr}\,\mathrm{I}^{0,a}$, and the maps α, β are just the Gr of maps $\mathrm{I}^{-a,0} \hookrightarrow \mathrm{I}^{-b,0}$ and $\mathrm{I}^{-b,0} \twoheadrightarrow \mathrm{I}^{-b,a-b}$. To obtain the statement about $\mathrm{D}(\alpha_{a,b})$, we see before passing to Gr that $\mathrm{D}(\alpha_{a,b})(-1)[-2] : \mathrm{I}^{0,b} \to \mathrm{I}^{0,a}$ is just reduction modulo I^a. Passing to Gr, this is the same as

$$\beta_{a,b}(b) : \mathscr{L}_b(b) = (\operatorname{Gr} \mathrm{I}^{-b,0})(b) \twoheadrightarrow (\operatorname{Gr} \mathrm{I}^{-a,0})(a) = \mathscr{L}_a(a).$$

3. Beilinson's construction of Ψ_f^{un}. Everything is clear except perhaps at first glance the expression $\ker(\mathrm{N}^a, \Psi_f^{\mathrm{un}}\mathrm{K})$, as N is not an endomorphism of Ψ_f^{un} but rather sends it to $\Psi_f^{\mathrm{un}}(-1)$. Write N as a linear map $\Psi_f^{\mathrm{un}}(1) \to \Psi_f^{\mathrm{un}}$; what this means is that N assigns, linearly in $\mathbf{F}_\ell$, an endomorphism of Ψ_f^{un} to an element of $\mathbf{F}_\ell(1)$. Therefore there is an unambiguous meaning to $\ker(\mathrm{N}^a, \Psi_f^{\mathrm{un}}\mathrm{K})$: if $t_1, \ldots, t_a$ are a nonzero elements of $\mathbf{F}_\ell(1)$, $\ker(\mathrm{N}t_a \ldots \mathrm{N}t_1, \Psi_f^{\mathrm{un}}\mathrm{K})$ is independent of the choice of t_i.

(3.3) Returning to the various ways of defining Ψ_f, we settled upon $\tilde{\Psi}_f^{\mathrm{un}}$. This functor with $f = \mathrm{id}$, as a direct factor of tame cycles, no longer has the the property of sending every shifted local system $\mathscr{L}[1]$ on $\mathbf{G}_{m,k}$ to $\mathrm{L}_{\{0\}}$ (where L the representation of $\pi_1(\mathbf{G}_{m,k}, 1)$ corresponding to $\mathscr{L}$). But it still does for these $\mathscr{L}_a$, as $\pi_1(\mathbf{G}_{m,\overline{k}}, 1)$ acts on $\mathscr{L}_a$ through the tame quotient, and moreover T acts unipotently by construction.

(3.4) is indeed obvious, but it appears to have a typo. Start with $\Psi_f^{\mathrm{un}}(\mathrm{K})$, express the latter as $\ker(\mathrm{N}^{a+1}, \Psi_f^{\mathrm{un}}(\mathrm{K}))$ for $a \gg 0$ and then via the isomorphism γ as $\ker(\mathrm{N}, \Psi_f^{\mathrm{un}}(\mathrm{K} \otimes \mathscr{L}_a))$; i.e. for $a \gg 0$ the map

$$\begin{aligned} \Psi_f^{\mathrm{un}}(\mathrm{K}) &\to \ker(\mathrm{N}, \Psi_f^{\mathrm{un}}(\mathrm{K} \otimes \mathscr{L}_a)) \\ x &\mapsto (x, -\mathrm{N}x, \ldots, (-\mathrm{N})^a x) \end{aligned}$$

is an embedding. The target is identified with ${}^p\mathrm{H}^{-1} i^* j_*(\mathrm{K} \otimes \mathscr{L}_a))$, which now has N acting only by $1 \otimes \mathrm{N} = 1 \otimes (\beta_{a,a+1} \circ \alpha_{a,a+1})$; i.e. by

$$(x, -\mathrm{N}x, \ldots, (-\mathrm{N})^a x) \mapsto (-\mathrm{N}x, \ldots, (-\mathrm{N})^a x, 0);$$

via the above embedding, this means that $1 \otimes \mathrm{N} = 1 \otimes (\beta_{a,a+1} \circ \alpha_{a,a+1})$ acts by $-\mathrm{N}$, not N, on $\Psi_f^{\mathrm{un}}(\mathrm{K})$.

(3.5) Beilinson's construction of Ψ_f^{un} has the merit of behaving simply with respect to duality and admitting a description in terms of basic functors, but, as remarked above, whether it lands in Y or $\mathrm{Y}_{\overline{k}}$ is insignificant since it lands in $\mathrm{Y}_{\overline{k}}$ with action of $\mathrm{Gal}(\overline{k}/k)$

coming from the split-exact arithmetic-geometric sequence of π_1, and therefore can be regarded as a sheaf on Y_k to begin with.

4. Duality. As before, $\mathbf{F}_\ell = \mathbf{Q}_\ell, E_\lambda, \overline{\mathbf{Q}}_\ell$, etc.

Lemma. — Let $f : X \to S$ *be a separated morphism of finite type between schemes of finite type over* k *and* $\mathscr{L}$ *a lisse* $\mathbf{F}_\ell$*-sheaf on* S. *Then* $f^!(\mathscr{L}) \xleftarrow{\sim} f^*\mathscr{L} \otimes f^!\mathbf{F}_\ell$.

Proof. — The counit of adjunction $f_!f^! \to \mathrm{id}$ defines an arrow in the H^0 of

$$\mathrm{R\,Hom}(\mathscr{L} \otimes f_!f^!\mathbf{F}_\ell, \mathscr{L}) = \mathrm{R\,Hom}(f_!(f^*\mathscr{L} \otimes f^!\mathbf{F}_\ell), \mathscr{L}) = \mathrm{R\,Hom}(f^*\mathscr{L} \otimes f^!\mathbf{F}_\ell, f^!(\mathscr{L}))$$

which is evidently locally an isomorphism. □

The lemma allows one to write, for $K \in D^b_c(X, \mathbf{F}_\ell)$,

$$\begin{aligned} D(K \otimes f^*\mathscr{L}_a) &= \mathrm{R\,Hom}(K, D(f^*\mathscr{L}_a^\vee)) = \mathrm{R\,Hom}(K, f^!D(\mathscr{L}_a^\vee)) \\ &= \mathrm{R\,Hom}(K, f^*\mathscr{L}_a^\vee \otimes f^!\mathbf{F}_\ell)) = D(K) \otimes f^*\mathscr{L}_a^\vee \simeq D(K) \otimes f^*\mathscr{L}_a(a). \end{aligned}$$

(4.1) As D is involutive and $D(X(a)) = D(X)(-a)$, the diagram

$$\begin{tikzcd}
{}^pH^0 i^* j_*(K \otimes \mathscr{L}_a) \arrow[r, "\alpha_{a,a+b+1}"] & {}^pH^0 i^* j_*(K \otimes \mathscr{L}_{a+b+1}) \arrow[r, "\beta_{b,a+b+1}"] & {}^pH^0 i^* j_*(K \otimes \mathscr{L}_b) \\
\mathrm{coker}_a(K) \arrow[u, "\wr"] \arrow[r, "\alpha_{a,a+b+1}"] \arrow[d, equal] & \mathrm{coker}_{a+b+1}(K) \arrow[u, "\wr"] \arrow[r, "\beta_{b,a+b+1}"] \arrow[d, equal] & \mathrm{coker}_b(K)(-a-1) \arrow[u, "\wr"] \arrow[d, equal] \\
D(\ker_a(DK)(a)) \arrow[r, "D(\beta_{a,a+b+1}(a))"] & D(\ker_{a+b+1}(DK)(a))(-b-1) \arrow[r, "D(\alpha_{b,a+b+1}(a))(-b-1)"] & D(\ker_b(DK)(a))(-b-1) \\
D(\Psi_f^{\mathrm{un}}(DK))(-a) \arrow[u, "\wr"] \arrow[r, "D(N^{b+1})(-a)"] & D(\Psi_f^{\mathrm{un}}(DK))(-a-b-1) \arrow[u, "\wr"] \arrow[r, "D(\mathrm{id})(-a-b-1)"] & D(\Psi_f^{\mathrm{un}}(DK))(-a-b-1) \arrow[u, "\wr"]
\end{tikzcd}$$

commutes, where the first line is isomorphic to the second by [**BBD**, 4.1.12.4], and we have used that the map $\alpha_{a,b}$ induces the map $\mathscr{L}_b^\vee = \mathscr{L}_b(b) \xrightarrow{\beta_{a,b}(b)} \mathscr{L}_a(a) = \mathscr{L}_a^\vee$ and vice versa. We need that $N^{b+1} = 0$ on $\Psi_f^{\mathrm{un}}(DK)$, and I don't see why this is secured if $N^{b+1} = 0$ on $\Psi_f^{\mathrm{un}}(K)$, so let's modify the hypothesis to say '$N^{a+1} = 0$ on $\Psi_f^{\mathrm{un}}(K)$ and $N^{b+1} = 0$ on $\Psi_f^{\mathrm{un}}(DK)$.'

5. The maximal extension functor. The βs in the commutative diagram defining $\gamma_{a,a-1}$ are $\beta_{a-1,a}$, not $\beta_{a,a+1}$.

(5.1) It is of course the map $\beta_{a-1,a} : \mathscr{L}_a \to \mathscr{L}_{a-1}(-1)$ that is surjective. The snake lemma gives an isomorphism

$$\operatorname{coker} \gamma_{a,a-1} \simeq i_*^p \mathrm{H}^0 i^* j_*(\mathrm{K} \otimes \mathscr{L}_{a-1})(-1).$$

To see that the last vertical arrow in the last commutative diagram of the proof is an isomorphism, it is easier to identify it with

$$\begin{aligned} i_* \Psi_f^{\mathrm{un}} \mathrm{K}(-1) &= \ker(j_!(\mathrm{K} \otimes f^* \mathscr{L}_{a-1}(-1)) \to j_*(\mathrm{K} \otimes f^* \mathscr{L}_{a-1}(-1))) \\ &\xrightarrow{\alpha_{a-1,a}(-1)} \ker(j_!(\mathrm{K} \otimes f^* \mathscr{L}_a(-1)) \to j_*(\mathrm{K} \otimes f^* \mathscr{L}_a(-1))) = i_* \Psi_f^{\mathrm{un}} \mathrm{K}(-1), \end{aligned}$$

which is the identity by the results of §3, noting that the square below commutes

$$\begin{tikzcd} \mathscr{L}_a \arrow[r, "\alpha_{a,a+1}"] \arrow[d, "\beta_{a-1,a}"] & \mathscr{L}_{a+1} \arrow[d, "\beta_{a,a+1}"] \\ \mathscr{L}_{a-1}(-1) \arrow[r, "\alpha_{a-1,a}(-1)"] & \mathscr{L}_a(-1). \end{tikzcd}$$

The remaining claims are now clear; given $0 \to \mathrm{K}_1 \to \mathrm{K}_2 \to \mathrm{K}_3 \to 0$ exact, one deduces exactness of $0 \to \Xi_f \mathrm{K}_1 \to \Xi_f \mathrm{K}_2 \to \Xi_f \mathrm{K}_3$ by finding an integer a such that $\ker(j_!(\mathrm{K}_i \otimes f^* \mathscr{L}_a) \xrightarrow{\gamma_{a,a-1}} j_*(\mathrm{K}_i \otimes f^* \mathscr{L}_{a-1})(-1)) = \Xi_f \mathrm{K}_i$ for $i = 1, 2, 3$ and then applying the snake lemma to the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & j_!(\mathrm{K}_1 \otimes f^* \mathscr{L}_a) \arrow[r] \arrow[d, "\gamma_{a,a-1}"] & j_!(\mathrm{K}_2 \otimes f^* \mathscr{L}_a) \arrow[r] \arrow[d, "\gamma_{a,a-1}"] & j_!(\mathrm{K}_3 \otimes f^* \mathscr{L}_a) \arrow[r] \arrow[d, "\gamma_{a,a-1}"] & 0 \\ 0 \arrow[r] & j_*(\mathrm{K}_1 \otimes f^* \mathscr{L}_a) \arrow[r] & j_*(\mathrm{K}_2 \otimes f^* \mathscr{L}_a) \arrow[r] & j_*(\mathrm{K}_3 \otimes f^* \mathscr{L}_a) \arrow[r] & 0. \end{tikzcd}$$

(5.4) Typo: it should read

$$\operatorname{coker}(j_!(\mathrm{K} \otimes f^* \mathscr{L}_b)(1) \to j_*(\mathrm{K} \otimes f^* \mathscr{L}_{b-1})) = \operatorname{coker}(j_!(\mathrm{K} \otimes f^* \mathscr{L}_{b-1}) \to j_*(\mathrm{K} \otimes f^* \mathscr{L}_{b-1}))$$

The point is, writing

$$\gamma_{a,a+1} = j_!(\mathrm{K} \otimes \mathscr{L}_a) \to j_*(\mathrm{K} \otimes \mathscr{L}_a) \xrightarrow{\alpha_{a,a+1}} j_*(\mathrm{K} \otimes \mathscr{L}_{a+1})$$

shows that $\ker\gamma_{a,a+1} = \ker(j_!(\mathrm{K}\otimes\mathscr{L}_a) \to j_*(\mathrm{K}\otimes\mathscr{L}_a)) = \Psi_f^{\mathrm{un}}(\mathrm{K})$ and the snake lemma begins with

$$0 \to \Psi_f^{\mathrm{un}} \xrightarrow{\alpha_{a,a+b+1}} \Psi_f^{\mathrm{un}} \to \ker(\gamma_{b,b-1})(-a-1) \to \cdots$$

where $\alpha_{a,a+b+1}$ induces an isomorphism of Ψ_f^{un}. The dual statement about cokernels, after writing

$$\gamma_{b,b-1}(a-1) = j_!(\mathrm{K}\otimes\mathscr{L}_b)(a-1) \xrightarrow{\beta_{b,b-1}(a-1)} j_!(\mathrm{K}\otimes\mathscr{L}_{b-1})(a-2) \to j_*(\mathrm{K}\otimes\mathscr{L}_{b-1})(a-2),$$

finds that

$$\begin{aligned}\operatorname{coker}\gamma_{b,b-1}(a-1) &\simeq \operatorname{coker}(j_!(\mathrm{K}\otimes\mathscr{L}_{b-1})(-a-2) \to j_*(\mathrm{K}\otimes\mathscr{L}_{b-1})(a-2))\\ &\simeq \Psi_f^{\mathrm{un}}\mathrm{K}(-a-2)(-b+1)\end{aligned}$$

so that, denoting $\operatorname{coker}(j_!(\mathrm{K}\otimes\mathscr{L}_{a+b+1}) \to j_*(\mathrm{K}\otimes\mathscr{L}_{a+b+1}))$ by $\operatorname{coker}(j_! \to j_*)$, the snake lemma ends in

$$\begin{tikzcd}
\operatorname{coker}(\gamma_{a,a+1}) \arrow[r] \arrow[d, equal] & \underset{a+b+1}{\operatorname{coker}}(j_! \to j_*) \arrow[r, "\beta_{b,a+b+1}"] \arrow[d, equal] & \operatorname{coker}(\gamma_{b,b-1}(-a-1)) \to 0 \arrow[d, "\wr"] \\
\operatorname{coker}(\gamma_{a,a+1}) \arrow[r] \arrow[d, equal] & \underset{a+b+1}{\operatorname{coker}}(j_! \to j_*) \arrow[r, "\beta_{b-1,a+b+1}"] \arrow[d, "\wr"] & \underset{b-1}{\operatorname{coker}}(j_! \to j_*)(-a-1) \to 0 \arrow[d, "\wr"] \\
\operatorname{coker}(\gamma_{a,a+1}) \arrow[r] & \Psi_f^{\mathrm{un}}\mathrm{K}(-a-b-1) \arrow[r, "\sim"] & \Psi_f^{\mathrm{un}}\mathrm{K}(-a-2)(-b+1) \to 0,
\end{tikzcd}$$

using (4.1). Of course

$$\beta_{b-1,a+b+1} : \underset{a+b+1}{\operatorname{coker}}(j_! \to j_*) \xrightarrow{\sim} \underset{b-1}{\operatorname{coker}}(j_! \to j_*)(-a-1)$$

is an isomorphism from the dual statement about α.

(5.6) Just apply $\beta_{a,a+1}$ to the exact sequence of (5.5) for $a \gg 0$.

(5.7) If we denote the map $i_*\Psi_f^{\mathrm{un}} \to \Xi_f$ by υ, then the composition coincides with

$$i_*\Psi_f^{\mathrm{un}}\mathrm{D} \xrightarrow{\upsilon} \Xi_f\mathrm{D} = \mathrm{D}\Xi_f \xrightarrow{\mathrm{D}\upsilon} \mathrm{D}i_*\Psi_f^{\mathrm{un}} = i_*\Psi_f^{\mathrm{un}}(-1)\mathrm{D}, \qquad \text{so that}$$

$$\mathrm{D}(\mathrm{D}\upsilon\circ\upsilon) = \mathrm{D}\upsilon\circ\mathrm{D}\mathrm{D}\upsilon = \mathrm{D}\upsilon\circ\upsilon.$$

The morphism $\Xi_f \to i_*\Psi_f^{\mathrm{un}}(-1)$ is induced by $\beta_{a-1,a} : j_!(\mathrm{K}\otimes\mathscr{L}_a) \to j_!(\mathrm{K}\otimes\mathscr{L}_{a-1})(-1)$ for $a \gg 0$. The dual morphism fits into a commutative diagram ($a \gg 0$)

$$\begin{tikzcd}
\mathrm{coker}(j_!(\mathrm{K}\otimes\mathscr{L}_{a-1})(a) \to j_*(\mathrm{K}\otimes\mathscr{L}_{a-1})(a)) \arrow[r, "\alpha_{a-1,a}(a)"] & \mathrm{coker}(\gamma_{a-1,a}(a)) \\
\mathrm{ker}(j_!(\mathrm{K}\otimes\mathscr{L}_{a-1}) \to j_*(\mathrm{K}\otimes\mathscr{L}_{a-1})) \arrow[r, "\alpha_{a-1,a}"] \arrow[u, "\wr"] & \mathrm{ker}(\gamma_{a,a-1}) \arrow[u, "\wr"'] \\
i_*\Psi_f^{\mathrm{un}}(\mathrm{K}) \arrow[r, hook] \arrow[u, "\wr"] & \Xi_f(\mathrm{K}). \arrow[u, "\wr"']
\end{tikzcd}$$

To see that the upper square commutes, it helps to recall that both vertical arrows are coboundaries in the snake lemma applied to two similar diagrams, and a morphism between these two diagrams can be built on $\alpha_{a-1,a}$ (applied to the upper right corners of the diagrams) and its twist (applied to the lower left corners). Therefore $\mathrm{D}\upsilon \circ \upsilon = \mathrm{N} : i_*\Psi_f^{\mathrm{un}} \to \Psi_f^{\mathrm{un}}(-1)$ and $\mathrm{DN} = \mathrm{N}$.

We can also study the composition $\Xi_f \to i_*\Psi_f^{\mathrm{un}}(-1) \to \Xi_f(-1)$. The first map is induced by $\beta_{a,a-1}$ and the second by $\alpha_{a-1,a}(-1)$, as already remarked. We have (5.2) that

$$\mathrm{ker}(\gamma_{a,a-1}) \xrightarrow{\beta_{a-1,a}} \mathrm{ker}(\gamma_{a-1,a-2}(-1))$$

induces N on Ξ_f for $a \gg 0$. Since $\gamma_{a,a-1}$ factors as $\beta_{a-1,a}$ followed by $j_!(\mathrm{K}\otimes\mathscr{L}_{a-1})(-1) \to j_*(\mathrm{K}\otimes\mathscr{L}_{a-1})(-1)$, $\beta_{a-1,a}$ applied to $\mathrm{ker}(\gamma_{a,a-1})$ must actually factor as

$$\mathrm{ker}(\gamma_{a,a-1}) \xrightarrow{\beta_{a-1,a}} \mathrm{ker}(j_!(\mathrm{K}\otimes\mathscr{L}_{a-1}) \to j_*(\mathrm{K}\otimes\mathscr{L}_{a-1}))(-1) \hookrightarrow \mathrm{ker}(\gamma_{a-1,a-2}(-1)).$$

The middle is $i_*\Psi_f^{\mathrm{un}}(\mathrm{K})(-1)$, and if we postcompose the above morphism by $\alpha_{a-1,a}(-1)$ we don't change it, as $\alpha_{a-1,a}$ induces an isomorphism $\Xi_f\mathrm{K} \xrightarrow{\sim} \Xi_f\mathrm{K}$ for $a \gg 0$. We have shown that $\mathrm{N} : \Xi_f \to \Xi_f$ factors as

$$\Xi_f \twoheadrightarrow i_*\Psi_f^{\mathrm{un}}(-1) \hookrightarrow \Xi_f(-1)$$

and the former map induces via the inclusion $i_*\Psi_f^{\mathrm{un}} \hookrightarrow \Xi_f$ the morphism N on $i_*\Psi_f^{\mathrm{un}}$; i.e.

$$i_*\Psi_f^{\mathrm{un}} \hookrightarrow \Xi_f \twoheadrightarrow i_*\Psi_f^{\mathrm{un}}(-1) \qquad \text{also composes to N.}$$

Last, it is clear that the square

$$\begin{tikzcd}
j_! \arrow[rr, hook] \arrow[dd] \arrow[dr] & & \ker(\gamma_{a-1,a-2}) \arrow[dd, "\wr"] \arrow[dl, equal] \\
& \Xi_f \arrow[dl] \arrow[dr, equal] & \\
j_* & & \operatorname{coker}(\gamma_{a-1,a}(a)) \arrow[ll, two heads]
\end{tikzcd}$$

commutes, where the leftmost down arrow is the canonical morphism, which shows that this morphism is also sent to itself by D.

6. The unipotent vanishing cycles functor.

REMARK. The first paragraph discusses the construction of vanishing cycles. However, (6.2) shows that the functor Φ_f^{un} constructed is a direct factor of the the tame vanishing cycles which appears in the distinguished triangle

$$\Psi_f^{\mathrm{un}} j^* \xrightarrow{\mathrm{can}} \Phi_f^{\mathrm{un}} \to i^* \to$$

where as before $\Psi_f^{\mathrm{un}} = \tilde{\Psi}_f^{\mathrm{un}}$.

In light of [**BBD**, 4.1.10.1], Ψ_f^{un} sends $\mathrm{M}(\mathrm{X})$ to $\mathrm{M}(\mathrm{Y})$ because it has support in Y. (5.6) $\rightsquigarrow$ $\varepsilon(-1) \circ \mathrm{N} = 0$. (5.2) $\rightsquigarrow$ $\mathrm{N} \circ \delta = 0$. It is easy to see that $\mathrm{can}(-1) \circ \mathrm{var}$ induces the map $\Xi_f \twoheadrightarrow \Psi_f^{\mathrm{un}}(-1) \hookrightarrow \Xi_f$. We studied this composition in the note to the previous section and showed that it coincides with N.

(6.2) There is a slight inaccuracy in the second part of the proof that doesn't change the conclusion. The following diagram is commutative with exact rows and columns.

$$
\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & i_* \, {}^p\mathrm{H}^{-1} i^* \mathrm{K} \arrow[d] & \\
 & & j_! j^* \mathrm{K} \arrow[r, equal] \arrow[d, "d^{-1}"] & j_! j^* \mathrm{K} \arrow[d, "\mathrm{adj}"] & \\
0 \arrow[r] & i_* \Psi_f^{\mathrm{un}} j^* \mathrm{K} \arrow[r] \arrow[d, "\kappa"] & \ker d^0 \arrow[r] \arrow[d] & \mathrm{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \nu \arrow[r] \arrow[d] & i_* \Phi_f^{\mathrm{un}} \mathrm{K} \arrow[r, dashed, "\nu"] \arrow[d] & i_* \, {}^p\mathrm{H}^0 i^* \mathrm{K} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{$\dagger$}
$$

The snake lemma gives the exact sequence

$$0 \to \ker \kappa \to j_! j^* \mathrm{K} \to j_! j^* \mathrm{K} / i_* \, {}^p\mathrm{H}^{-1} i^* \mathrm{K} \to \operatorname{coker} \kappa \to 0$$

so that κ is surjective and $\ker \kappa = i_* \, {}^p\mathrm{H}^{-1} i^* \mathrm{K}$. This doesn't change the conclusion that $i_* \operatorname{coker} \mathrm{can} \xrightarrow{\sim} i_* \, {}^p\mathrm{H}^0 i^* \mathrm{K}$. Moreover these conclusions for $\ker(i_* \,\mathrm{can})$, $i_* \operatorname{coker} \mathrm{can}$ imply the results without the i_* because i_* is t-exact and fully faithful.

7. The functor Ω_f. Ω_f can be defined as $\ker(\varepsilon + \mathrm{adj})$ or as $\ker(\varepsilon - \mathrm{adj})$, and the isomorphism of complexes $\mathrm{C}^{\bullet}\mathrm{K} \xrightarrow{\sim} \mathrm{C}'^{\bullet}\mathrm{K}$ (6.1) carries the latter isomorphically onto the former. In particular, this gives an isomorphism $\Omega_f \simeq \ker d^0$, and, in light of the middle row of the diagram ($\dagger$) above, the second short exact sequence. The first short exact sequence is (6.1) on the nose.

The argument that gives the quasi-isomorphism $(\Psi_f^{\mathrm{un}} j^* \xrightarrow{\mathrm{can}} \Phi_f^{\mathrm{un}}) \to i^*$ actually establishes the existence of a distinguished triangle

$$\Psi_f^{\mathrm{un}} j^* \xrightarrow{\mathrm{can}} \Phi_f^{\mathrm{un}} \to i^* \to$$

which is the one we expect from the 'usual' definition of Φ_f^{un}. But while the first two functors take perverse sheaves to perverse sheaves, i^* has perverse amplitude in

$[-1,0]$. One should take care when interpreting the word 'quasi-isomorphism,' since $\Psi_f^{\mathrm{un}} j^* \xrightarrow{\mathrm{can}} \Phi_f^{\mathrm{un}}$ is a complex of perverse sheaves (not an object of $\mathrm{D}_c^b(\mathrm{Y})$), while $i^*\mathrm{K}$ is usually not a perverse sheaf.

8. Gluing. $\mathrm{D}^{\boldsymbol{\cdot}}(c)$ is a complex because the composition of the two differentials gives $\mathrm{N}-\mathrm{N}=0$.

(8.1) I can't get either direction of the proof to work as written. The first part poses a problem because $\Phi_f^{\mathrm{un}}\mathrm{D}^{\boldsymbol{\cdot}}(c)$ cannot have the stated form: $b \circ a = 0$ but $-vu = -\mathrm{N}$ so that what is written is not a complex. The issue with the second part is that the square on the right in the morphism of exact sequences doesn't commute: restricting our attention to the second factor $\Omega_f\mathrm{K}$, tracing the square right-down gives the null morphism, while tracing it down-right gives $\Omega_f \twoheadrightarrow i_*\Phi_f^{\mathrm{un}}\mathrm{K} \xrightarrow{\mathrm{var}} i_*\Psi_f^{\mathrm{un}} j^*\mathrm{K}(-1)$, at least up to sign.

So, to complete the proof, we turn to Beilinson's device of monads and diads developed in his appendix, which gets the job done. Conforming to the language of the next section, we consider a category $\mathrm{M}(\mathrm{X})_1^\sharp$ of diads of perverse sheaves on X. An object of this category is a diagram

$$\mathrm{C}_- \xhookrightarrow{\alpha_-=(\alpha_-^1,\alpha_-^2)} \mathrm{A}\oplus\mathrm{B} \xrightarrow{\alpha_+=(\alpha_+^1,\alpha_+^2)} \!\!\!\!\rightarrow \mathrm{C}_+$$

in which α_-^1 is a monomorphism and α_+^1 an epimorphism. Morphisms are defined pointwise and such a morphism of diads is a (mono/epi/iso)morphism if it is pointwise; one has $(\mathrm{M}(\mathrm{X})_1^\sharp)^\circ = (\mathrm{M}(\mathrm{Y})^\circ)_1^\sharp$ and exact functors $\mathrm{M}(\mathrm{X}) \to \mathrm{M}(\mathrm{Y})$ induce exact functors $\mathrm{M}(\mathrm{X})_1^\sharp \to \mathrm{M}(\mathrm{Y})_1^\sharp$. The category $\mathrm{M}(\mathrm{X})_1^\sharp$ is endowed with an involutive autoequivalence r called the reflection functor, which we compute explicitly in the next section. The category of perverse sheaves embeds into $\mathrm{M}(\mathrm{X})_1^\sharp$ via the functor $\mathrm{C} : \mathrm{M}(\mathrm{X}) \hookrightarrow \mathrm{M}(\mathrm{X})_1^\sharp$ which sends a perverse sheaf K to the diad

$$j_!j^*\mathrm{K} \xrightarrow{\delta\oplus\mathrm{adj}} \Xi j^*\mathrm{K}\oplus\mathrm{K} \xrightarrow{\varepsilon-\mathrm{adj}} j_*j^*\mathrm{K}.$$

Likewise the category of gluing data embeds into $\mathrm{M}(\mathrm{X})_1^\sharp$ via the functor D which sends a gluing datum $c = (\mathrm{K}_\mathrm{U}, \mathrm{K}_\mathrm{V}, u, v)$ to the diad

$$i_*\Psi_f^{\mathrm{un}}\mathrm{K}_\mathrm{U} \xrightarrow{\mathrm{can}\oplus u} \Xi_f\mathrm{K}_\mathrm{U}\oplus i_*\mathrm{K}_\mathrm{Y} \xrightarrow{\mathrm{can}-v} i_*\Psi_f^{\mathrm{un}}\mathrm{K}_\mathrm{U}(-1).$$

Identifying the category of perverse sheaves and the category of gluing data with their respective essential images in $\mathrm{M}(\mathrm{X})_1^\sharp$ via C (resp. D), r exchanges these two subcategories of $\mathrm{M}(\mathrm{X})_1^\sharp$, and in doing so coincides with the functors F and G. To verify this, we use our explicit computation of r in the next section to check that

$$r(\mathrm{C}(\mathrm{K})) = i_*\Psi_f^{\mathrm{un}} j^*\mathrm{K} \xrightarrow{\mathrm{can}\oplus\mathrm{can}} \Xi_f j^*\mathrm{K} \oplus i_*\Psi_f^{\mathrm{un}}\mathrm{K} \xrightarrow{\mathrm{can}-\mathrm{var}} i_*\Psi_f^{\mathrm{un}} j^*\mathrm{K}(-1),$$
$$r(\mathrm{D}(c)) = j_!\mathrm{K}_\mathrm{U} \xrightarrow{\delta\oplus\mathrm{adj}} \Xi_f\mathrm{K}_\mathrm{U} \oplus \mathrm{H}^0(\mathrm{D}(c)) \xrightarrow{\varepsilon-\mathrm{adj}} j_*\mathrm{K}_\mathrm{U}.$$

As r is an involution, it induces an equivalence between the category of perverse sheaves and the category of gluing data, embedded as subcategories of $\mathrm{M}(\mathrm{X})_1^\sharp$, completing the proof.

We now provide details of the proofs of the statements of Beilinson's appendix on monads and diads. Reich provides details for the key statement about the equivalence of diads of type $\mathrm{A}_1^\#$ and $\mathrm{A}_2^\#$ but his proof is wrong (he claims certain maps obtained from a diad of type $\mathrm{A}_2^\#$ should yield an exact sequence when they don't form a complex).

Beilinson's Appendix A1 & A2. Let $\mathscr{A}$ denote an exact category.

Proof of Beilinson's Lemma A1. — Let $\mathscr{A}^\flat =: \mathscr{A}_0^\flat$ denote the category of monads and $\mathscr{A}_i^\flat$ Beilinson's $\mathrm{A}_i^\sim$. Let $\mathrm{T}_i : \mathscr{A}_i^\flat \to \mathscr{A}_{i+1}^\flat$ be the functors described, where $i \in \mathbf{Z}/3\mathbf{Z}$.

Define the functor $\mathrm{T}_0^{-1} : \mathscr{A}_1^\flat \to \mathscr{A}_0^\flat$ by

$$(\mathrm{P}_{-1} \xhookrightarrow{\gamma_{-1}} \mathrm{P}_0 \xhookrightarrow{\gamma_0} \mathrm{P}_1) \mapsto (\mathrm{P}_{-1} \xhookrightarrow{\gamma_0\circ\gamma_{-1}} \mathrm{P}_1 \twoheadrightarrow \mathrm{P}_1/\mathrm{P}_0).$$

T_0^{-1} is clearly an inverse to T_0.

Turning to $\mathscr{A}_2^\flat$, first note that the following square is a pushout and pullback square, and therefore ε_- is an admissible epimorphism and ε_+ an admissible monomorphism.

$$\begin{tikzcd} \mathrm{L}_- \arrow[r, two heads, "\varepsilon_-"] \arrow[d, hook, "\delta_-"] & \mathrm{B} \arrow[d, hook, "-\varepsilon_+"] \\ \mathrm{A} \arrow[r, two heads, "\delta_+"] & \mathrm{L}_+ \end{tikzcd}$$

This is true because to give a morphism to A and B which agrees on L_+ via δ_+ and $-\varepsilon_+$ is the same as giving a kernel for $\mathrm{A}\oplus\mathrm{B} \xrightarrow{\delta_++\varepsilon_+} \mathrm{L}_+$, and dually. We therefore have a commutative diagram in which the right square is bicartesian, and claim that the dashed arrow is an isomorphism.

$$\begin{tikzcd}
\ker\varepsilon_- \arrow[r, hook] \arrow[d, dashed] & L_- \arrow[r, two heads, "\varepsilon_-"] \arrow[d, hook, "\delta_-"] & B \arrow[d, hook, "-\varepsilon_+"] \\
\ker\delta_+ \arrow[r, hook] & A \arrow[r, two heads, "\delta_+"] & L_+
\end{tikzcd}$$

To give a morphism to A which δ_+ sends to zero is (since ε_+ is a monomorphism) the same as giving a morphism to $A \oplus B$ which is in the kernel of $\delta_+ + \varepsilon_+$ and with null projection to B. This is the same as giving a morphism to $L-$ which ε_- sends to zero, which is the same as giving a morphism to $\ker\varepsilon_-$. Therefore every monad in $\mathscr{A}_2^\flat$ has ε_- an admissible epimorphism and ε_+ an admissible monomorphism, and

$$(L_- \xhookrightarrow{(\delta_-,\varepsilon_-)} A\oplus B \xrightarrow{(\delta_+,\varepsilon_+)} L_+) \simeq (L_- \xhookrightarrow{(\delta_-,\varepsilon_-)} A\oplus L_-/\ker\varepsilon_- \xrightarrow{(\delta_+,\varepsilon_+)} A/\ker\varepsilon_-),$$

where ε_+ is induced by δ_-. The composition of functors $T_0 \circ T_2$ sends

$$(L_- \xhookrightarrow{(\delta_-,\varepsilon_-)} A \oplus B \xrightarrow{(\delta_+,\varepsilon_+)} L_+) \mapsto (\ker\varepsilon_- \hookrightarrow L_- \hookrightarrow A),$$

and by the above isomorphism is clearly an equivalence with inverse

$$T_1 : (\ker\varepsilon_- \hookrightarrow L_- \hookrightarrow A) \mapsto (L_- \hookrightarrow A \oplus L_-/\ker\varepsilon_- \twoheadrightarrow A/\ker\varepsilon_-).$$

Therefore T_1 is also an equivalence with inverse $T_0 \circ T_2$ and T_2 is an equivalence with inverse $T_1 \circ T_0$:

$$\begin{array}{c}
\mathscr{P} := P_- \hookrightarrow P \xrightarrow{\alpha_+} P_+ \\
\big\downarrow T_0 \\
P_- \hookrightarrow \ker\alpha_+ \hookrightarrow P \\
\big\downarrow T_1 \\
\ker\alpha_+ \hookrightarrow P \oplus H(\mathscr{P}) \twoheadrightarrow P/P_- \\
\big\downarrow T_2 \\
\mathscr{P}.
\end{array}$$

□

In §A2 the arrows $A \oplus B^i \to D_+$ are admissible epimorphisms, not monomorphisms.

Proof of Beilinson's Lemma A2. — Let $\mathscr{A}^\sharp =: \mathscr{A}_0^\sharp$ denote the category of diads and $\mathscr{A}_i^\sharp$ the related categories. Let $T_0^\sharp : \mathscr{A}_0^\sharp \to \mathscr{A}_1^\sharp$ and $T_1^\sharp : \mathscr{A}_1^\sharp \to \mathscr{A}_2^\sharp$ be as described; $T_0^\sharp$ is obviously an equivalence. Note that $\mathscr{A}_1^\sharp$ is a full subcategory of $\mathscr{A}_0^\flat$ and $\mathscr{A}_2^\sharp$ is a full subcategory of $\mathscr{A}_2^\flat$. Therefore $T_1^\sharp$ is just the restriction of $T_1 \circ T_0$ to $\mathscr{A}_1^\sharp$. We have to check that $T_1^\sharp$ actually lands in $\mathscr{A}_2^\sharp \subset \mathscr{A}_2^\flat$. Once that is established, if we can show that T_2 (the inverse of $T_1 \circ T_0$), restricted to $\mathscr{A}_2^\sharp$, lands in $\mathscr{A}_1^\sharp$, Lemma A1 then implies that $T_1^\sharp$ is an equivalence with inverse $T_2|_{\mathscr{A}_2^\sharp}$. We have

$$\begin{tikzcd} \mathbb{Q} := C_- \arrow[r, hook, "\alpha_-=(\alpha_-^1,\alpha_-^2)"] & A \oplus B \arrow[r, two heads, "\alpha_+=(\alpha_+^1,\alpha_+^2)"] \arrow[d, squiggly, "T_1\circ T_0=T_1^\sharp"] & C_+ \\ \ker(\alpha_+) \arrow[r, hook] & A \oplus B \oplus H(\mathbb{Q}) \arrow[r, two heads] & (A \oplus B)/C_- \end{tikzcd}$$

and to show that $T_1^\sharp$ lands in $\mathscr{A}_2^\sharp$ we must secure that $\ker(\alpha_+) \hookrightarrow A \oplus H(\mathbb{Q})$ and $A \oplus H(\mathbb{Q}) \twoheadrightarrow (A \oplus B)/C-$. The commutative diagram below has exact rows.

$$\begin{tikzcd} C- \arrow[r, hook] \arrow[d, hook] & \ker(\alpha_+) \arrow[r, two heads] \arrow[d] & H(\mathbb{Q}) \arrow[d, equal] \\ A \arrow[r, leftrightarrow] \arrow[d, two heads] & A \oplus H(\mathbb{Q}) \arrow[r, leftrightarrow] \arrow[d] & H(\mathbb{Q}) \arrow[d, equal] \\ C_+ & (A \oplus B)/C_- \arrow[l, two heads] & \ker(\alpha_+)/C_- \arrow[l, hook'] \end{tikzcd}$$

Both claims follow from the five lemma for exact categories.

It remains only to show that $T_2|_{\mathscr{A}_2^\sharp}$ lands in $\mathscr{A}_1^\sharp \subset \mathscr{A}_0^\flat$; then it will be an inverse to $T_1^\sharp$. We have

$$\begin{tikzcd} D_- \arrow[r, hook, "(\gamma_-,\delta_-^1,\delta_-^2)"] & A\oplus B^1 \oplus B^2 \arrow[r, two heads, "(\gamma_+,\delta_+^1,\delta_+^2)"] \arrow[d, squiggly, "T_2"] & D_+ \\ \ker(\delta_-^2) \arrow[r, hook] & A \oplus B^1 \arrow[r, two heads] & \operatorname{coker}((\gamma_-, \delta_-^1) : D_1 \to A \oplus B^1) \end{tikzcd}$$

and we need to verify that $\ker\delta_-^2 \to A$ is admissible monomorphism and $A \to \operatorname{coker}((\gamma_-, \delta_-^1) : D_1 \to A \oplus B^1)$ is admissible epimorphism. We know from A1 that δ_-^1 and δ_-^2 are admissible epimorphisms while δ_+^1 and δ_+^2 are admissible monomorphisms. If $\ker\delta_-^2 \to A$ admitted a cokernel, then it would follow from the obscure axiom applied to the commutative diagram with exact rows

$$\begin{tikzcd}
\ker\delta_-^2 \arrow[r, hook] \arrow[d] & D_- \arrow[r, two heads] \arrow[d, hook] & B^2 \arrow[d, equal] \\
A \arrow[r, hook] & A \oplus B^2 \arrow[r, two heads] & B^2
\end{tikzcd}$$

that $\ker\delta_-^2 \to A$ would be an admissible monomorphism. En effet,

$$\operatorname{coker}(\ker\delta_-^2 \to A) = \operatorname{coker}(D_- \xhookrightarrow{(\gamma_-,\delta_-^2)} A \oplus B^2),$$

as both can be described as the colimit of the following diagram.

$$\begin{tikzcd}
D_- \arrow[r, two heads, "-\delta_-^2"] \arrow[d, "\gamma_-"] & B^2 \\
A &
\end{tikzcd}$$

This identification additionally shows that the diagram

$$\begin{tikzcd}
\ker\delta_-^2 \arrow[r, hook] \arrow[d, hook] & A \arrow[r, two heads] \arrow[d, hook] & \operatorname{coker}(\ker\delta_-^2 \to A) \arrow[d, equal] \\
D_- \arrow[r, hook] & A \oplus B^1 \arrow[r, two heads] & \operatorname{coker}((\gamma_-, \delta_-^1) : D_- \to A \oplus B^1)
\end{tikzcd}$$

commutes so that $A \twoheadrightarrow \operatorname{coker}((\gamma_-, \delta_-^1) : D_- \to A \oplus B^1)$ is an admissible epimorphism. (Moreover the square to the left in this diagram is bicartesian [**Bühler**, 2.12].) We've proved that the functors

$$\begin{tikzcd}
\mathscr{A}_1^\sharp \arrow[r, shift left, "T_1^\sharp"] & \mathscr{A}_2^\sharp \arrow[l, shift left, "T_2|_{\mathscr{A}_2^\sharp}"]
\end{tikzcd}$$

are mutually inverse equivalences. □

The reflection functor. We wish to compute $T_2|_{\mathscr{A}_2^\sharp} \circ r \circ T_1^\sharp$, which is an involution of the category $\mathscr{A}_1^\sharp$ (since r is an involution of $\mathscr{A}_2^\sharp$ and $T_2|_{\mathscr{A}_2^\sharp} = (T_1^\sharp)^{-1}$) that we will also denote by r; the diagram below illustrates the composition.

$$\begin{tikzcd}
\mathbb{Q} := C_- \arrow[r, hook, "\alpha_-=(\alpha_-^1,\alpha_-^2)"] & A \oplus B \arrow[r, two heads, "\alpha_+=(\alpha_+^1,\alpha_+^2)"] \arrow[d, rightsquigarrow, "T_1\circ T_0=T_1^\sharp"] & C_+ \\
\ker(\alpha_+) \arrow[r, hook, "(\gamma_-,\delta_-^1,\delta_-^2)"] & A \oplus B \oplus H(\mathbb{Q}) \arrow[r, two heads, "(\gamma_+,\delta_+^1,\delta_+^2)"] \arrow[d, leftrightsquigarrow, "r"] & (A\oplus B)/C_- \\
\ker(\alpha_+) \arrow[r, hook, "(\gamma_-,\delta_-^2,\delta_-^1)"] & A \oplus H(\mathbb{Q}) \oplus B \arrow[r, two heads, "(\gamma_+,\delta_+^2,\delta_+^1)"] \arrow[d, rightsquigarrow, "T_2"] & (A\oplus B)/C_- \\
\ker(\delta_-^1) \arrow[r, hook] & A \oplus H(\mathbb{Q}) \arrow[r, two heads] & \operatorname{coker}(\ker\alpha_+ \hookrightarrow A \oplus H(\mathbb{Q}))
\end{tikzcd}$$

It is easy to see that $\ker(\delta_-^1 : \ker\alpha_+ \to B) = \ker(\alpha_+^1)$ because to give a morphism to $\ker(\alpha_+)$ which goes to zero under $\ker\alpha_+ \hookrightarrow A \oplus B \twoheadrightarrow B$ is the same as giving a morphism to A which goes to zero under α_+^1. As for $\operatorname{coker}(\ker\alpha_+ \hookrightarrow A \oplus H(\mathbb{Q}))$, to give a map from $A \oplus H(\mathbb{Q})$ that kills $\ker(\alpha_+)$ is to give a map a_1 from A and a_2 from $H(\mathbb{Q})$ such that $a_1 - a_2$ kills $\ker(\alpha_+)$; as $C_- \hookrightarrow \ker(\alpha_+) \twoheadrightarrow H(\mathbb{Q})$ is short exact, to give a map from A which coincides on $\ker(\alpha_+)$ with a map from $H(\mathbb{Q})$ is to give a map from A which annihilates C_-. Therefore $\operatorname{coker}(\ker\alpha_+ \hookrightarrow A \oplus H(\mathbb{Q})) = \operatorname{coker}(C- \hookrightarrow A)$, so

$$r(\mathbb{Q}) = \ker(\alpha_+^1) \hookrightarrow A \oplus H(\mathbb{Q}) \twoheadrightarrow \operatorname{coker}(\alpha_-^1) = A/C_-.$$

The map $\ker(\alpha_+^1) \hookrightarrow A$ is the natural inclusion and $A \twoheadrightarrow A/C_-$ the natural projection; the map $\ker(\alpha_+^1) \to H(\mathbb{Q})$ coincides with a in the below commutative diagram (with rows that are not exact)

$$\begin{tikzcd}
\ker(\alpha_+^1) \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "a"] & \ker(\alpha_+) \arrow[r, two heads] \arrow[d, hook] & H(\mathbb{Q}) \arrow[d, hook, "\iota"] \arrow[dd, bend left, "b"] \\
A \arrow[r, hook] & A \oplus B \arrow[r, two heads] & (A \oplus B)/C_- \arrow[d, "\pi"] \\
 & & A/C_-.
\end{tikzcd}$$

Finally, the map $H(\mathbb{Q}) \to A/C_-$ coincides with $-b$ in the diagram above, the negative sign appearing because we have $-\gamma_0 : P_0/P_{-1} \to P_1/P_{-1}$ in the definition of the functor T_1 so that $\delta_+^2 : H(\mathbb{Q}) \to (A \oplus B)/C_-$ coincides with $-\iota$ in the composition

$$H(\mathbb{Q}) \xrightarrow{\delta_+^2} (A \oplus B)/C_- \to A/C_-.$$

Bibliography

[B] *How to glue perverse sheaves* par Beilinson

[BBD] *Faisceaux Pervers* par Beilinson, Bernstein, Deligne (& Gabber!)

[M] *Beilinson's construction of nearby cycles and gluing* par Sophie Morel

12. *Categorical traces and a relative Lefschetz-Verdier formula*

These are notes on the article [**LZ**]. For background on bicategories, pseudofunctors, and symmetric monoidal bicategories (i.e. symmetric Gray monoid), see [**DS**]. For the definition of the Lefshetz-Verdier pairing, see [**SGA5**, III]. For another discussion of traces, see [**V**].

1.3. If $\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{Y})$ denotes the internal mapping object if it exists, this essentially means that for all Z in $\mathscr{C}$, there is an equivalence of categories

$$\mathrm{Hom}(\mathrm{Z}\otimes\mathrm{X},\mathrm{Y}) \simeq \mathrm{Hom}(\mathrm{Z},\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{Y}));$$

i.e. if $\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{Y})$ exists for every $\mathrm{Y}\in\mathrm{ob}\,\mathscr{C}$, $-\otimes\mathrm{X}$ is left 2-adjoint to $\underline{\mathrm{Hom}}(\mathrm{X},-)$.

1.4. For the 'only if' part, the morphism m is an equivalence. For the 'if' part, let's say L is the pseudofunctor $\mathscr{C}\to\mathscr{C}$ given by $-\otimes\mathrm{X}$ and R the pseudofunctor $\underline{\mathrm{Hom}}(\mathrm{X},-)$ (where defined). Let the section of m be denoted s and e the morphism obtained by adjunction from id_X. Then the map that m is adjoint to is given by ${}_\mathrm{X}\mathrm{R}m$ where ${}_\mathrm{X}$ now also denotes the counit obtained by adjuction from $\mathrm{id}_{\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{X})}$. The first diagram in the Definition 1.1 of dual can be written

$$\mathrm{X}\xrightarrow{\mathrm{R}e}\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{X})\otimes\mathrm{X}\xrightarrow{\mathrm{R}s}\mathrm{X}\otimes\underline{\mathrm{Hom}}(\mathrm{X},1_\mathscr{C})\otimes\mathrm{X}\xrightarrow{\mathrm{R}m}\mathrm{Hom}(\mathrm{X},\mathrm{X})\otimes\mathrm{X}\xrightarrow{\mathrm{X}}\mathrm{X}$$
$$\simeq\mathrm{X}\xrightarrow{\mathrm{R}e}\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{X})\otimes\mathrm{X}\xrightarrow{\mathrm{X}}\mathrm{X},$$

since $\mathrm{R}m\mathrm{R}s\simeq\mathrm{R}(ms)\simeq\mathrm{R}\,\mathrm{id}_{\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{X})}\simeq\mathrm{id}_{\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{X})\otimes\mathrm{X}}$. The second composite ${}_\mathrm{X}\mathrm{R}e$ is where e goes under the adjunction

$$\mathrm{Hom}(\mathrm{X},\mathrm{X})\simeq\mathrm{Hom}(\mathrm{id}_\mathrm{X},\underline{\mathrm{Hom}}(\mathrm{X},\mathrm{X})).$$

As we know, e was obtained from $\mathrm{id}_\mathrm{X}\in\mathrm{ob}\,\mathrm{Hom}(\mathrm{X},\mathrm{X})$ by adjunction, so that's where it goes (up to isomorphism).

12.1. Notes on Lu & Zheng.

The paper is [**LZ**].

1.10. The tensor product of 2-morphisms is defined in the obvious way: given 2-morphisms $p:(c,u)\Rightarrow(d,v)$ and $p':(c',u')\Rightarrow(d',v')$, $p\otimes p'$ gives a 2-morphism

$c \otimes c' \Rightarrow d \otimes d'$ in $\mathscr{B}$. We have to check that this makes a commutative diagram

$$\begin{tikzcd} \mathrm{F}(c\otimes c')(\mathrm{L}\boxtimes\mathrm{L}') \arrow[r, "\mathrm{F}_{c,c'}"] & \mathrm{F}(c)\mathrm{L}\boxtimes\mathrm{F}(c')(\mathrm{L}') \arrow[r, "u\boxtimes u'"] & \mathrm{M}\boxtimes\mathrm{M}' \arrow[d, equal] \\ \mathrm{F}(d\otimes d')(\mathrm{L}\boxtimes\mathrm{L}') \arrow[u, "\mathrm{F}(p\otimes p')(\mathrm{L}\boxtimes\mathrm{L}')"] \arrow[r, "\mathrm{F}_{d,d'}"] & \mathrm{F}(d)(\mathrm{L})\boxtimes\mathrm{F}(d')(\mathrm{L}') \arrow[u, "\mathrm{F}(p)(\mathrm{L})\boxtimes\mathrm{F}(p')(\mathrm{L}')"] \arrow[r, "v\boxtimes v'"] & \mathrm{M}\boxtimes\mathrm{M}'. \end{tikzcd}$$

Indeed it does: the left square commutes since $\boxtimes$ is a pseudonatural transformation $\mathrm{F}(-)\times\mathrm{F}(-)\to\mathrm{F}(-\otimes-)$.

1.12. I'd like to check the claim that the diagram defining the right-lax symmetric monoidal structure commutes 'on the nose'; i.e. with identity 2-cell. The arrow gotten by tracing right-down is

$$\mathrm{G}(c\otimes c')(\alpha_{\mathrm{X}}(\mathrm{L})\boxtimes\alpha_{\mathrm{X}'}(\mathrm{L}')) \xrightarrow{\alpha_{\mathrm{X,X}'}} \mathrm{G}(c\otimes c')(\alpha_{\mathrm{X,X}'}(\mathrm{L}\boxtimes\mathrm{L}')) \xrightarrow{\alpha_{c\otimes c'}}$$

$$\alpha_{\mathrm{Y}\otimes\mathrm{Y}'}(\mathrm{F}(c\otimes c')(\mathrm{L}\boxtimes\mathrm{L}')) \xrightarrow{\mathrm{F}_{c,c'}} \alpha_{\mathrm{Y}\otimes\mathrm{Y}'}((\mathrm{F}(c)\mathrm{L})\boxtimes(\mathrm{F}(c')\mathrm{L}')) \xrightarrow{u\boxtimes u'} \alpha_{\mathrm{Y}\otimes\mathrm{Y}'}(\mathrm{M}\boxtimes\mathrm{M}'),$$

while the arrow gotten by tracing down-right is

$$\mathrm{G}(c\otimes c')(\alpha_{\mathrm{X}}(\mathrm{L})\boxtimes\alpha_{\mathrm{X}'}(\mathrm{L}')) \xrightarrow{\mathrm{G}_{c,c'}} \mathrm{G}(c)(\alpha_{\mathrm{X}}\mathrm{L})\boxtimes\mathrm{G}(c')(\alpha_{\mathrm{X}'}\mathrm{L}') \xrightarrow{\alpha_c\boxtimes\alpha_{c'}}$$

$$\alpha_{\mathrm{Y}}(\mathrm{F}(c)\mathrm{L})\boxtimes\alpha_{\mathrm{Y}'}(\mathrm{F}(c')\mathrm{L}') \xrightarrow{u\boxtimes u'} (\alpha_{\mathrm{Y}}\mathrm{M})\boxtimes(\alpha_{\mathrm{Y}'}\mathrm{M}') \xrightarrow{\alpha_{\mathrm{Y,Y}'}} \alpha_{\mathrm{Y}\otimes\mathrm{Y}'}(\mathrm{M}\boxtimes\mathrm{M}').$$

We can exchange $u\boxtimes u'$ with $\alpha_{\mathrm{Y,Y}'}$ above by naturality of $\alpha_{\mathrm{Y,Y}'}$. We have two natural transformations obtained from the pasting diagrams

$$\begin{tikzcd} \mathrm{F}(\mathrm{X})\times\mathrm{F}(\mathrm{X}') \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{F}(c)\times\mathrm{F}(c')"'] \arrow[dr, phantom, "\Leftarrow \mathrm{F}_{c,c'}"] & \mathrm{F}(\mathrm{X}\otimes\mathrm{X}') \arrow[r, "\alpha_{\mathrm{X}\otimes\mathrm{X}'}"] \arrow[d, "\mathrm{F}(c\otimes c')"] \arrow[dr, phantom, "\Leftarrow \alpha_{c\otimes c'}"] & \mathrm{G}(\mathrm{X}\otimes\mathrm{X}') \arrow[d, "\mathrm{G}(c\otimes c')"] \\ \mathrm{F}(\mathrm{Y})\times\mathrm{F}(\mathrm{Y}') \arrow[r, "\boxtimes"] & \mathrm{F}(\mathrm{Y}\otimes\mathrm{Y}') \arrow[r, "\alpha_{\mathrm{Y}\otimes\mathrm{Y}'}"] & \mathrm{G}(\mathrm{Y}\otimes\mathrm{Y}'), \end{tikzcd} \qquad \text{and}$$

$$\begin{tikzcd} \mathrm{F}(\mathrm{X})\times\mathrm{F}(\mathrm{X}') \arrow[r, "\alpha_{\mathrm{X}}\times\alpha_{\mathrm{X}'}"] \arrow[d, "\mathrm{F}(c)\times\mathrm{F}(c')"'] \arrow[dr, phantom, "\Leftarrow \alpha_c\times\alpha_{c'}"] & \mathrm{G}(\mathrm{X})\times\mathrm{G}(\mathrm{X}') \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{G}(c)\times\mathrm{G}(c')"] \arrow[dr, phantom, "\Leftarrow \mathrm{G}_{c,c'}"] & \mathrm{G}(\mathrm{X}\otimes\mathrm{X}') \arrow[d, "\mathrm{G}(c\otimes c')"] \\ \mathrm{F}(\mathrm{Y})\times\mathrm{F}(\mathrm{Y}') \arrow[r, "\alpha_{\mathrm{Y}}\times\alpha_{\mathrm{Y}'}"] & \mathrm{G}(\mathrm{Y})\times\mathrm{G}(\mathrm{Y}') \arrow[r, "\boxtimes"] & \mathrm{G}(\mathrm{Y}\otimes\mathrm{Y}') \end{tikzcd}$$

and the point is that $\alpha_{X,X'}$ is a *modification* between these two natural transformations. This means that

$$\begin{aligned} &G(c \otimes c')(\alpha_X(L) \boxtimes \alpha_{X'}(L')) \xrightarrow{\alpha_{X,X'}} G(c \otimes c')(\alpha_{X,X'}(L \boxtimes L')) \xrightarrow{\alpha_{c\otimes c'}} \\ &\qquad \alpha_{Y\otimes Y'}(F(c \otimes c')(L \boxtimes L')) \xrightarrow{F_{c,c'}} \alpha_{Y\otimes Y'}((F(c)L) \boxtimes (F(c')L')) \end{aligned}$$

coincides with

$$\begin{aligned} &G(c \otimes c')(\alpha_X(L) \boxtimes \alpha_{X'}(L')) \xrightarrow{G_{c,c'}} G(c)(\alpha_X L) \boxtimes G(c')(\alpha_{X'} L') \xrightarrow{\alpha_c \boxtimes \alpha_{c'}} \\ &\qquad \alpha_Y(F(c)L) \boxtimes \alpha_{Y'}(F(c')L') \xrightarrow{\alpha_{Y,Y'}} \alpha_{Y\otimes Y'}((F(c)L) \boxtimes (F(c')L')). \end{aligned}$$

2.2. I'd like to check that the tensor product of 2-morphisms given by product of morphisms of schemes over S is well-defined. To this end, suppose given $(c,u) \otimes (c',u') \Rightarrow (d,v) \otimes (d',v')$, giving rise to the diagrams below.

$$\begin{tikzcd} & C \arrow[dl, "\overleftarrow{c}"'] \arrow[d, "p"] \arrow[dr, "\overrightarrow{c}"] & \\ X & D \arrow[l, "\overleftarrow{d}"] \arrow[r, "\overrightarrow{d}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & C \times_S C' \arrow[dl, "\overleftarrow{c}\times_S\overleftarrow{c}'"'] \arrow[dd, "p\times_S p'"] \arrow[dr, "\overrightarrow{c}\times_S\overrightarrow{c}'"] & \\ X \times_S X' & & Y \times_S Y' \\ & D \times_S D' \arrow[ul, "\overleftarrow{d}\times_S\overleftarrow{d}'"] \arrow[ur, "\overrightarrow{d}\times_S\overrightarrow{d}'"'] & \end{tikzcd}$$

$$\begin{tikzcd} & C' \arrow[dl, "\overleftarrow{c}'"'] \arrow[d, "p'"] \arrow[dr, "\overrightarrow{c}'"] & \\ X' & D' \arrow[l, "\overleftarrow{d}'"] \arrow[r, "\overrightarrow{d}'"'] & Y' \end{tikzcd}$$

Here, $u \otimes u'$ is

$$\overleftarrow{c}^* L \boxtimes \overleftarrow{c}'^* L' \xrightarrow{u\boxtimes u'} \overrightarrow{c}^! \boxtimes \overrightarrow{c}'^! M' \xrightarrow{\alpha} (\overrightarrow{c} \times \overrightarrow{c}')^!(M \boxtimes M')$$

and $v \otimes v'$ is

$$\overleftarrow{d}^* L \boxtimes \overleftarrow{d}'^* L' \xrightarrow{v\boxtimes v'} \overrightarrow{d}^! \boxtimes \overrightarrow{d}'^! M' \xrightarrow{\alpha} (\overrightarrow{d} \times \overrightarrow{d}')^!(M \boxtimes M'),$$

and to be a 2-morphism, $v \otimes v'$ must coincide with

$$(\overleftarrow{d} \times \overleftarrow{d}')^*(\mathrm{L} \boxtimes \mathrm{L}') \xrightarrow{\eta} (p \times p')_*(p \times p')^*(\overleftarrow{d} \times \overleftarrow{d}')^*(\mathrm{L} \boxtimes \mathrm{L}')$$
$$= (p \times p')_!(\overleftarrow{c} \times \overleftarrow{c}')^*(\mathrm{L} \boxtimes \mathrm{L}') \xrightarrow{u \boxtimes u'} (p \times p')_!(\vec{c}^{\,!}\mathrm{M} \boxtimes \vec{c}'^{\,!}\mathrm{M}')$$
$$\xrightarrow{\alpha} (p \times p')_!(\vec{c} \times \vec{c}')^!(\mathrm{M} \boxtimes \mathrm{M}') = (p \times p')_!(p \times p')^!(\vec{d} \times \vec{d}')^!(\mathrm{M} \boxtimes \mathrm{M}')$$
$$\xrightarrow{\varepsilon} (\vec{d} \times \vec{d}')^!(\mathrm{M} \boxtimes \mathrm{M}'),$$

provided that v, v' coincide with

$$\overleftrightarrow{d}^*\mathrm{L} \xrightarrow{\eta} p_*p^*\overleftrightarrow{d}^*\mathrm{L} = p_!\overleftrightarrow{c}^*\mathrm{L} \xrightarrow{u} p_!\vec{c}^{\,!}\mathrm{M} = p_!p^!\vec{d}^{\,!}\mathrm{M} \xrightarrow{\varepsilon} \vec{d}^{\,!}\mathrm{M}, \quad \text{and}$$
$$\overleftrightarrow{d}'^*\mathrm{L}' \xrightarrow{\eta} p'_*p'^*\overleftrightarrow{d}'^*\mathrm{L} = p'_!\overleftrightarrow{c}'^*\mathrm{L} \xrightarrow{u'} p'_!\vec{c}'^{\,!}\mathrm{M}' = p'_!p'^!\vec{d}'^{\,!}\mathrm{M}' \xrightarrow{\varepsilon} \vec{d}'^{\,!}\mathrm{M}',$$

respectively. Keeping in mind that p and p' are proper and the Künneth formula, this boils down to checking the commutativity of the square

$$\begin{tikzcd}
p_!p^!\vec{d}^{\,!}\mathrm{M} \boxtimes p'_!p'^!\vec{d}'^{\,!}\mathrm{M}' \arrow[r, "\varepsilon\boxtimes\varepsilon"] \arrow[d, "\alpha"] & \vec{d}^{\,!}\mathrm{M} \boxtimes \vec{d}'^{\,!}\mathrm{M}' \arrow[d, "\alpha"] \\
(p \times p')_!(p \times p')^!(\vec{d} \times \vec{d}')^!(\mathrm{M} \boxtimes \mathrm{M}') \arrow[r, "\varepsilon"] & (\vec{d} \times \vec{d}')^!(\mathrm{M} \boxtimes \mathrm{M}').
\end{tikzcd}$$

Tracing the square two different ways gives two morphisms, and to check they're the same it suffices to check that their adjoints are the same. The adjoint of [line width=0.35mm] is given by

$$(\vec{c} \times \vec{c}')_!(\vec{c}^{\,!}\mathrm{M} \boxtimes \vec{c}'^{\,!}\mathrm{M}') \xrightarrow{\alpha} (\vec{c} \times \vec{c}')_!(\vec{c} \times \vec{c}')^!(\mathrm{M} \boxtimes \mathrm{M}') \xrightarrow{\varepsilon} \mathrm{M} \boxtimes \mathrm{M}',$$

which, by the definition of α, coincides with

$$\vec{c}_!\vec{c}^{\,!}\mathrm{M} \boxtimes \vec{c}'_!\vec{c}'^{\,!}\mathrm{M}' \xrightarrow{\varepsilon\boxtimes\varepsilon} \mathrm{M} \boxtimes \mathrm{M}'. \tag{†}$$

Keeping in mind the definition of α, the adjoint of [line width=0.35mm] is given by

$$(\vec{d}\times\vec{d}')_!(p_!p^!\vec{d}^{\,!}\mathrm{M}\boxtimes p'_!p'^!\vec{d}'^{\,!}\mathrm{M}') \xrightarrow{\varepsilon\boxtimes\varepsilon} (\vec{d}\times\vec{d}')_!(\vec{d}^{\,!}\mathrm{M}\boxtimes\vec{d}'^{\,!}\mathrm{M}') \xrightarrow{\sim} \vec{d}_!\vec{d}^{\,!}\mathrm{M}\boxtimes\vec{d}'_!\vec{d}'^{\,!}\mathrm{M} \xrightarrow{\varepsilon\boxtimes\varepsilon} \mathrm{M}\boxtimes\mathrm{M}',$$

and indeed

$$\vec{d}_!p_!p^!\vec{d}^{\,!}\mathrm{M} \boxtimes \vec{d}'_!p'_!p'^!\vec{d}'^{\,!}\mathrm{M}' \xrightarrow{\varepsilon\boxtimes\varepsilon} \vec{d}_!\vec{d}^{\,!}\mathrm{M} \boxtimes \vec{d}'_!\vec{d}'^{\,!}\mathrm{M}' \xrightarrow{\varepsilon\boxtimes\varepsilon} \mathrm{M} \boxtimes \mathrm{M}'$$

coincides with (†).

2.7. Given morphisms $(c,u) : (\mathrm{X},\mathrm{L}) \to (\mathrm{Y},\mathrm{M})$ and $(d,v) : (\mathrm{Y},\mathrm{M}) \to (\mathrm{Z},\mathrm{N})$, let's check that the composition $(d,v) \circ (c,u)$ of the Grothendieck construction agrees with the formula of Constructions 1.10 and 2.6. Namely, we must check that the adjoint of

$$\vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} = \vec{d}_!\overleftarrow{d}^{*}\vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{\dot{u}} \vec{d}_!\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{\dot{v}} \mathrm{N} \quad \text{coincides with}$$

$$\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \overleftarrow{d}'^{*}\vec{c}^{!}\mathrm{M} \xrightarrow{\alpha} \vec{c}'^{!}\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{v} \vec{c}'^{!}\vec{d}^{!}\mathrm{N}, \tag{$*$}$$

where α is adjoint to

$$\vec{c}'_!\overleftarrow{d}'^{*}\vec{c}^{!}\mathrm{M} = \overleftarrow{d}^{*}\vec{c}_!\vec{c}^{!}\mathrm{M} \xrightarrow{\varepsilon} \overleftarrow{d}^{*}\mathrm{M}. \tag{‡}$$

Of course, this means that

$$\vec{c}'_!\overleftarrow{d}'^{*}\vec{c}^{!}\mathrm{M} \xrightarrow{\alpha} \vec{c}'_!\vec{c}'^{!}\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{\varepsilon} \overleftarrow{d}^{*}\mathrm{M}$$

coincides with (‡). The adjoint of $(*)$ is

$$\vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\vec{c}^{!}\mathrm{M} \xrightarrow{\alpha} \vec{d}_!\vec{c}'_!\vec{c}'^{!}\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{v} \vec{d}_!\vec{c}'_!\vec{c}'^{!}\vec{d}^{!}\mathrm{N} \xrightarrow{\varepsilon} \mathrm{N}.$$

As ε is a natural transformation,

$$\begin{tikzcd}
\vec{d}_!\vec{c}'_!\vec{c}'^{!}\overleftarrow{d}^{*}\mathrm{M} \arrow[r, "v"] \arrow[d, "\varepsilon"] & \vec{d}_!\vec{c}'_!\vec{c}'^{!}\vec{d}^{!}\mathrm{N} \arrow[d, "\varepsilon"] \\
\vec{d}_!\overleftarrow{d}^{*}\mathrm{M} \arrow[r, "v"] & \vec{d}_!\vec{d}^{!}\mathrm{N}
\end{tikzcd}$$

commutes, so we can rewrite the adjoint of $(*)$ as

$$\begin{aligned}
&\vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\vec{c}^{!}\mathrm{M} \xrightarrow{\alpha} \vec{d}_!\vec{c}'_!\vec{c}'^{!}\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{\varepsilon} \vec{d}_!\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{v} \vec{d}_!\vec{d}^{!}\mathrm{N} \xrightarrow{\varepsilon} \mathrm{N} \\
&= \vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\vec{c}^{!}\mathrm{M} = \vec{d}_!\overleftarrow{d}^{*}\vec{c}_!\vec{c}^{!}\mathrm{M} \xrightarrow{\varepsilon} \vec{d}_!\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{v} \vec{d}_!\vec{d}^{!}\mathrm{N} \xrightarrow{\varepsilon} \mathrm{N} \\
&= \vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} = \vec{d}_!\overleftarrow{d}^{*}\vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \vec{d}_!\overleftarrow{d}^{*}\vec{c}_!\vec{c}^{!}\mathrm{M} \xrightarrow{\varepsilon} \vec{d}_!\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{v} \vec{d}_!\vec{d}^{!}\mathrm{N} \xrightarrow{\varepsilon} \mathrm{N} \\
&= \vec{d}_!\vec{c}'_!\overleftarrow{d}'^{*}\overleftarrow{c}^{*}\mathrm{L} = \vec{d}_!\overleftarrow{d}^{*}\vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{\dot{u}} \vec{d}_!\overleftarrow{d}^{*}\mathrm{M} \xrightarrow{\dot{v}} \mathrm{N}.
\end{aligned}$$

Now we check that the 2-morphisms from the Grothendieck construction agree with the ones described in Construction 2.6. It amounts to showing that the adjoint of

$$\overleftrightarrow{d}^{*}\mathrm{L} \xrightarrow{\eta} p_* p^* \overleftrightarrow{d}^{*}\mathrm{L} = p_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} p_!\vec{c}^{!}\mathrm{M} = p_! p^! \vec{d}^{!}\mathrm{M} \xrightarrow{\varepsilon} \vec{d}^{!}\mathrm{M}$$

coincides with

$$\vec{d}_!\overleftrightarrow{d}^{*}\mathrm{L} \xrightarrow{\eta} \vec{d}_! p_* p^* \overleftrightarrow{d}^{*}\mathrm{L} = \vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{\dot{u}} \mathrm{M},$$

where $\dot{u}$ is the morphism obtained by adjunction from u. Indeed,

$$\begin{aligned}
&\vec{d}_!\overleftrightarrow{d}^{*}\mathrm{L} \xrightarrow{\eta} \vec{d}_! p_* p^* \overleftrightarrow{d}^{*}\mathrm{L} = \vec{d}_! p_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \vec{d}_! p_!\vec{c}^{!}\mathrm{M} = \vec{d}_! p_! p^! \vec{d}^{!}\mathrm{M} \xrightarrow{\varepsilon} \vec{d}_!\vec{d}^{!}\mathrm{M} \xrightarrow{\varepsilon} \mathrm{M} \\
&= \vec{d}_!\overleftrightarrow{d}^{*}\mathrm{L} \xrightarrow{\eta} \vec{d}_! p_* p^* \overleftrightarrow{d}^{*}\mathrm{L} = \vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{u} \vec{c}_!\vec{c}^{!}\mathrm{L} \xrightarrow{\varepsilon} \mathrm{L} \\
&= \vec{d}_!\overleftrightarrow{d}^{*}\mathrm{L} \xrightarrow{\eta} \vec{d}_! p_* p^* \overleftrightarrow{d}^{*}\mathrm{L} = \vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{\dot{u}} \mathrm{L}.
\end{aligned}$$

The last thing to verify is that the tensor product of morphisms agrees between Constructions 1.10 and 2.6. It amounts to showing that the morphisms

$$(\vec{c} \times \vec{c}')_!(\overleftarrow{c} \times \overleftarrow{c}')^*(\mathrm{L} \boxtimes \mathrm{L}') \xrightarrow{\sim} \vec{c}_!\overleftarrow{c}^{*}\mathrm{L} \boxtimes \vec{c}'_!\overleftarrow{c}'^{*}\mathrm{L}' \xrightarrow{\dot{u}\boxtimes\dot{u}'} \mathrm{M} \boxtimes \mathrm{M}'$$

$$\overleftarrow{c}^{*}\mathrm{L} \boxtimes \overleftarrow{c}'^{*}\mathrm{L}' \xrightarrow{u\boxtimes u'} \vec{c}^{!}\mathrm{M} \boxtimes \vec{c}'^{!}\mathrm{M}' \xrightarrow{\alpha} (\vec{c} \times \vec{c}')^{!}(\mathrm{M} \boxtimes \mathrm{M}')$$

are adjoint, where α is adjoint to

$$(\vec{c} \times \vec{c}')_!\vec{c}^{!}\mathrm{M} \boxtimes \vec{c}'^{!}\mathrm{M}' \xrightarrow{\sim} \vec{c}_!\vec{c}^{!}\mathrm{M} \boxtimes \vec{c}'_!\vec{c}'^{!}\mathrm{M}' \xrightarrow{\varepsilon\boxtimes\varepsilon} \mathrm{M} \boxtimes \mathrm{M}',$$

so that the adjoint of the second morphism coincides with

$$(\vec{c}\times\vec{c}')_!\overleftarrow{c}^{*}\mathrm{L}\boxtimes\overleftarrow{c}'^{*}\mathrm{L}' \xrightarrow{u\boxtimes u'} (\vec{c}\times\vec{c}')_!\vec{c}^{!}\mathrm{M}\boxtimes\vec{c}'^{!}\mathrm{M}' \xrightarrow{\sim} \vec{c}_!\vec{c}^{!}\mathrm{M}\boxtimes\vec{c}'_!\vec{c}'^{!}\mathrm{M}' \xrightarrow{\varepsilon\boxtimes\varepsilon} \mathrm{M}\boxtimes\mathrm{M}'.$$

Modulo the Künneth formula, this is the definition of $\dot{u} \boxtimes \dot{u}'$.

As for $\Omega\mathscr{C}$, the equation $\beta = p_*\alpha$ is obtained as

$$\Lambda_{\mathrm{Y}} = \overleftrightarrow{d}^{*}\Lambda \xrightarrow{\eta} p_* p^* \overleftrightarrow{d}^{*}\Lambda = p_!\overleftarrow{c}^{*}\mathrm{L} \xrightarrow{\alpha} p_!\vec{c}^{!}\Lambda = p_! p^! \vec{d}^{!}\Lambda \xrightarrow{\varepsilon} \vec{d}^{!}\Lambda = \mathrm{K}_{\mathrm{Y}}.$$

The isomorphism $\mathrm{Hom}(\Lambda_{\mathrm{Y}}, -) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Y}, -)$ as functors defined on $\mathrm{D}^+(\mathrm{Y}, \Lambda)$ is obtained from $\Lambda_{\mathrm{Y}} \to \mathrm{K}$ in $\mathrm{D}^+(\mathrm{Y}, \Lambda)$ by taking $\mathrm{H}^0\mathrm{R}\Gamma =: \mathrm{H}^0(\mathrm{Y}, -)$ and precomposing by the diagonal map $\Lambda \to \mathrm{H}^0(\mathrm{Y}, \Lambda)$, where the latter group is free on generators in bijection with the connected components of Y. Applying this recipe to the above, we find $p_*\alpha$, since $H^0(Y, \eta)$ is the diagonal map.

2.8. There remains to check that this bijection of objects actually gives an isomorphism of categories; i.e. is compatible with morphisms. To check this, it's easier to use the perspective given by the Grothendieck construction, using Remark 2.7. In other words, suppose we have a proper morphism p making the diagram

$$\begin{tikzcd} & C \arrow[d, "p"] \arrow[ddl, "x"'] \arrow[ddr, "z"] & \\ & D \arrow[dl, "x'"] \arrow[d, "y'"] \arrow[dr, "z'"'] & \\ X & Y & Z \end{tikzcd}$$

commute, and let $y := y' \circ p : C \to Y$. Suppose given morphisms

$$u : z_! x^* L \otimes z_! y^* M \to N \longleftrightarrow \dot{u} : (y \times z)_! x^* L \to R\mathscr{H}om(p_Y^* M, p_Z^! N) \quad \text{and}$$
$$v : z'_! x'^* L \otimes z'_! y'^* M \to N \longleftrightarrow \dot{v} : (y' \times z')_! x'^* L \to R\mathscr{H}om(p_Y^* M, p_Z^! N),$$

where $p_Y, p_Z : X \times_S Y \rightrightarrows X, Y$ are the projections, $\longleftrightarrow$ means 'in correspondance under adjunction,' and all products are over S. The condition that p be a 2-morphism between the undotted or dotted morphisms is that the left or right diagram below commute, respectively.

$$\begin{tikzcd} z_!(x \times y)^*(L \boxtimes M) \arrow[r, "u"] \arrow[d, equal] & N \\ z'_! p_* p^* (x' \times y')^* (L \boxtimes M) & \\ z'_!(x' \times y')^*(L \boxtimes M) \arrow[u, "\eta"] \arrow[uur, bend right, "v"'] & \end{tikzcd} \qquad \begin{tikzcd} (y \times z)_! x^* L \arrow[r, "\dot{u}"] \arrow[d, equal] & R\mathscr{H}om(p_Y^* M, p_Z^! N) \\ (y' \times z')_! p_* p^* x'^* L & \\ (y' \times z')_! x'^* L \arrow[u, "\eta"] \arrow[uur, bend right, "\dot{v}"'] & \end{tikzcd}$$

To check the isomorphism of categories, we must have that one cannot commute without the other commuting, and this is what we will show straightaway. More precisely, we will show that the adjoint of the diagram on the right is the diagram on the left. We follow the following recipe:

$$\begin{aligned}&(y \times z)_! x^* \mathrm{L} \to \mathrm{R}\mathscr{H}\!om(p_{\mathrm{Y}}^* \mathrm{M}, p_{\mathrm{Z}}^! \mathrm{N}) \\ &\rightsquigarrow (y \times z)_! x^* \mathrm{L} \otimes p_{\mathrm{Y}}^* \mathrm{M} = (y \times z)_! (x^* \mathrm{L} \otimes y^* \mathrm{M}) \to p_{\mathrm{Z}}^! \mathrm{N} \rightsquigarrow z_! (x^* \mathrm{L} \otimes y^* \mathrm{M}) \to \mathrm{N},\end{aligned}$$

where here we've used the projection formula for the equality. We apply the same recipe to the above-right diagram, noting that we also have $p_* p^*(x'^* \mathrm{L} \otimes y'^* \mathrm{M}) = p_* x^* \mathrm{L} \otimes y'^* \mathrm{M}$

by the projection formula.

$$\begin{tikzcd}
(y\times z)_! x^*\mathrm{L}\otimes p_{\mathrm{Y}}^*\mathrm{M} \arrow[r, "\dot u"] \arrow[d, equal] & \mathrm{R}\mathscr{H}\!om(p_{\mathrm{Y}}^*\mathrm{M}, p_{\mathrm{Z}}^!\mathrm{N})\otimes p_{\mathrm{Y}}^*\mathrm{M} \arrow[r, "\mathrm{ev}"] & p_{\mathrm{Z}}^!\mathrm{N} \\
(y'\times z')_! p_*p^*x'^*\mathrm{L}\otimes p_{\mathrm{Y}}^*\mathrm{M} & & \\
(y'\times z')_! x'^*\mathrm{L}\otimes p_{\mathrm{Y}}^*\mathrm{M} \arrow[u, "\eta"] \arrow[uur, bend right, "\dot v"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
(y\times z)_!(x^*\mathrm{L}\otimes y^*\mathrm{M}) \arrow[r, "\dot u"] \arrow[d, equal] & \mathrm{R}\mathscr{H}\!om(p_{\mathrm{Y}}^*\mathrm{M}, p_{\mathrm{Z}}^!\mathrm{N})\otimes p_{\mathrm{Y}}^*\mathrm{M} \arrow[r, "\mathrm{ev}"] & p_{\mathrm{Z}}^!\mathrm{N} \\
(y'\times z')_! p_*p^*(x'^*\mathrm{L}\otimes y'^*\mathrm{M}) \arrow[d, equal] & & \\
(y'\times z')_!(p_*p^*x'^*\mathrm{L}\otimes y'^*\mathrm{M}) & & \\
(y'\times z')_!(x'^*\mathrm{L}\otimes y'^*\mathrm{M}) \arrow[u, "\eta"] \arrow[uuuur, bend right, "\dot v"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
z_!(x^*\mathrm{L}\otimes y^*\mathrm{M}) \arrow[r, "\dot u"] \arrow[d, equal] \arrow[rrr, bend left, "u"] & p_{\mathrm{Z}!}\mathrm{R}\mathscr{H}\!om(p_{\mathrm{Y}}^*\mathrm{M}, p_{\mathrm{Z}}^!\mathrm{N})\otimes p_{\mathrm{Y}}^*\mathrm{M} \arrow[r, "\mathrm{ev}"] & p_{\mathrm{Z}!}p_{\mathrm{Z}}^!\mathrm{N} \arrow[r, "\varepsilon"] & \mathrm{N} \\
z'_! p_*p^*(x'^*\mathrm{L}\otimes y'^*\mathrm{M}) & & & \\
z'_!(x'^*\mathrm{L}\otimes y'^*\mathrm{M}) \arrow[u, "\eta"] \arrow[uur, bend right, "\dot v"'] \arrow[uurrr, bend right, "v"'] & & &
\end{tikzcd}$$

We're done, as this is the diagram on the left we started with.

2.9. We should verify that f and Δ satisfy the necessary compatibilities to be 2-morphisms $f_\natural f^\natural \Rightarrow \mathrm{id}_{(\mathrm{X}', f_*\mathrm{L})}$ and $\mathrm{id}_{(\mathrm{X},\mathrm{L})} \Rightarrow f^\natural f_\natural$, respectively.

$\boxed{f_\natural f^\natural \Rightarrow \mathrm{id}_{(X',f_*L)}}$ The composition diagram for $f_\natural f^\natural$ and the 2-morphism f look like

$$
\begin{tikzcd}[column sep=small]
 & & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"'] & & X \arrow[dl, "\mathrm{id}"] \arrow[dr, "f"] & \\
X' & & X & & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[d, "f"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "f"] & \\
 & X' \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \\
X' & & X'.
\end{tikzcd}
$$

The map on L corresponding to $f_\natural f^\natural$ is $f^* f_* L \xrightarrow{\varepsilon} L \xrightarrow{\eta} f^! f_! L$, while of course $\mathrm{id}_{(X',f_*L)}$ corresponds to $\mathrm{id} : f_*L \to f_*L$. So, for f to provide a 2-morphism, the composition

$$
\begin{tikzcd}
f_*L \arrow[r, "\eta"] \arrow[rr, bend left, "\mathrm{id}"] & f_*f^*f_*L \arrow[r, "\varepsilon"] & f_*L = f_!L \arrow[r, "\eta"] \arrow[rr, bend right, "\mathrm{id}"'] & f_!f^!f_!L \arrow[r, "\varepsilon"] & f_!L
\end{tikzcd}
$$

must be the identity, which it is.

$\boxed{\mathrm{id}_{(X,L)} \Rightarrow f^\natural f_\natural}$ To establish that Δ provides a 2-morphism in this situation, it's easier to work with the Grothendieck set-up, where $f_\natural$ and $f^\natural$ correspond to

$$
\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\
X & & X' \\
 & f_!L \xrightarrow{\mathrm{id}} f_!L &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
 & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\
X' & & X \\
 & f^*f_*L \xrightarrow{\varepsilon} L, &
\end{tikzcd}
$$

respectively. The composition $f^\natural f_\natural$ is given by $f^* f_* L \xrightarrow{\mathrm{id}} f^* f_* L \xrightarrow{\varepsilon} L$, keeping in mind that $f^* f_* = a_* a^*$ in the diagram below.

$$
\begin{tikzcd}
 & & X \arrow[d, "\Delta"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & & \\
 & & X \times_{X'} X \arrow[dl, "a"] \arrow[dr, "a"'] & & \\
 & X \arrow[dl, "\mathrm{id}"] \arrow[dr, "f"] & & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"'] & \\
X & & X' & & X
\end{tikzcd}
$$

In the 2-category of correspondences, Δ is a 2-morphism $\mathrm{id}_X \Rightarrow f^{\natural} f_{\natural}$. In the notation of Remark 2.7, $F(\Delta)$ is given by $a_* a^* \xrightarrow{\eta} a_* \Delta_* \Delta^* a^* L = L$. To be a 2-morphism in the category of cohomological correspondences, the diagram

$$\begin{tikzcd}
a_* a^* L \arrow[r, "\eta"] & a_* \Delta_* \Delta^* a^* L \arrow[d, equal] \\
f^* f_* L \arrow[u, "\sim"] \arrow[r, "\varepsilon"] & L
\end{tikzcd}$$

must commute. To verify that it does, we take the adjoint of both morphisms. The adjoint of the map along the bottom is $\mathrm{id} : f_* L \to f_* L$, while the adjoint of the other map is given by the composition

$$f_* L \xrightarrow{\eta} f_* f^* f_* L \xrightarrow{\sim} f_* a_* a^* L \xrightarrow{\eta} f_* a_* \Delta_* \Delta^* a^* L = f_* L. \tag{$*$}$$

Now, the base change morphism $f^* f_* \to a_* a^*$ is the map obtained by adjunction from $\eta : f_* \to f_* a_* a^*$, hence coincides with $f^* f_* \xrightarrow{\eta} f^* f_* a_* a^* \xrightarrow{\varepsilon} a_* a^*$, and the square

$$\begin{tikzcd}
f^* f_* \arrow[r, "\eta"] \arrow[d, "\varepsilon"] & f^* f_* a_* a^* \arrow[d, "\varepsilon"] \\
\mathrm{id} \arrow[r, "\eta"] & a_* a^*
\end{tikzcd}$$

commutes by naturality of either natural transformation. Therefore $(*)$ coincides with

$$\begin{tikzcd}
f_* L \arrow[r, "\eta"] \arrow[rr, bend left, "\mathrm{id}"] & f_* f^* f_* L \arrow[r, "\varepsilon"] & f_* L \arrow[r, "\eta"] & f_* a_* a^* L \arrow[r, "\eta"] & f_* a_* \Delta_* \Delta^* a^* L \arrow[r, equal] & f_* L
\end{tikzcd}$$

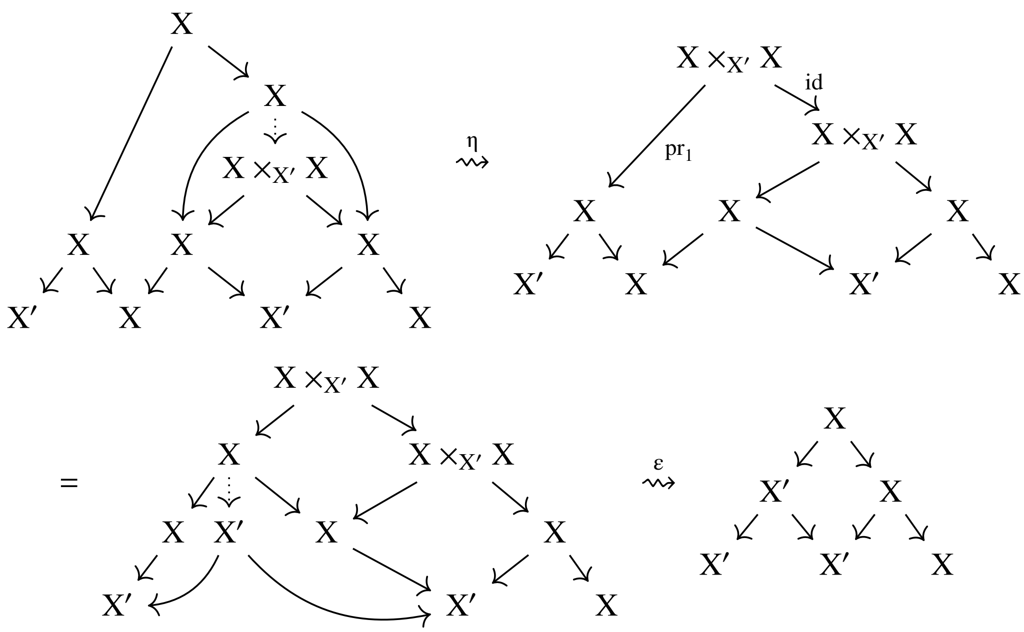
The composition of the two units above coincides with the unit of the adjunction $\mathrm{id} \to (a \circ \Delta)_* (a \circ \Delta)^* = \mathrm{id}_* \, \mathrm{id}^*$ and is thus also the identity. This verifies that Δ indeed provides a 2-morphism $\mathrm{id}_{(X,L)} \Rightarrow f^{\natural} f_{\natural}$.

It remains to verify the triangle identities; i.e.

$$\begin{tikzcd} f^{\natural} \arrow[r, "\eta"] \arrow[rr, bend left, "\mathrm{id}"] & f^{\natural} f_{\natural} f^{\natural} \arrow[r, "\varepsilon"] & f^{\natural}. \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} f_{\natural} \arrow[r, "\eta"] \arrow[rr, bend left, "\mathrm{id}"] & f_{\natural} f^{\natural} f_{\natural} \arrow[r, "\varepsilon"] & f_{\natural} \end{tikzcd} \tag{2}$$

Verifying these identities amounts to finding the corresponding maps between the relevent composite roofs and showing that they compose to the identity. For (1), we write (with all maps the obvious ones and the 2-morphisms given by dotted arrows)



The map on the peaks of the roofs are $X \xrightarrow{\Delta} X \times_{X'} X \underset{\mathrm{pr}_2}{\overset{f \circ \mathrm{pr}_1}{\rightrightarrows}} Y \times_Y X \xrightarrow{\mathrm{pr}_2} X$. The universal property assures that there are unique isomorphisms $X' \times_{X'} X \leftrightarrows X$, and these isomorphisms are given by pr_2 and the vertical arrow in the diagram

$$
\begin{tikzcd}
 & X \arrow[d] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
 & X' \times_{X'} X \arrow[dl] \arrow[dr] & \\
X' \arrow[dr, "\mathrm{id}"] & & X \arrow[dl, "f"'] \\
 & X'. &
\end{tikzcd}
$$

To check this, one verifies that the diagram below commutes.

$$
\begin{tikzcd}
 & X' \arrow[rr, "\mathrm{id}"] & & X' \\
X' \times_{X'} X \arrow[ur, "\mathrm{pr}_1"] \arrow[rr, "\mathrm{pr}_2"] \arrow[drrr, "\mathrm{pr}_2"'] & & X \arrow[ur, "f"] \arrow[r] \arrow[dr, "\mathrm{id}"] & X' \times_Y X \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
 & & & X
\end{tikzcd}
$$

Therefore the diagram (1) commutes.

The verification of (2) is the same. The diagrams look like

$$
\begin{tikzcd}[column sep=small]
 & & & & & X \arrow[dll, "\mathrm{id}"] \arrow[dddrr, "\mathrm{id}"] & & & \\
 & & & X \arrow[d, dotted] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & & & & \\
 & & & X \times_{X'} X \arrow[dll] \arrow[drr] & & & & & \\
 & X \arrow[dl] \arrow[drr] & & & & X \arrow[dll] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\
X & & & X' & & & X & & X'
\end{tikzcd}
\overset{\eta}{\rightsquigarrow}
\begin{tikzcd}[column sep=small]
 & & & & & X \times_{X'} X \arrow[dll, "\mathrm{id}"] \arrow[ddrr, "\mathrm{pr}_2"] & & & \\
 & & & X \times_{X'} X \arrow[dll] \arrow[drr] & & & & & \\
 & X \arrow[dl] \arrow[drr] & & & & X \arrow[dll] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\
X & & & X' & & & X & & X'
\end{tikzcd}
$$

$$
=
\begin{tikzcd}[column sep=small]
 & & & & & X \times_{X'} X \arrow[dll, "\mathrm{id}"] \arrow[drr, "\mathrm{pr}_2"] & & & \\
 & & & X \times_{X'} X \arrow[dll] \arrow[drr] & & & & X \arrow[dll] \arrow[d, dotted] \arrow[ddr] & \\
 & X \arrow[dl] \arrow[drr] & & & & X \arrow[dll] & & X' \arrow[dllll, bend left] \arrow[dr, bend right] & \\
X & & & X' & & & & & X'
\end{tikzcd}
\overset{\varepsilon}{\rightsquigarrow}
\begin{tikzcd}[column sep=small]
 & & X \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr] & & X' \arrow[dl] \arrow[dr] & \\
X & & X' & & X',
\end{tikzcd}
$$

so the map on roofs is $\mathrm{X} \xrightarrow{\Delta} \mathrm{X} \times_{\mathrm{X}'} \mathrm{X} \underset{f\circ \mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathrm{X} \times_{\mathrm{X}'} \mathrm{X}' \xrightarrow{\mathrm{pr}_1} \mathrm{X}$, which composes to the identity as before.

2.10. The proper morphism q is a 2-morphism between the two composite correspondences

$$\begin{tikzcd}[column sep=small] & & \mathrm{C} \arrow[dl] \arrow[dr] & & \\ & \mathrm{C} \arrow[dl] \arrow[dr] & & \mathrm{Y} \arrow[dl] \arrow[dr, "g"] & \\ \mathrm{X} & & \mathrm{Y} & & \mathrm{Y}' \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd}[column sep=small] & & \mathrm{X} \times_{\mathrm{X}'} \mathrm{C}' \arrow[dl] \arrow[dr] & & \\ & \mathrm{X} \arrow[dl] \arrow[dr, "f"] & & \mathrm{C}' \arrow[dl] \arrow[dr] & \\ \mathrm{X} & & \mathrm{X}' & & \mathrm{Y}'. \end{tikzcd}$$

2.11. (c) One replaces S by an affine neighborhood S′ of s and then passes to the limit along maps of the sort $t' \to \mathrm{U} \to \mathrm{S}$ where U is affine, étale and of finite type over S′ and t' is the spectrum of a finite field extension of the residue field of its image.

(d) Λ_{X} is a compact object of $\mathrm{D}(\mathrm{X}, \Lambda)$ $\rightsquigarrow$ see the note to 2.14 below, in light of [**SGAA**, VII 3.3]. (Recall 'coherent' means quasi-compact & quasi-separated.) Implicitly, the definition of compact object being used here is that Homs out of such an object commute with arbitrary direct sums (07LS), as this is the definition in Neeman's paper that is used to complete the proof of Lemma 2.14.

2.13. The case of a smooth morphism of relative dimension n follows via the formula $f^! = f^*(n)[2n]$ from the case of f^*, which follows since left adjoints commute with colimits. For the same reason, when f is a closed immersion, f_* commutes with colimits. For K_i in $\mathrm{D}(\mathrm{Y}, \Lambda)$ indexed by some small set, there is a map of distinguished triangles

$$\begin{tikzcd} f_*(\oplus_i f^! \mathrm{K}_i) \arrow[r] \arrow[d] & \oplus_i \mathrm{K}_i \arrow[r] \arrow[d, equal] & \oplus_i j_* j^* \mathrm{K}_i \arrow[r] \arrow[d] & {} \\ f_* f^!(\oplus_i \mathrm{K}_i) \arrow[r] & \oplus_i \mathrm{K}_i \arrow[r] & j_*(\oplus_i j^* \mathrm{K}_i) \arrow[r] & {}, \end{tikzcd}$$

so it suffices to show that j_* commutes with small direct sums. Note that in fact the reference gives more, that j_* has a right adjoint, therefore commutes with colimits. So the conclusion of Lemma 2.13 can be that $f^!$ commutes with colimits, when they exist.

2.14. The argument for why $j_!\Lambda_U[n]$ is compact for $U \to X$ étale of finite presentation under the hypothesis for each such U, (U, Λ) has finite cohomological dimension, is given in `094D`. Recall the definition of compactly generated: `09SQ` & `09SJ`. The condition that $H^{-n}(U, L) = 0$ for all j and all n (not just $n \geq 0$) implies that $L_x = 0$ for every geometric point $x \to X$ by passage to the limit.

'It follows that c-perfect complexes are compact' $\rightsquigarrow$ `09JC`, thanks to the hypothesis that (V, Λ) is of finite cohomological dimension. The proof goes by two steps: $\mathcal{H}om$ followed by global sections. Global sections commute with arbitrary direct sums thanks to the hypothesis of finite cohomological dimension. Meanwhile, $\mathcal{H}om(L, -)$ commutes with direct sums since its stalks are computed pointwise and $\mathcal{L}$ is a perfect complex, hence its stalks are perfect complexes of Λ-modules, hence are compact (`07LT`).

To apply Neeman's version of Thomason localization, we must check that perfect complexes are closed under suspension (shift), the formation of cones, and direct summands. The first is obvious, the second is `066R`, the third is `066S`.

2.15. To find $\mathrm{ev} : X \otimes \mathcal{H}om(X, 1_{\mathcal{C}}) \to 1_{\mathcal{C}}$, we use the description of the isomorphism

$$\mathrm{Hom}((X, L) \otimes (Y, M), (Z, N)) \simeq \mathrm{Hom}((X, L), \mathcal{H}om((Y, M), (Z, N))$$

in the proof of Lemma 2.8, where $(X, L) = \mathcal{H}om((X, L), 1_{\mathcal{C}})$, $(Y, M) = (X, L)$, and $(Z, N) = (S, \Lambda)$, so that the morphism on the right is $\mathrm{id} : \mathcal{H}om((X, L), 1_{\mathcal{C}}) \to \mathcal{H}om((X, L), 1_{\mathcal{C}})$ corresponding to the correspondence $X \leftarrow X \to X$ or, on the left side, to the correspondence $X \times_S X \xleftarrow{\Delta} X \to S$. Applying $c^! = \Delta^!$ to α and letting $a : X \to S$ denote the structural morphism gives the isomorphism

$$R\mathcal{H}om(D_X L \otimes L, a^!\Lambda) = R\mathcal{H}om(D_X L, D_X L),$$

and we recover from the identity on the right the usual evaluation map $D_X L \otimes L = R\mathcal{H}om(L, K_X) \otimes L \to K_X$.

The coevaluation map is a map $1_{\mathcal{C}} = (S, \Lambda) \to (X \times_S X, L \boxtimes D_X L) = (X, L) \otimes (X, L)^\vee$ corresponding by adjunction to $\mathrm{id} : (X, L) \to (X, L)$. As the equivalence $X^\vee \otimes X \simeq \mathcal{H}om(X, X)$ of Remark 1.3 (b) identifies the $(- \otimes X, - \otimes X^\vee)$ adjunction with the $(- \otimes X, \mathcal{H}om(X, -))$ adjunction, we need only ask what $\mathrm{id} : (X, L) \to (X, L)$

corresponds to in $\mathrm{Hom}(1_{\mathscr{C}}, \mathscr{H}om((X,L),(X,L)))$. We get the correspondence $S \leftarrow X \xrightarrow{\Delta} X \times_S X$ and the map $\Lambda \to R\mathscr{H}om(L,L)$ coming by adjunction from $\mathrm{id} : L \to L$.

2.16. We only have to check that the morphism of Proposition 2.5 is the one required by Lemma 1.4. In other words, we have to interpret the morphism $Y \otimes X^\vee \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} Y$ in our category $\mathscr{C}$ and show it coincides with the morphism adjoint to the morphism in Proposition 2.5. We just need to unspool what is $(Y,M) \otimes (X,L)^\vee \otimes (X,L) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} (Y,M)$. As we've just seen, the map $\mathrm{ev} : (X,L)^\vee \otimes (X,L) \to 1_{\mathscr{C}}$ is given by the correspondence $X \times_S X \xleftarrow{\Delta} X \xrightarrow{a} S$ and the map $\mathrm{ev} : L \otimes D_X L \to K_X$. The symmetric monoidal structure on $\mathscr{C}$ tells us that $- \otimes (Y,M)$ applied to this morphism gives us the morphism

$$D_X L \otimes L \boxtimes M \xrightarrow{\mathrm{ev} \boxtimes \mathrm{id}} K_X \boxtimes M = a^! \Lambda \boxtimes M \to p_Y^! M,$$

where the last map is adjoint to $p_{Y!}(a^! \Lambda \boxtimes M) \to a_! a^! \Lambda \boxtimes M \to M$. This is indeed the map adjoint to the arrow in Proposition 2.5.

2.19. Implicit is the claim that if $S = \mathrm{Spec}\, k$, $\overline{S} = \mathrm{Spec}\, k^{\mathrm{sep}}$, X is of finite type over S and Λ is a Noetherian ring, then L in $D(X, \Lambda)$ is constructible of finite tor-dimension if its inverse image on $X_{\overline{S}}$ is. The k^{sep}-rational geometric points of X factor through $X_{\overline{S}}$ and finite tor-dimension can be checked on a conservative set of points (`0DJJ`). As for constructibility, as inverse image commutes with cohomology, one need only show that a sheaf on X is constructible if its reciprocal image on $X_{\overline{S}}$ is. This is easily seen thanks to the characterization of constructible sheaves as the noetherian objects in the category of étale sheaves of Λ-modules [**SGAA**, IX 2.9] and the fact that the reciprocal image to $X_{\overline{S}}$ is conservative.

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13. SGA 4

I. Préfaisceaux.

1.4. For the isomorphism $\mathrm{Hom}_{\hat{\mathrm{C}}}(h(\mathrm{X}), \mathrm{F}) \xleftarrow{\sim} \mathrm{F}(\mathrm{X})$, note that a map belonging to the left hand side is determined by what it does to the section $\mathrm{id} : \mathrm{X} \to \mathrm{X}$ of $h(\mathrm{X})(\mathrm{X})$.

3.4. Given (X, f) in C/F, the maps $f : \mathrm{X} \to \mathrm{F}$ determine the map

$$\varinjlim_{\mathrm{C}/\mathrm{F}} \mathrm{source}(.) \longrightarrow \mathrm{F}.$$

To check this map is surjective on sections over Z, recall (1.4), which says that for a presheaf F on a $\mathcal{U}$-category C and an object Z of C, $\mathrm{Hom}_{\hat{\mathrm{C}}}(h(\mathrm{Z}), \mathrm{F}) \xleftarrow{\sim} \mathrm{F}(\mathrm{Z})$. It also says that $\mathrm{Hom}_{\hat{\mathrm{C}}}(h(\mathrm{Z}), h(\mathrm{X})) = \mathrm{Hom}_{\mathrm{C}}(\mathrm{Z}, \mathrm{X})$ when X is also an object of C. Given a section $s \in \mathrm{F}(\mathrm{Z})$, let f be the corresponding map $\mathrm{Z} \to \mathrm{F}$. Then (Z, f) is in C/F, and $\mathrm{id} : \mathrm{Z} \to \mathrm{Z}$, considered as a section in $h(\mathrm{Z})(\mathrm{Z})$, is sent to s under f.

To check this map is injective on sections over Z, suppose (X, f) and (Y, g) are in C/F, and $s \in h(\mathrm{X})(\mathrm{Z})$ and $t \in h(\mathrm{Y})(\mathrm{Z})$ are sent to the same section of $\mathrm{F}(\mathrm{Z})$. This means that the maps $s : \mathrm{Z} \to \mathrm{X}$ and $t : \mathrm{Z} \to \mathrm{Y}$ compose with f and g, respectively, to give the same map $f \circ s = g \circ t : \mathrm{Z} \to \mathrm{Y}$. Then $(\mathrm{Z}, f \circ s)$ is an object of C/F and we have maps $\mathrm{X} \xleftarrow{s} \mathrm{Z} \xrightarrow{t} \mathrm{Y}$ in C/F. Then $\mathrm{id} : \mathrm{Z} \to \mathrm{Z}$ as a section of Z is sent to s and t in $h(\mathrm{X})(\mathrm{Z})$ and $h(\mathrm{Y})(\mathrm{Z})$, respectively. Therefore $s \sim t$ in $\varinjlim_{\mathrm{C}/\mathrm{F}} \mathrm{source}(.)(\mathrm{Z})$, and the colimit commutes with 'sections over Z.'

4.2. Restriction of the presheaf corresponds to the special property of the crible.

4.3.1. The fiber product $\mathrm{R} \times_{\mathrm{X}} \mathrm{Y}$ is a fiber product in $\hat{\mathrm{C}}$. Namely, on Z in C, $(\mathrm{R} \times_{\mathrm{X}} \mathrm{Y})(\mathrm{Z})$ returns the set of morphisms from Z to Y so that, composed with f, belong to $\mathrm{R}(\mathrm{Z})$.

5.0. 'Le foncteur u^* commmute aux limites inductives et projectives' $\rightsquigarrow$ every time this is written, it means 'limites inductives *représentables*' and 'limites projectives *représentables*.' When limits and colimits in D are representable, the commutation of u^* with limits and colimits is a corollary of the fact that these are computed pointwise.

5.4. 3) The commutativity of the diagram up to isomorphism follows from

$$\mathrm{Hom}(u_!\mathrm{X},\mathrm{F}) \simeq \mathrm{Hom}(\mathrm{X},u^*\mathrm{F}) \xrightarrow{\sim} u^*\mathrm{F}(\mathrm{X}) := \mathrm{F}(u(\mathrm{X})) \xleftarrow{\sim} \mathrm{Hom}(u(\mathrm{X}),\mathrm{F}).$$

Choose a natural isomorphism $\chi : u_!h \xrightarrow{\sim} h'u$. We modify $u_!$ to make the square commute on the nose. As h is fully faithful, it induces an isomorphism of C with its image, which is a subcategory of $\hat{\mathrm{C}}$. Let

$$u_?(h\mathrm{X}) := h'u\mathrm{X}$$

$$u_?(\mathrm{F}) := \mathrm{F} \text{ if F is not in the image of } h.$$

If $f : h\mathrm{X} \to \mathrm{F}$ is any morphism with source but not target in the image of h, set $u_?(f) = u_!(f)\chi_{\mathrm{X}}^{-1}$. If $g : \mathrm{F} \to h\mathrm{X}$ is any morphism with target but not source in the image of h, set $u_?(g) = \chi_{\mathrm{X}}u_!(g)$. If $m : h\mathrm{X} \to h\mathrm{Y}$ has both source and target in the image of h, set $u_?(m) = \chi_{\mathrm{Y}}u_!(m)\chi_{\mathrm{X}}^{-1}$. Then $u_?$ is naturally isomorphic to $u_!$ and $u_?h = h'u$.

5.6. The commutativity of the diagram amounts to showing that if you start with $f : u^*u_!\mathrm{H} \to \mathrm{K}$, the two compositions

$$\mathrm{H} \xrightarrow{\eta} u^*u_!\mathrm{H} \xrightarrow{f} \mathrm{K}$$

$$\mathrm{H} \xrightarrow{\eta} u^*u_!\mathrm{H} \xrightarrow{u^*\eta} u^*u_*u^*u_!\mathrm{H} \xrightarrow{u^*u_*f} u^*u_*\mathrm{K} \xrightarrow{\varepsilon} \mathrm{K}$$

are equal. But thanks to naturality of ε we can write the commutative diagram

$$\begin{tikzcd}
\mathrm{H} \arrow[r, "\eta"] & u^*u_!\mathrm{H} \arrow[r, "u^*\eta"] \arrow[dr, "\mathrm{id}"] & u^*u_*u^*u_!\mathrm{H} \arrow[r, "u^*u_*f"] \arrow[d, "\varepsilon"] & u^*u_*\mathrm{K} \arrow[d, "\varepsilon"] \\
& & u^*u_!\mathrm{H} \arrow[r, "f"] & \mathrm{K}
\end{tikzcd}$$

and it's obvious.

5.11. If C is any category and f is the final object of $\hat{\mathrm{C}}$, then since limits are computed pointwise, f is the presheaf that returns a singleton for each object of C. Since colimits are computed pointwise, the isomorphism $f \xrightarrow{\sim} \mathrm{colim}_{\mathrm{Y}\in\mathrm{ob}\,\mathrm{C}}\,\mathrm{Y}$ follows from the bijection

$$\mathrm{colim}_{\mathrm{Y}\in\mathrm{ob}\,\mathrm{C}}\,\mathrm{Hom}(\mathrm{Z},\mathrm{Y}) = \{*\}.$$

'Or $j_{X!}(f) = X$' $\rightsquigarrow$ this follows from the description of $j_{X!}(f)$ given in (5.4) (3). Namely, given any category D and final object f of $\hat{D}$, the functor source : $D/f \to D$ is an equivalence. In the case $D = C/X$, (5.4) (3) and (3.4) give that

$$j_{X!}(f) = \operatorname*{colim}_{C/X} \text{source} = X.$$

This fact is the only one needed to conclude that $j_{X!} : \widehat{C/X} \to \hat{C}$ factors via $\hat{C}/X$.

`00Y1` has some more explanation about why e_X is an equivalence.

5.12. An object of $(C/X)/[m]$ is an object Z of C with arrow $(Z \to X)$, and map in $\widehat{C/X}$ from $(Z \to X)$ to $[m]$. The equivalence e_X restricts to representable presheaves to send $(Z \to X)$ to itself and the map $(Z \to X) \to [m]$ to $(Z \to X) \to m$. This functor induces an equivalence with C/Y since given a map $Z \to Y$, the composition with m determines the structural morphism to X.

II. Topologies et faisceaux.

1.1.1. Suppose $R \subset R'$ are cribles of X and $R \in J(X)$. Then $R' \in J(X)$ as well: since the map $Y \to X$ factors through R, $R' \times_X Y = R \times_X Y$, and the latter is in $J(Y)$ by (T1), so $R' \in J(X)$ by (T2).

Suppose now R_1 and R are in $J(X)$ and let $R' = R \cap R_1$. Then for all $Y \to R$, $R_1 \times_X Y = R' \times_X Y$, and the former is in $J(Y)$ by (T1), so R' is in $J(X)$ by (T2).

1.3.1. To prove the second claim, one has to check that the $\mathrm{Cov}(X)$ consisting of covering families of X for the topology T verifies (PT 1) and (PT 2).

(PT1) Given a covering family $(f_\alpha : X_\alpha \to X)_{\alpha\in A}$ and a morphism $Y \to X$, let $R \in J(X)$ be the crible generated by the f_α. Then the family $(f_\alpha \times_X Y)_{\alpha\in A}$ generates $R \times_X Y$, so is in $\mathrm{Cov}(Y)$ by (T1). To see this, note that $(R \times_X Y)(Z)$ are the morphisms $Z \to Y$ so that $Z \to Y \to X$ factors through one of the f_α. In other words, it's a morphism $Z \to Y$ factoring through $f_\alpha \times_X Y$ for some α.

(PT2) Let R be the crible of X generated by the $X_\alpha \to X$. Let R' be the crible of X generated by the $f_\gamma : X_\gamma \to X$. Let $Y \to R$ for $Y \in \mathrm{ob}\,C$; then $Y \to X$ factors through one of the X_α. Let R'' be the crible of X_α generated by the maps $X_{\beta_\alpha} \to X_\alpha \to X$;

$R'' \in J(X_\alpha)$ by hypothesis. Then $R'' \times_{X_\alpha} Y \subset R'' \times_X Y \subset R' \times_X Y$, where here we also use R'' to refer to its image in X. By (T1), $R'' \times_{X_\alpha} \in J(Y)$, so $R' \times_X Y$ is too. By (T2), $R' \in J(X)$.

Finally, every crible in T can be generated by a covering family: namely, if $R \in J(X)$, then the family of all $Y \to X$ in R generates R.

1.4. The topology T is the coarsest topology for which the families in the pretopology E are covering. Therefore the crible generated by a family in E is covering, and any crible containing such is covering, whence the inclusion $J'(X) \subset J_T(X)$. If $J'(X)$ is a topology, then it's a topology for which the families in E are covering, so $J_T(X) \subset J'(X)$ by definition of $J_T(X)$.

To conclude the proof, the image of $R' \times_X X_\alpha$ in X contains the image of X_{β_α}, so it suffices to show that the image of $R' \times_X X_\alpha$ in X is contained in R', but this is obvious: sections of the image are maps $Z \to R'$ with the property that the composition $Z \to R' \to X$ factors through X_α.

2.2. $R' = R' \times_X R$ is the inductive limit of the $R' \times_X Y$. The map

$$\mathrm{Hom}_{\hat{C}}(R, F_i) \to \mathrm{Hom}_{\hat{C}}(R', F_i)$$

is the limit of the bijections

$$\mathrm{Hom}_{\hat{C}}(Y, F_i) \to \mathrm{Hom}_{\hat{C}}(R' \times_X Y, F_i).$$

To check the stability of (1) under base change $Y \to X$, you have to write that if $Z \to R \times_X Y$ is any morphism, then $R \times_X Y \times_Y Z$ is in $J_{\mathcal{F}}(Z)$ since it equals $R \times_X Z$.

2.3. The topology described in (2.2) is finer than the topology generated by the $K(X)$.

2.4. We take the limit following (I 3.5) for the crible generated by the $(X_\alpha \to X)_{\alpha \in A}$. To compute this limit, we apply (I 2.12), taking $A = A$ and $i_\alpha = X_\alpha$.

2.6. The reference should be to (I 4.3), not (I 5.3).

3.1. 3) If $\ell(\mathrm{F}) \circ u = \ell(\mathrm{F}) \circ v$, that means (since $\mathrm{J}(\mathrm{Y})$ is cofiltered) there's a refinement $\mathrm{R} \hookrightarrow \mathrm{Y}$ of Y so that u and v go to the same element under the map

$$\mathrm{Hom}_{\hat{\mathrm{C}}}(\mathrm{Y}, \mathrm{F}) \to \mathrm{Hom}_{\hat{\mathrm{C}}}(\mathrm{R}, \mathrm{F}).$$

In other words, $\mathrm{R} \hookrightarrow \mathrm{F}$ factors through the equalizer of u and v. Therefore this equalizer is a sub-presheaf of Y containing the refinement R, and is therefore a refinement of Y.

2) is a simple consequence of Z_{R}, 4) of the fact that $\mathrm{J}(\mathrm{X})$ is cofiltered.

1) '$\mathrm{Z}_{\mathrm{R}\times_{\mathrm{X}}\mathrm{Y}}(u \circ g') = \ell(\mathrm{F}) \circ u \circ g$' $\rightsquigarrow$ The map $u \circ g' \in \mathrm{Hom}(\mathrm{R} \times_{\mathrm{X}} \mathrm{Y}, \mathrm{F})$ and $u \circ g \in \mathrm{Hom}(\mathrm{Y}, \mathrm{F})$ go to the same element of $\mathrm{LF}(\mathrm{Y}) = \mathrm{colim}_{\mathrm{J}(\mathrm{Y})}(-, \mathrm{F})$ since $\mathrm{R} \times_{\mathrm{X}} \mathrm{Y} \xrightarrow{\sim} \mathrm{Y}$.

3.2. 2) 'Par suite, u et v coïncident sur un raffinement $\mathrm{R}'' \hookrightarrow \mathrm{R}'$' $\rightsquigarrow$ by assumption f, g go to the same morphism under the map

$$\mathrm{Hom}(\mathrm{X}, \mathrm{LF}) \to \mathrm{Hom}(\mathrm{R}, \mathrm{LF}).$$

In particular, since $\mathrm{R} = \mathrm{colim}_{\mathrm{C}/\mathrm{R}}$ source, for each $h : \mathrm{Y} \to \mathrm{X}$ in $\mathrm{R}(\mathrm{Y})$, h factors through $\mathrm{R} \hookrightarrow \mathrm{X}$. Therefore f and g go to the same element of $\mathrm{Hom}(\mathrm{Y}, \mathrm{LF})$.

$$\begin{tikzcd}
u, v & \in & \mathrm{Hom}(\mathrm{R}', \mathrm{F}) \arrow[r, "\mathrm{Z}_{\mathrm{R}'}"] \arrow[d] & \mathrm{Hom}(\mathrm{X}, \mathrm{LF}) \arrow[d] & \ni & f, g \arrow[d, mapsto] \\
& & \mathrm{Hom}(\mathrm{R}' \times_{\mathrm{X}} \mathrm{Y}, \mathrm{F}) \arrow[r] & \mathrm{Hom}(\mathrm{Y}, \mathrm{LF}) & \ni & *
\end{tikzcd}$$

As this diagram commutes, the images of u and v in $\mathrm{Hom}(\mathrm{R}' \times_{\mathrm{X}} \mathrm{Y}, \mathrm{F})$ go to $*$, so we can find a refinement $\mathrm{R}_h \hookrightarrow \mathrm{R}' \times_{\mathrm{X}} \mathrm{Y}$ of Y so that the images of u and v in $\mathrm{Hom}(\mathrm{R}' \times_{\mathrm{X}} \mathrm{Y}, \mathrm{F})$ coincide on R_h by (3.1 4). Let $\mathrm{R}'' \hookrightarrow \mathrm{X}$ denote the full subcategory of C/X consisting of arrows $h \circ g$, with h an arrow of R and g an arrow of R_h. It's easy to check that R'' is a crible of X, since given $\mathrm{Z} \to \mathrm{X}$ factoring as $\mathrm{Z} \to \mathrm{Z}' \to \mathrm{Y} \xrightarrow{h} \mathrm{X}$ with the last arrow in $\mathrm{R}(\mathrm{Y})$ and the middle arrow in $\mathrm{R}_h(\mathrm{Z}')$, then $\mathrm{Z} \to \mathrm{Y}$ is in $\mathrm{R}_h(\mathrm{Z})$, so $\mathrm{Z} \to \mathrm{X}$ is in R''. Using (T3) we see that $\mathrm{R}'' \in \mathrm{J}(\mathrm{X})$, since for every arrow $h : \mathrm{Y} \to \mathrm{X}$ in $\mathrm{R}(\mathrm{Y})$, $\mathrm{R}_h \subset \mathrm{R}'' \times_{\mathrm{X}} \mathrm{Y}$.

Clearly $R'' \hookrightarrow R'$, and u and v go to the same element of $\mathrm{Hom}(R'', F)$, since every $Z \to X$ in $R''(Z)$ factors as

$$\begin{tikzcd}
Z \arrow[r] \arrow[drr, "g"] & R_h \arrow[r, hook] & R' \times_X Y \arrow[r] \arrow[d] & R' \arrow[r, hook] & X \\
& & Y \arrow[urr, "h"] \arrow[rr] & & R \arrow[u, hook]
\end{tikzcd}$$

and u and v go to the same element of $\mathrm{Hom}(R_h, F)$.

3) 'La projection $R'' \xrightarrow{p_2} Y$ fait de R'' un crible couvrant Y' $\rightsquigarrow$ it will suffice to show that R'' contains a covering sieve of Y. The map $u \circ m$ corresponds via (3.1 2) to a sieve $R''' \in J(Y)$ and a morphism $q : R''' \to F$, with the property that $Z_{R'''}(q) = u \circ m$. By (3.1 1), $\ell(F) \circ q = Z_{R'''}(q) \circ p_2 = u \circ m \circ p_2$, so we get the desired inclusion $R''' \hookrightarrow R''$.

'par suite $v \circ i \circ \mathrm{pr}_2 = u \circ m \circ p_2$' $\rightsquigarrow$ Should say $v \circ i \circ \mathrm{pr}_2 \circ p_1$.

3.4. One must show that every morphism from a presheaf F to a sheaf G factors uniquely through $\ell(LF)\ell(F) : F \to LLF$; it suffices to show the same for $\ell(F) : L \to LF$ instead. As ℓ is a natural transformation, any $m : F \to G$ factors as $F \xrightarrow{\ell(F)} LF \xrightarrow{Lm} LG$ followed by the inverse of the isomorphism $G \xrightarrow{\sim} LG$. As for uniqueness, given an element of $LF(X)$, we can represent it as $Z_R(u)$ for some $u : R \to F$. The square in the diagram

$$\begin{tikzcd}
R \arrow[r, hook, "i"] \arrow[d, "u"] & X \arrow[d, "Z_R(u)"] \\
F \arrow[r, "\ell(F)"] \arrow[d, "m"] & LF \arrow[dl] \\
G
\end{tikzcd}$$

commutes; assuming the triangle does, we must show that the corresponding element s of $G(X)$ is uniquely determined. As G is a sheaf, the map

$$\mathrm{Hom}_{\hat{C}}(X, G) \to \mathrm{Hom}_{\hat{C}}(R, G)$$

induced by i is a bijection, and s is uniquely determined by the property that it goes to $m \circ u$.

4.2. 'u est un monomorphisme de faisceaux' $\rightsquigarrow$ i commutes with projective limits.

'on verifie immédiatement que le diagramme ci-après est cartésien et co-cartésien' $\rightsquigarrow$ one can check that the diagram is cartesian on sections, where it follows from the following argument about sets. Suppose that $\mathrm{Y} \xleftarrow{g} \mathrm{Z} \xrightarrow{f} \mathrm{X}$ is a correspondence of sets with g injective. The pushout $\mathrm{Y} \coprod_{\mathrm{Z}} \mathrm{X}$ is the quotient of $\mathrm{Y} \coprod \mathrm{X}$ by the equivalence relation generated by $g(z) \sim f(z)$. So $x, x' \in \mathrm{X}$ are equivalent in $\mathrm{Y} \coprod_{\mathrm{Z}} \mathrm{X}$ iff they can be related by a 'zig-zag' of the form

$$x = f(z_0) \sim g(z_0) = g(z_1) \sim f(z_1) = f(z_2) \sim \cdots \sim f(z_n) = x'.$$

If g is injective, there are no non-trivial such zig-zags, because $g(z_0) = g(z_1)$ iff $z_0 = z_1$. So, $x \sim x'$ if and only if $x = x'$, and $\mathrm{X} \hookrightarrow \mathrm{Y} \coprod_{\mathrm{Z}} \mathrm{X}$ is a monomorphism. Then the fiber product of Y and X over $\mathrm{Y} \coprod_{\mathrm{Z}} \mathrm{X}$ is the subset of Y that gets sent within the image of X in $\mathrm{Y} \coprod_{\mathrm{Z}} \mathrm{X}$; this is $\mathrm{Z} \xrightarrow{g} \mathrm{Y}$.

The morphism $\underline{a}i_1$ is an isomorphism since u is an epimorphism and the diagram is cocartesian.

4.3. The exactness of the diagram

$$\mathrm{Hom}(\mathrm{X}', \mathrm{Z}) \to \prod_{i \in \mathrm{I}} \mathrm{Hom}(\mathrm{X}_i, \mathrm{Z}) \rightrightarrows \prod_{(i,k) \in \mathrm{I} \times \mathrm{I}} \mathrm{Hom}(\mathrm{X}_i \times_{\mathrm{X}'} \mathrm{X}_k, \mathrm{Z})$$

for every Z in $\hat{\mathrm{C}}$ is equivalent to the property that the family u'_i is effective epimorphic (I 10.3). In $\hat{\mathrm{C}}$, every epimorphic family is universal effective epimorphic.

4.4. The following is is a corollary of Theorem 4.4.

Corollary. — Let F *be in* $\hat{\mathrm{C}}$, X *in* C, *and* $\mathrm{F} \to \mathrm{X}$ *be a morphism. Then the image of* F *is a covering sieve of* X *if and only if the associated map* $\underline{a}\mathrm{F} \to \underline{a}\mathrm{X}$ *is an epimorphism.*

Proof. — Let $\mathrm{R} \hookrightarrow \mathrm{X}$ be the (presheaf) image of $f : \mathrm{F} \to \mathrm{X}$. As $\mathrm{F} = \mathrm{colim}_{\mathrm{C}/\mathrm{F}}\, \mathrm{source}(-)$, the family of maps $f \circ u_i : \mathrm{X}_i \to \mathrm{F}$ ($u_i \in \mathrm{ob}\, \mathrm{C}/\mathrm{F}$) generates the sieve R.

($\Rightarrow$) Assume R is a covering sieve of X. Then $f \circ u_i$ is a covering family of X, so $\underline{a}f \circ \underline{a}u_i : \underline{a}\mathrm{X}_i \to \underline{a}\mathrm{X}$ is an epimorphic family of $\tilde{\mathrm{C}}$ by (4.4), and $\underline{a}f : \underline{a}\mathrm{F} \to \underline{a}\mathrm{X}$ is an epimorphism.

($\Leftarrow$) Assume $\underline{a}f$ is an epimorphism. As $\underline{a}\mathrm{F} = \mathrm{colim}_{\mathrm{C/F}}\, \underline{a}\,\mathrm{source}(-)$ (4.1.1), given any sheaf G,

$$\mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{X}, \mathrm{G}) \hookrightarrow \mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{F}, \mathrm{G}) = \lim_{\mathrm{X}_i \in \mathrm{C/F}} \mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{X}_i, \mathrm{G}) \subset \prod_{\mathrm{X}_i \in \mathrm{C/F}} \mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{X}_i, \mathrm{G}).$$

This shows that the maps $\underline{a}f \circ \underline{a}u_i$ ($u_i \in \mathrm{ob}\,\mathrm{C/F}$) form an epimorphic family. By (4.4), the $f \circ u_i$ form a covering family; i.e. R is a covering sieve of X. $\square$

4.10. The morphism of exact sequences of sets is similar to the one in (2.4), the difference being that the products $\mathrm{X}_i \times_{\mathrm{X}} \mathrm{X}_j$ are replaced by X_{ijk}. But a section of F or F′ $\mathrm{X}_i \times_{\mathrm{X}} \mathrm{X}_j$ is known once it's known over all X_{ijk}, since G is topologically generating, again by (2.4). In other words,

$$\ker\Big(\prod_i \mathrm{F}(\mathrm{X}_i) \rightrightarrows \prod_{ij} \mathrm{F}(\mathrm{X}_i \times_{\mathrm{X}} \mathrm{X}_j)\Big) = \ker\Big(\prod_i \mathrm{F}(\mathrm{X}_i) \rightrightarrows \prod_{ijk} \mathrm{F}(\mathrm{X}_{ijk})\Big).$$

5.1. For the equivalence of (i) with the property that $\underline{a}(\mathrm{H} \times_{\mathrm{K}} \mathrm{X}) \to \underline{a}\mathrm{X}$ is an epimorphism, see the corollary in the note to 4.4.

'il en est de même de la famille $\underline{a}\mathrm{X}_i \to \underline{a}\mathrm{K}$ dans $\tilde{\mathrm{C}}$' $\rightsquigarrow$ the property that the family $\mathrm{X}_i \to \mathrm{K}$ is epimorphic is the same as that the map $\coprod_i \mathrm{X}_i \to \mathrm{K}$ is epimorphic, which is the same as $\mathrm{id} : \mathrm{K} \rightrightarrows \mathrm{K}$ being the pushout of the diagram $\mathrm{K} \leftarrow \coprod_i \mathrm{X}_i \to \mathrm{K}$; as $\underline{a}$ commutes with inductive limits, $\coprod_i \underline{a}\mathrm{X}_i \to \underline{a}\mathrm{K}$ is an epimorphism.

'il suffit de le voir après tout changement de base du type précédent' $\rightsquigarrow$ for any sheaf G,

$$\begin{array}{ccc}
\mathrm{Hom}_{\tilde{\mathrm{C}}}(\coprod_i \underline{a}\mathrm{X}_i, \mathrm{G}) & \longrightarrow & \mathrm{Hom}_{\tilde{\mathrm{C}}}(\coprod_i \underline{a}\mathrm{X}_i \times_{\underline{a}\mathrm{K}} \underline{a}\mathrm{H}, \mathrm{G}) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{K}, \mathrm{G}) & \longrightarrow & \mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{H}, \mathrm{G})
\end{array}$$

commutes, and the bottom arrow is injective if the either composition is. As the top arrow can be rewritten as a product of $\mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{X}_i, \mathrm{G}) \to \mathrm{Hom}_{\tilde{\mathrm{C}}}(\underline{a}\mathrm{X}_i \times_{\underline{a}\mathrm{K}} \underline{a}\mathrm{H}, \mathrm{G})$, it's injective if each of these components are. As we've already seen, the left vertical arrow is injective, so we're done.

5.3. (i) $\Leftrightarrow$ (i bis) is equivalent to the statement that $\mathrm{H} \times_{\mathrm{H}\times_{\mathrm{K}}\mathrm{H}} \mathrm{X} \to \mathrm{X}$ coincides with the equalizer of $\mathrm{X} \rightrightarrows \mathrm{H}$. This is clear from looking at the diagram

$$\mathrm{H} \times_{\mathrm{H}\times_{\mathrm{K}}\mathrm{H}} \mathrm{X} \to \mathrm{X} \to \mathrm{H} \times_{\mathrm{K}} \mathrm{H} \rightrightarrows \mathrm{H} \to \mathrm{K}.$$

Therefore a map $\mathrm{Y} \to \mathrm{X}$ belongs to the kernel of u and v iff it factors through $\mathrm{H} \times_{\mathrm{H}\times_{\mathrm{K}}\mathrm{H}} \mathrm{X}$.

i) $\Rightarrow$ ii) The point is that in order to check a map of sheaves is a monomorphism, it suffices to check on sections (4.1 3). Since the diagonal $\underline{a}\mathrm{H} \to \underline{a}\mathrm{H} \times_{\underline{a}\mathrm{K}} \underline{a}\mathrm{H}$ is an isomorphism, and again by (4.3 3) the formation of this fiber product commutes with taking sections, $\underline{a}f$ is a monomorphism.[5]

5.4. 'Les familles couvrantes de la topologie canonique sur $\hat{\mathrm{C}}$ sont les familles épimorphiques sur $\hat{\mathrm{C}}$' $\rightsquigarrow$ We have the following implications (I 10.3):

$$\begin{aligned} &\text{épimorphique effective universelle} \Rightarrow \text{épimorphique stricte universelle} \\ &\qquad\Rightarrow \text{épimorphique universelle.} \end{aligned}$$

Moreover, every epimorphic family of morphisms in $\hat{\mathrm{C}}$ is universally effectively epimorphic (4.3 2). Therefore a family of morphisms is a covering family for the canonical topology (i.e. is universally strictly epimorphic) iff it's epimorphic.

'la topologie $\mathrm{T_C}$ est plus fine que la topologie canonique de $\hat{\mathrm{C}}$' $\rightsquigarrow$ the functor $\underline{a}$ preserves epimorphisms since it preserves finite colimits, so by (5.1 ii), an epimorphism $\mathrm{G} \to \mathrm{F}$ in $\hat{\mathrm{C}}$ is a covering map in the sense of (5.2).

'les familles couvrantes de C sont des familles couvrantes de $\mathrm{T_C}$' $\rightsquigarrow$ since $\underline{a}$ commutes with coproducts, this is the 'easy direction' of (4.4) in light of (5.1 ii).

'La famille des s_i est couvrante pour T′' $\rightsquigarrow$ the s_i form an epimorphic family and T′ is finer than the canonical topology.

'il suffit de montrer que le morphisme $f : \mathrm{H} \to \mathrm{K}$ est couvrant pour T′' $\rightsquigarrow$ once this is established, then to show that the f_i generate a covering sieve R, it suffices to show after base change by f, but $\mathrm{R} \times_{\mathrm{K}} \mathrm{H}$ contains the sieve generated by the s_i, which is covering.

[5] For the simple reason that if you have a map of sets $\mathrm{S} \to \mathrm{T}$, $s, s' \in \mathrm{S}$ go to the same element of T, and $s \neq s'$, then $(s, s') \in \mathrm{S} \times_{\mathrm{T}} \mathrm{S}$ is not in the image of the diagonal.

'Pour montrer que $f : \mathrm{H} \to \mathrm{K}$ est couvrant pour T′, il suffit de montrer que pour tout $\lambda \in \Lambda$, le morphisme $\mathrm{pr}_2 : \mathrm{H} \times_{\mathrm{K}} \mathrm{X}_\lambda \to \mathrm{X}_\lambda$ est couvrant' ⇝ let R denote the crible generated by f; then for all $\mathrm{G} \to \mathrm{K}$ ($\mathrm{G} \in \mathrm{ob}\,\hat{\mathrm{C}}$), $\mathrm{R} \times_{\mathrm{K}} \mathrm{G}$ is generated by $\mathrm{H} \times_{\mathrm{K}} \mathrm{G} \to \mathrm{G}$.[6] Since u_λ, $\lambda \in \Lambda$, are covering, it suffices by (T2) to check that for each $g : \mathrm{G} \to \mathrm{X}_\lambda$, $\mathrm{R} \times_{\mathrm{K}} \mathrm{G}$ is a covering sieve of G. Since $\mathrm{R} \times_{\mathrm{K}} \mathrm{G} = \mathrm{R} \times_{\mathrm{K}} \mathrm{X}_\lambda \times_{\mathrm{X}_\lambda} \mathrm{G}$, by (T1) it suffices to check when $\mathrm{G} = \mathrm{X}_\lambda$. Therefore it indeed suffices to check that $\mathrm{H} \times_{\mathrm{K}} \mathrm{G} \to \mathrm{G}$ is covering.

'La famille $(\mathrm{pr}_2 \circ v_j, j \in \mathrm{J})$ est couvrante pour $\mathrm{T_C}$' ⇝ here we can use that the covering families for the topology $\mathrm{T_C}$ determine a pretopology (1.3.1). Then (PT2) gives that pr_2 is covering and (PT3) gives that $(\mathrm{pr}_2 \circ v_j, j \in \mathrm{J})$ is covering.

It remains only to check that the covering families of (5.2) are indeed the covering families of a topology $\mathrm{T_C}$. (T3) is evident. *'A sieve of* H *is covering if and only if it's generated by a family f_i such that $\coprod_i f_i$ is covering in the sense of (5.2). Note that the sieve generated by the f_i is not the same as the sieve generated by $\coprod_i f_i$, because in particular the map $\coprod_i f_i$ doesn't factor through any one of the f_i.'*

(T1): Suppose given a covering family $f_i : \mathrm{H}_i \to \mathrm{K}$ in $\hat{\mathrm{C}}$ in the sense of (5.2) generating a sieve R of K, and a map $\mathrm{F} \to \mathrm{K}$ in $\hat{\mathrm{C}}$. Then $f_i \times_{\mathrm{K}} \mathrm{F}$ generates $\mathrm{R} \times_{\mathrm{K}} \mathrm{F}$. By hypothesis, for every $\mathrm{X} \to \mathrm{K}$, with X in C, if $\mathrm{H} := \coprod_i \mathrm{H}_i$ and $f := \coprod_i f_i : \mathrm{H} \to \mathrm{K}$, $\mathrm{H} \times_{\mathrm{K}} \mathrm{X} \to \mathrm{X}$ has image a covering sieve of X. We have to show that for every $\mathrm{Y} \to \mathrm{F}$, with $\mathrm{Y} \in \mathrm{ob}\,\mathrm{C}$, $(\coprod_i (\mathrm{H}_i \times_{\mathrm{K}} \mathrm{F})) \times_{\mathrm{F}} \mathrm{Y} \to \mathrm{Y}$ has image a covering sieve of Y. As colimits in $\hat{\mathrm{C}}$ are universal, this is the same as saying that $\mathrm{H} \times_{\mathrm{K}} \mathrm{Y} \to \mathrm{Y}$ has image a covering sieve of Y, which is true by hypothesis.

(T2): Suppose given a sieve R′ of K and a covering family $(f_i : \mathrm{H}_i \to \mathrm{K}, i \in \mathrm{I})$ generating a sieve R of K with the property that for every map $g : \mathrm{G} \to \mathrm{R}$ with G in $\hat{\mathrm{C}}$, $\mathrm{R}' \times_{\mathrm{K}} \mathrm{G}$ is generated by a covering family. We must show that R′ is generated by a covering family.

For each $i \in \mathrm{I}$, $\mathrm{R}' \times_{\mathrm{K}} \mathrm{H}_i$ is generated by a covering family $(f_{ij} : \mathrm{H}_{ij} \to \mathrm{H}_i, j \in \mathrm{J}_i)$. Consider the family $(f_i \circ f_{ij} : \mathrm{H}_{ij} \to \mathrm{K}, i \in \mathrm{I}, j \in \mathrm{J}_i)$. By definition, each $f_i \circ f_{ij}$ factors through R′. The composition $\coprod_{ij} \mathrm{H}_{ij} \to \coprod_i \mathrm{H}_i \to \mathrm{K}$ has the property that for every

[6]Both sieves consist of those morphisms $\mathrm{F} \to \mathrm{G}$ in $\hat{\mathrm{C}}$ that when composed with $\mathrm{G} \to \mathrm{K}$ factor through f.

sheaf F on C,

$$\mathrm{Hom}(\mathrm{K},\mathrm{F}) \to \prod_i \mathrm{Hom}(\mathrm{H}_i,\mathrm{F}) \to \prod_{ij} \mathrm{Hom}(\mathrm{H}_{ij},\mathrm{K})$$

is a composition of injections. Therefore by (5.1 ii bis), the family of $f_i \circ f_{ij}$ is a covering family of K that factors through R′. Let $j : \mathrm{R}' \hookrightarrow \mathrm{K}$ in $\hat{\hat{\mathrm{C}}}$. The arrows $j \circ u_k$, u_k in $\hat{\mathrm{C}}/\mathrm{R}'$, generate R′, and $\underline{a}(\coprod_k j \circ u_k) = \coprod_k (\underline{a} j \underline{a} u_k)$ is an epimorphism since the $f_i \circ f_{ij}$ are among the $j \circ u_k$ and $\underline{a}(\coprod_{ij} f_i \circ f_{ij}) = \coprod_{ij} \underline{a} f_i \underline{a} f_{ij}$ is already an epimorphism. Therefore R′ is generated by a covering family.

5.5. Let's check the axioms for the set of cribles $\mathrm{J}_e(\mathrm{X})$: (T3) is trivial and (T1) follows from left-exactness of $\underline{a}'$. For (T2), we're given cribles $\mathrm{R}, \mathrm{R}' \hookrightarrow \mathrm{X}$ with $\underline{a}'\mathrm{R} \xrightarrow{\sim} \underline{a}'\mathrm{X}$, and that for every $\mathrm{Y} \to \mathrm{R}$, Y in C, $\underline{a}'(\mathrm{R}' \times_{\mathrm{X}} \mathrm{Y}) \to \underline{a}'\mathrm{Y}$ is an isomorphism. As $\underline{a}'\mathrm{R}' \hookrightarrow \underline{a}'\mathrm{X}$ is a monomorphism, it suffices to check that it's an epimorphism (II 4.2). Since $\mathrm{colim}_{\mathrm{C}/\mathrm{R}}\,\mathrm{source}(.) \xrightarrow{\sim} \mathrm{R}$, $\underline{a}'$ preserves colimits, and $\underline{a}'\mathrm{R} \to \underline{a}'\mathrm{X}$ is an epimorphism, $\coprod_{\mathrm{C}/\mathrm{R}} \underline{a}'\mathrm{Y} \to \underline{a}'\mathrm{X}$ is, too. Now we can follow the note to 5.1 (with $\underline{a}'$ in lieu of $\underline{a}$, H = R′, K = X, and X_i the objects of C/R) to conclude that $\underline{a}'\mathrm{R}' \to \underline{a}'\mathrm{X}$ is an epimorphism.

ii) ⇒ i) Assume $u : \mathrm{H} \to \mathrm{K}$ is a map of $\hat{\mathrm{C}}$ so that $\underline{a}'u$ is an isomorphism. To check that u is covering, we have to show that for every $(\mathrm{X} \to \mathrm{K}) \in \mathrm{ob}\,\mathrm{C}/\mathrm{K}$, the image of the base change of u, $\mathrm{H} \times_{\mathrm{K}} \mathrm{X} \to \mathrm{X}$, is a covering sieve of X. Let R denote this image; then we have to show $\underline{a}'\mathrm{R} \to \underline{a}'\mathrm{X}$ is an isomorphism, by the definition of the topology T_e. It's a monomorphism since $\underline{a}'$ is left-exact, and it's an epimorphism since the base change of the isomorphism $\underline{a}'u$ factors through it.

To check that u is bicovering, by (5.3 i bis) we must show that for every X in C and pair of maps $f, g : \mathrm{X} \rightrightarrows \mathrm{H}$ so that $uf = ug$, $\ker(f, g)$ is a covering sieve of X. As $\underline{a}'$ is left-exact, $\underline{a}' \ker(f, g) = \ker(\underline{a}' f, \underline{a}' g)$, and as $\underline{a}'u$ is an isomorphism, the condition that $uf = ug$ implies that $\underline{a}' f = \underline{a}' g$, so $\ker(\underline{a}' f, \underline{a}' g) \hookrightarrow \underline{a}'\mathrm{H}$ is indeed an isomorphism.

i) ⇒ ii) First, we show that given a covering morphism $u : \mathrm{H} \to \mathrm{K}$ in $\hat{\mathrm{C}}$, $\underline{a}u'$ is an epimorphism. For every $(\mathrm{X} \to \mathrm{K}) \in \mathrm{C}/\mathrm{K}$, $\underline{a}'u$ is an epimorphism by hypothesis. Using that coproducts in $\hat{\mathrm{C}}$ are universal and that $\underline{a}'$ commutes with them and is left-exact, one deduces from the logic of (5.1 i⇒ii) and its note (with $\underline{a}'$ in lieu of $\underline{a}$) that $\underline{a}'u$ is an epimorphism. From this, one deduces by the same argument as (5.3 i⇒ii) (with $\underline{a}'$ in lieu of $\underline{a}$)[7] that $\underline{a}'u$ is an isomorphism.

'Il est clair que C′ est une sous-catégorie de C_e' ⇝ if $G \in \operatorname{ob} C'$, $X \in C$ and $R \in J_e(X)$, the diagram below commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\hat{C}}(X, G) \arrow[r] \arrow[d, dash, "\wr"] & \mathrm{Hom}_{\hat{C}}(R, G) \arrow[d, dash, "\wr"] \\
\mathrm{Hom}_{C'}(\underline{a}'X, G) \arrow[r, "\sim"] & \mathrm{Hom}_{C'}(\underline{a}'R, G)
\end{tikzcd}$$

III. Fonctorialité des catégories de faisceaux.

1.2. iii) ⇒ ii) follows from the fact that $u_!X \simeq uX$ (I 5.4 3) and that a crible couvrant $R \hookrightarrow X$ is bicouvrant: it is couvrant by (II 5.1 i), so induces a surjection after applying $\underline{a}$, but this map is also an injection since $\underline{a}$ is exact, hence an isomorphism (II 4.2).

1.9.1. 'Le foncteur $(f^{-1})^s$ commute aux limites projectives finies, grâce à 1.3 5)' ⇝ one needs, by I 5.4 4) to check that finite limits in $\mathrm{Ouv}(Y)$ are representable and that f^{-1} commutes with them. The product of U and V in $\mathrm{Ouv}(Y)$ is represented by $U \cap V$. As $\mathrm{Ouv}(Y)$ is a poset, it has equalizers. Therefore it has all finite limits and f^{-1} commutes with them.

2.2. (⇐) Let q denote the monomorphism $R \hookrightarrow u(Y)$. A map $s : Z \to Y$ factors through $\hat{u}^*R \times_{\hat{u}^*u(Y)} Y \hookrightarrow Y$ iff there exists a map $t : Z \to \hat{u}^*R$ so that $ps = \hat{u}^*(q)t$. The adjoint of ps is just the map $u(s)$. By naturality of the counit, the adjoint of $\hat{u}^*(q)t$ is the adjoint of t composed with $R \hookrightarrow u(Y)$. Therefore if there exists such a t, the map $u(s)$ belongs to R. Conversely, suppose that $u(s)$ factors as $u(Z) \xrightarrow{v} R \overset{q}{\hookrightarrow} u(Y)$. Then ps, the adjoint of $u(s)$, factors as the adjoint of v, which is a map $t : Z \to \hat{u}^*R$, followed by $\hat{u}^*(q)$. This shows that a $t : Z \to \hat{u}^*R$ exists with $ps = \hat{u}^*(q)t$ iff $u(s)$ factors through R.

[7]This argument uses (4.2), but the analogue of (4.2) for C′ is true with the same proof ($\underline{a}'$ in lieu of $\underline{a}$); i.e. a morphism of C′ that is a monomorphism and an epimorphism is an isomorphism. Similarly, one checks that $\underline{a}'u$ is a monomorphism on sections, which is enabled by the fact that i is left-exact.

($\Rightarrow$) Let R' denote the crible $R \times_S u(Y)$ of $u(Y)$. The proof shows that the square

$$\begin{tikzcd} \hat{u}^*R \times_{\hat{u}^*S} Y \arrow[r, hook] \arrow[d] & Y \arrow[d, "p"] \\ \hat{u}^*R' \arrow[r, hook] & \hat{u}^*u(Y) \end{tikzcd}$$

is cartesian. We've just verified in the previous paragraph that if R' is a sieve of $u(Y)$, the sieve $\hat{u}^*R' \times_{\hat{u}^*u(Y)} Y \hookrightarrow Y$ coincides with the one defined by (COC) applied to R'. By hypothesis, R' is covering. Therefore by (COC), $\hat{u}^*R' \times_{\hat{u}^*u(Y)} Y \simeq \hat{u}^*R \times_{\hat{u}^*S} Y$ is too.

2.3. If u is cocontinuous, this means by (2.2) that $\hat{u}_* : \hat{C} \to \hat{C}'$ restricts to a functor $u_* : \tilde{C} \to \tilde{C}'$ so that $i'u_* = \hat{u}_* i$. To see that u_* is right adjoint to $u^* := \underline{a}\hat{u}^*i'$, write for sheaves F, G on C', C, respectively,

$$\begin{aligned} \mathrm{Hom}_{\tilde{C}}(u^*F, G) &= \mathrm{Hom}_{\hat{C}}(\hat{u}^*i'F, iG) = \mathrm{Hom}_{\hat{C}'}(i'F, \hat{u}_*iG) \\ &= \mathrm{Hom}_{\hat{C}'}(i'F, i'u_*G) = \mathrm{Hom}_{\tilde{C}'}(F, u_*G). \end{aligned}$$

Therefore the functor u^* commutes with inductive limits, and since $\underline{a}$, $\hat{u}^*$, and i' commute with finite projective limits, u^* is exact. The isomorphism $u^*\underline{a}' \simeq \underline{a}\hat{u}^*$ comes via adjunction from $i'u_* = \hat{u}_*i$: Given $F \in \mathrm{ob}\,\hat{C}', G \in \mathrm{ob}\,\tilde{C}$,

$$\begin{aligned} \mathrm{Hom}_{\tilde{C}}(u^*\underline{a}F, G) &= \mathrm{Hom}_{\tilde{C}'}(\underline{a}F, u_*G) = \mathrm{Hom}_{\hat{C}'}(F, i'u_*G) \\ &= \mathrm{Hom}_{\hat{C}'}(F, \hat{u}_*iG) = \mathrm{Hom}_{\hat{C}}(\hat{u}^*F, iG) = \mathrm{Hom}_{\hat{C}}(\underline{a}\hat{u}^*F, G). \end{aligned}$$

2.5. The reference is rather to I 5.5.

2.6. In light of the fact that u is continuous, the functor u_s coincides with u^*, and the functor u^s coincides with $u_!$. The reference to (I 5.7) should rather be to (I 5.6).

3.4. A covering family is said to 'majorize' another (of the same object) if every arrow in the former family factors through one of the latter.

The reference to (I 3.0.2) is rather to (II 3.0.2). That the induced topology on C is a $\mathcal{U}$-topology is equivalent to i b) in light of 3.3 (uses fullness).

Assume u is cocontinuous. By property (COC), this means that given any covering family $(Y_j \to X)_{j \in J}$ in C', the family of those maps $(X_i \to X)_{i \in I}$, $X_i \in \mathrm{ob}\,C$, that factor through a $Y_j \to X$, is covering for the induced topology on C, and hence, by (3.3), also for the topology on C′. In other words, for every covering sieve R of X for the topology on C′, there is a covering sieve $R' \subset R$ generated by arrows with their sources in C.

If, conversely, for every such R (generated by the $Y_j \to X$)[8] there is such an R′,[9] then the sieve generated by the family of maps $X_i \to X$ that factor through a $Y_j \to X$ as in (COC) contains R′, so is covering for the topology on C′, and so also for the induced topology on C by (3.3).

3.5. A) 'par suite $(\epsilon_C X_i \to \epsilon_C X, i \in I)$ est couvrante pour la topologie canonique de $\tilde{C}$' $\rightsquigarrow$ The equivalence of i) and ii bis) in (II 5.1) (for $C = \tilde{C}$) shows that $\coprod_i \epsilon_C X_i \to \epsilon_C X$ is a covering map for the canonical topology on $\tilde{C}$. Therefore it is an epimorphism. The maps $\epsilon_C X_i \to \coprod_i \epsilon_C X_i$ form an epimorphic family, so the maps $(\epsilon_C X_i \to \epsilon_C X)_{i \in I}$ do too. Now use (II 4.4) to conclude that the $(X_i \to X)_{i \in I}$ form a covering family.

B) Suppose F, as a sheaf on $\tilde{C}$, is represented by a sheaf $\mathcal{F} \in \operatorname{ob} \tilde{C}$. Then given $\mathcal{G} \in \operatorname{ob} \tilde{C}$, $F(\mathcal{G}) = \operatorname{Hom}_{\tilde{C}}(\mathcal{G}, \mathcal{F})$, so

$$(F \circ \epsilon_C)(X) = \operatorname{Hom}_{\tilde{C}}(\epsilon_C(X), \mathcal{F}) = \operatorname{Hom}_{\hat{C}}(X, \mathcal{F}) = \mathcal{F}(X),$$

so $F \circ \epsilon_C$ is a sheaf on C, as $\mathcal{F}$ is.

3.6.1. The reference is to II 1.1.6.

4.1.1. (i) $\Rightarrow$ (ii) Premier pas The category C/u^*H is isomorphic to the category written C/H, since the objects of the former category are arrows $Y \to u^*H$, which are sections $(u^*H)(Y) = H(u(Y))$, which are morphisms $u(Y) \to H$. Arrows of C/u^*H are maps $Y \to Y'$ over u^*H, which is the same condition as asking $u(Y) \to u(Y')$ to be over H.

Checking that φ coincides with the stated map is done below. The condition $(*)$ follows from the fact that the factorization is given by going back and forth between the adjunction $\operatorname{Hom}_{\hat{C}'}(u(Y), H) \simeq \operatorname{Hom}_{\hat{C}}(Y, u^*H)$. Given arrows $b, d : u(Y) \rightrightarrows u_! u^*H$, then as u, hence $u_!$, is fully faithful (I 5.6), there are unique arrows $\beta, \delta : Y \rightrightarrows u^*H$ so

[8]Note that this puts no restriction on R, since every sieve is generated by the arrows that factor through it.
[9]Covering for the topology on C′ and generated by arrows with their sources in C that factor through R; i.e. through some $Y_j \to X$.

that $b = u_!(\beta)$ and $d = u_!(\delta)$, and the adjoints of β, δ are $\varphi \circ b$ and $\varphi \circ d$, respectively. So if $\varphi \circ b = m = \varphi \circ d$, then $\beta = \delta$, and $b = d$.

Let's check that φ coincides with the obvious map $\varinjlim_{Y \in \mathrm{ob}\, C/H} u(Y) \to H$. To do so, we'll show that $u^*H \xrightarrow{\eta} \varinjlim_{Y \in \mathrm{ob}\, C/H} u^*u(Y) \to u^*H$ composes to the identity; i.e.

$$\varinjlim_{Y \in \mathrm{ob}\, C/H} Y \xrightarrow{\eta} \varinjlim_{Y \in \mathrm{ob}\, C/H} u^*u(Y) \to u^*H$$

composes to the identity. It will suffice to look on sections, so suppose given a section in $u^*H(Y)$ corresponding to an object Y of C/H.

$$\begin{tikzcd}
Y \arrow[r, "\eta"] \arrow[d] & u^*u(Y) \arrow[d] \arrow[dr] & \\
\varinjlim_{Y \in \mathrm{ob}\, C/H} Y \arrow[r, "\eta"] & \varinjlim_{Y \in \mathrm{ob}\, C/H} u^*u(Y) \arrow[r] & u^*H
\end{tikzcd}$$

The above diagram commutes and the map $Y \xrightarrow{\eta} u^*u(Y) \to u^*H$ corresponds to the adjoint of the map $u(Y) \to H$ that is part of the data of Y as an object of C/H. The isomorphism $u^*H(Y) = H(u(Y))$ is expressed by the isomorphism of adjunction

$$\mathrm{Hom}_{\hat{C}'}(u(Y), H) \simeq \mathrm{Hom}_{\hat{C}}(Y, u^*H).$$

So, tracing the diagram down right gives the map which is adjoint to the map $u(Y) \to H$ corresponding to $Y \in \mathrm{ob}\, C/H$, which is just the section in $u^*H(Y)$ that we started with.

To see that $u^*H(Y) = H(u(Y))$ corresponds to the isomorphism of adjunction, reprenons the notation of the proof of (II 5.1). The map $X \to u^*F$ corresponds to a section $\xi \in F(u(X))$. Let $Y = u(X)$ and $(X, m : Y = u(X) \xrightarrow{\mathrm{id}} u(X))$ specify an object of I_u^Y. Let id : $X \to X$ be the section of $X \circ \mathrm{pr}_{u(X)}$ applied to the object (X, m). Then the morphism

$$X \circ \mathrm{pr}_Y \to u^*F \circ \mathrm{pr}_Y$$

on sections over X sends $\mathrm{id} \in X(X)$ to the desired section of $u^*F(X) = F(u(X))$. Next we compose with the map $u^*F \circ \mathrm{pr}_{u(X)} \to F(u(X))$ via the identity (corresponding to the object of $\mathrm{I}_u^{u(\mathrm{X})}$ that we chose, in particular, m). In the end, we get the map

$$\mathrm{X}(\mathrm{X}) \to u_!(\mathrm{X})(u(\mathrm{X})) = \operatorname{colim}_{\mathrm{I}_u^{\mathrm{Y}}} \mathrm{X} \circ \mathrm{pr}_{\mathrm{Y}}(-) \to u^*\mathrm{F}(\mathrm{X}) \to \mathrm{F}(u(\mathrm{X})),$$

which sends $\mathrm{id} \in \mathrm{X}(\mathrm{X}) \mapsto \mathrm{id} \in u(\mathrm{X})(u(\mathrm{X})) \mapsto \xi$. Therefore the resulting map $u(\mathrm{X}) \to \mathrm{F}$ corresponds to ξ.

(For the last claim, that $\mathrm{id} \in \mathrm{X}(\mathrm{X})$ maps to $\mathrm{id} \in u(\mathrm{X})(u(\mathrm{X}))$, let's see how $u_!\mathrm{X}$ is identified with $u(\mathrm{X})$. We get a map

$$u_!\mathrm{X}(\mathrm{Y}) = \varinjlim_{\mathrm{Z}\in\mathrm{I}_u^{\mathrm{Y}}} \mathrm{X}(\mathrm{Z}) \to \mathrm{Hom}(\mathrm{Y}, u(\mathrm{X})) = u(\mathrm{X})(\mathrm{Y})$$

by considering that an object of $\mathrm{I}_u^{\mathrm{Y}}$ corresponds to a map $\mathrm{Y} \to u(\mathrm{Z})$, and this composed with u applied to the map $\mathrm{Z} \to \mathrm{X}$ coming from a section in $\mathrm{X}(\mathrm{Z})$ gives a map $\mathrm{Y} \to u(\mathrm{Z}) \to u(\mathrm{X})$; i.e. a section in $u(\mathrm{X})(\mathrm{Y})$. This map is functorial in Y and an isomorphism. The map $\mathrm{id} \in \mathrm{X}(\mathrm{X})$ is sent to itself in $\operatorname{colim}_{\mathrm{I}_u^{\mathrm{Y}}} \mathrm{X} \circ \mathrm{pr}_{\mathrm{Y}}(-)$ corresponding to the object (X, m) of $\mathrm{I}_u^{\mathrm{Y}}$. By the above, this corresponds to $\mathrm{id} \in u(\mathrm{X})(u(\mathrm{X}))$.)

Deuxième pas It's enough to check that $u^*(i) : u^*\mathrm{R} \to \mathrm{Y}$ is covering because, in light of the isomorphism $u^*u(\mathrm{Y}) \simeq \mathrm{Y}$, this crible is the same as the crible $u^*\mathrm{R}\times_{u^*u(\mathrm{Y})}\mathrm{Y} \to \mathrm{Y}$ of (2.2), which is the same as the crible of (COC).[10]

'$u^*u_!\mathrm{Y} \simeq \mathrm{Y}$' $\rightsquigarrow$ again by (I 5.6).

For the stated factorization of $u_!u^*(i)$: the adjoint of the factorization is the map $u^*(i) : u^*(\mathrm{R}) \to u^*u(\mathrm{Y})$. The adjoint of $u_!u^*(i) : u_!u^*(\mathrm{R}) \to u(\mathrm{Y})$ is found by tracing

$$\begin{tikzcd}
u^*(\mathrm{R}) \arrow[r, "\eta"] \arrow[d, "u^*(i)"] & u^*u_!u^*(\mathrm{R}) \arrow[d, "u^*u_!u^*(i)"] \\
\mathrm{Y} \arrow[r, "\eta"] & u^*u(\mathrm{Y})
\end{tikzcd}$$

right-down. The square commutes by naturality of η. The last thing to remark is that the down arrow on the left is actually the composition of $u^*(i) : u^*(\mathrm{R}) \to u^*u(\mathrm{Y})$ with the inverse of the isomorphism η_{Y}.

[10]With more details in the note to 2.2.

(ii) $\Rightarrow$ (i) We have a continuous, fully faithful functor $u : \mathrm{C} \to \mathrm{C}'$ and the topology on C' is subcanonical. Then the topology on C is necessarily subcanonical, since given any $\mathrm{X} \in \operatorname{ob} \mathrm{C}$, the sheaf $u^*(u(\mathrm{X}))$ coincides with X as a presheaf.

'$u_s u\mathrm{Y} = \mathrm{Y}$' $\rightsquigarrow$ we've already seen this above with u_s and $u\mathrm{Y}$ written as u^* and $u_!\mathrm{Y}$, respectively, since $u_!$ is fully faithful.

The reference to (II 5.1) should rather be to (II 4.4).

5.1. The functor $j_{\mathrm{X}!}$ commutes with all connected (nonempty) limits because, using the factorization of (I 5.11), an equivalence preserves all limits, and the forgetful functor from a slice category preserves all connected (nonempty) limits (`002T`).

The functor j_{X} induces an isomorphism $(\mathrm{C}/\mathrm{X})/(\mathrm{Y} \to \mathrm{X}) \simeq \mathrm{C}/\mathrm{Y}$ of categories, for any $(\mathrm{Y} \to \mathrm{X})$ in C/X, and sets up a bijection of cribles of $(\mathrm{Y} \to \mathrm{X})$ with cribles of Y. Since $j_{\mathrm{X}!}$ coincides with j_{X} on representable presheaves, and preserves colimits, given a crible R of $(\mathrm{Y} \to \mathrm{X})$,

$$j_{\mathrm{X}!}\mathrm{R} = j_{\mathrm{X}!}(\varinjlim_{(\mathrm{C}/\mathrm{X})/\mathrm{R}} \mathrm{source}(.)) = \varinjlim_{(\mathrm{C}/\mathrm{X})/\mathrm{R}} j_{\mathrm{X}} \circ \mathrm{source}(.).$$

This presheaf injects into Y and is the crible of Y corresponding to R.

5.2. 1) Using (T1) and the fact that $j_{\mathrm{X}!}$ commutes with fiber products and coincides with j_{X} on objects of C/X.

2) We have to check that for every $(\mathrm{Y} \to \mathrm{X})$ in C/X and crible $\mathrm{R} \hookrightarrow \mathrm{Y}$ in C, that the crible T of $(\mathrm{Y} \to \mathrm{X})$ generated by those arrows $a : (\mathrm{Z} \to \mathrm{X}) \to (\mathrm{Y} \to \mathrm{X})$ so that $u(a)$ belongs to R, covers Y. Since $j_{\mathrm{X}!}$ sets up a bijective correspondence of sieves of $(\mathrm{Y} \to \mathrm{X})$ and sieves of Y, we can assume that a sieve $\mathrm{R} \hookrightarrow \mathrm{Y}$ is of the form $j_{\mathrm{X}!}\mathrm{R}'$ for R' a sieve of some $(\mathrm{Y} \to \mathrm{X})$ in C/X. Then $\mathrm{R}' = \mathrm{T}$ and 1) gives the conclusion.

3) One can use (I 5.12) to write a diagram

$$\begin{tikzcd}
 & (\mathrm{C}/\mathrm{X})/[m] \arrow[dl, "e_m"'] \arrow[dr, "j_{[m]}"] & \\
\mathrm{C}/\mathrm{Y} \arrow[rr, "j_m"] \arrow[dr, "j_{\mathrm{Y}}"'] & & \mathrm{C}/\mathrm{X} \arrow[dl, "j_{\mathrm{X}}"] \\
 & \mathrm{C}. &
\end{tikzcd}$$

Letting e_m^{-1} denote a quasi-inverse for e_m, the functors j_m and $j_{[m]} \circ e_m^{-1}$ are naturally isomorphic. Therefore j_m^* and $(j_{[m]} \circ e_m^{-1})^*$ are naturally isomorphic, and so $j_{m!}$ and $(j_{[m]} \circ e_m^{-1})_!$ are too. As both $(e_m^{-1})_!$ and $j_{[m]!}$ preserve connected limits, $j_{m!}$ does too. Using the criterion of (3.2), we see that a crible R of $(\mathrm{Z} \to \mathrm{Y}) \in \mathrm{ob}\, \mathrm{C}/\mathrm{Y}$ is covering for the topology induced by j_m iff $j_{m!}(\mathrm{R}) \hookrightarrow (\mathrm{Z} \to \mathrm{Y} \xrightarrow{m} \mathrm{X})$ is, which by 1) is convering iff $j_{\mathrm{X}!}(j_{m!}(\mathrm{R})) = (j_{\mathrm{X}} \circ j_m)_!(\mathrm{R}) = j_{\mathrm{Y}!}(\mathrm{R}) \hookrightarrow \mathrm{Z}$ is.

4) Given an object $(\mathrm{Z} \to \mathrm{X})$ of C/X, there is a covering family of $\mathrm{G}_i \to \mathrm{Z}$ (for the topology on C) with arrows whose source is in the family $(\mathrm{G}_i)_{i \in \mathrm{I}}$. Call by R the sieve generated by these maps. The map $\mathrm{Z} \to \mathrm{X}$ allows us to endow each G_i that appears in the covering family with a structural arrow to X, and the resulting crible $\mathrm{R}' \hookrightarrow (\mathrm{Z} \to \mathrm{X})$ generated by these maps $(\mathrm{G}_i \to \mathrm{X}) \to (\mathrm{Z} \to \mathrm{X})$ has $j_!\mathrm{R}' = \mathrm{R}$, so it's covering by 1).

5.4. Decorating with a hat the corresponding functors on presheaves, the functor $\hat{j}_{\mathrm{X}!} : \widehat{\mathrm{C}/\mathrm{X}} \to \hat{\mathrm{C}}$ factors via the equivalence $e_{\mathrm{X}} : \widehat{\mathrm{C}/\mathrm{X}} \to \hat{\mathrm{C}}/\mathrm{X}$ and sends the terminal object ϵ of $\widehat{\mathrm{C}/\mathrm{X}}$ to X. The functor $j_!$ on sheaves is given by $\underline{a}\hat{j}_!i$ and therefore sends ϵ (which is a sheaf) to $\underline{a}\mathrm{X}$. It follows that $\tilde{e}_{\mathrm{X}}$ is naturally isomorphic to $\underline{a}e_{\mathrm{X}}i$.

The last assertion about the composite functor $\tilde{e}_{\mathrm{X}} \circ j_{\mathrm{X}}^*$ can be seen as follows. Given a sheaf F on C, the sheaf $\mathrm{F} \times \mathrm{X} \xrightarrow{\mathrm{pr}_2} \mathrm{X}$ has $\mathrm{Hom}_{\hat{\mathrm{C}}/\mathrm{X}}((\mathrm{Y} \to \mathrm{X}), (\mathrm{F} \times \mathrm{X} \xrightarrow{\mathrm{pr}_2} \mathrm{X})) = \mathrm{F}(\mathrm{Y})$ for any $\mathrm{Y} \to \mathrm{X}$ in C/X, so $j_{\mathrm{X}}^*\mathrm{F}$ is isomorphic to the sheaf $\mathscr{F}$ on C/X that's sent via the equivalence e_{X} to $\mathrm{F} \times \mathrm{X} \xrightarrow{\mathrm{pr}_2} \mathrm{X}$, and

$$\tilde{e}_{\mathrm{X}}(\mathscr{F}) = \underline{a}e_{\mathrm{X}}(\mathscr{F}) = \underline{a}(\mathrm{F} \times \mathrm{X} \xrightarrow{\mathrm{pr}_2} \mathrm{X}) = \mathrm{F} \times \underline{a}\mathrm{X} \xrightarrow{\mathrm{pr}_2} \underline{a}\mathrm{X}.$$

To show that $\tilde{e}_X$ is an equivalence (`00Y1`): Recall first and foremost the quasi-inverse to e_X which sends $(F \xrightarrow{\gamma} X)$ to the presheaf F_γ on C/X given by

$$F_\gamma(Y \xrightarrow{a} X) := \mathrm{Hom}_{\hat{C}/X}((Y \xrightarrow{a} X), (F \xrightarrow{\gamma} X)).$$

Given $(Y \xrightarrow{a} X) \in \mathrm{ob}\, C/X$, the unit $\eta : \epsilon \to \hat{j}_X^* \hat{j}_{X!}\epsilon = \hat{j}_X^*(X)$ sends $* \in \epsilon(Y \to X)$ to

$$a \in \mathrm{Hom}_{\hat{C}}(Y, X) = X(Y) = X(j_X(Y \xrightarrow{a} X)) = \hat{j}_X^* X(Y \xrightarrow{a} X).$$

Therefore the square

$$\begin{tikzcd} F_\gamma \arrow[r] \arrow[d] & \epsilon \arrow[d, "\eta"] \\ \hat{j}_X^* F \arrow[r, "\gamma"] & \hat{j}_X^* X \end{tikzcd}$$

is cartesian. Similarly, we define a functor $\tilde{C}/\underline{a}X \to \widetilde{C/X}$ by sending $F \xrightarrow{\delta} \underline{a}X$ to

$$\begin{tikzcd} F_\delta \arrow[r] \arrow[d] & \epsilon \arrow[d, "\eta"] \\ j_X^* F \arrow[r, "\delta"] & j_X^*(\underline{a}X), \end{tikzcd}$$

where $\eta : \epsilon \to j_X^* j_{X!}\epsilon = j_X^*(\underline{a}X)$.[11] Sections of F_δ are therefore given by

$$F_\delta(Y \xrightarrow{a} X) = \mathrm{Hom}_{\hat{C}/X}((Y \xrightarrow{a} X), (iF \times_{i\underline{a}X} X \xrightarrow{\mathrm{pr}_2} X)), \tag{$*$}$$

so F_δ coincides with the quasi-inverse of e_X applied to $(iF \times_{i\underline{a}X} X \xrightarrow{\mathrm{pr}_2} X)$. In other words, $F_\delta(Y \xrightarrow{a} X)$ is the set of dotted arrows making the diagram

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "a"] & iF \arrow[d, "\delta"] \\ X \arrow[r] & i\underline{a}X \end{tikzcd}$$

[11]Which is the composition of $\eta : \epsilon \to \hat{j}_X^* X$ with $\hat{j}_X^*(X \to i\underline{a}X)$.

commute. We get a square of functors

$$\begin{tikzcd} \hat{C}/X \arrow[r] \arrow[d] & \widehat{C/X} \arrow[d] \\ \tilde{C}/\underline{a}X \arrow[r] & \widetilde{C/X}, \end{tikzcd} \tag{†}$$

where the vertical arrows are induced by sheafification. To check that the diagram commutes up to natural isomorphism, as $\underline{a}$ is exact, it suffices to check that $j_X^*\underline{a} \simeq \underline{a}\hat{j}_X^*$. This is true since j_X is cocontinuous (2.3 2).

This implies i) ⇔ ii). For i) ⇒ ii), suppose $(F \xrightarrow{\gamma} X) \in \operatorname{ob} \hat{C}/X$ corresponds to a sheaf $\mathcal{F}$ on C/X. Then by commutativity of (†), $\mathcal{F} \simeq (\underline{a}F)_{\underline{a}\gamma}$, which we know from (∗) corresponds via e_X to the object $i\underline{a}F \times_{i\underline{a}X} X \xrightarrow{\mathrm{pr}_2} X \in \operatorname{ob} \hat{C}/X$, so indeed these two objects of $\hat{C}/X$ are isomorphic. For ii) ⇒ i), just note that given any $(F \xrightarrow{\delta} aX) \in \operatorname{ob} \tilde{C}/aX$, the sheaf $F_\delta \in \operatorname{ob} \widetilde{C/X}$ constructed above corresponds via e_X to $iF \times_{i\underline{a}X} X \xrightarrow{\mathrm{pr}_2} X$.

The equivalence of the two properties i) and ii) implies that $\tilde{e}_X$ is an equivalence as follows. Given a sheaf G in $\tilde{C}$ and a map $\gamma : G \to \underline{a}X$, the presheaf $G' := G \times_{i\underline{a}X} X \xrightarrow{\mathrm{pr}_2} X$ is in the image of the restriction of e_X to $\widetilde{C/X}$, since $\underline{a}(G' \xrightarrow{\mathrm{pr}_2} X) = G \xrightarrow{\gamma} \underline{a}X$ by exactness of $\underline{a}$, so the diagram

$$\begin{tikzcd} G' \arrow[r] \arrow[d, "\mathrm{pr}_2"] & i\underline{a}G' \simeq iG \arrow[d, "i\gamma"] \\ X \arrow[r] & i\underline{a}X \end{tikzcd}$$

is indeed cartesian. The image of the sheaf on C/X corresponding to $G' \xrightarrow{\mathrm{pr}_2} X$ under $\tilde{e}_X$ is $\underline{a}(G' \xrightarrow{\mathrm{pr}_2} X) = G \xrightarrow{\gamma} \underline{a}X$. Therefore $\tilde{e}_X$ is essentially surjective.

Given maps of presheaves $f : F \to X$ and $h : H \to X$ making the cartesian squares of ii), since e_X is an equivalence, if $\mathcal{F}, \mathcal{H}$ are the corresponding sheaves on C/X,

$$\operatorname{Hom}_{\widetilde{C/X}}(\mathcal{F}, \mathcal{H}) = \operatorname{Hom}_{\hat{C}/X}(f, h),$$

and $\tilde{e}_X$ induces the map

$$\underline{a} : \operatorname{Hom}_{\hat{C}/X}(f, h) \to \operatorname{Hom}_{\tilde{C}/\underline{a}X}(\underline{a}f, \underline{a}h).$$

We've already seen the inverse map: it associates to a map $q : \underline{a}f \to \underline{a}h$ the map

$$iq \times_{i\underline{a}\mathrm{X}} \mathrm{X} : \mathrm{F} \simeq i\underline{a}\mathrm{F} \times_{i\underline{a}\mathrm{X}} \mathrm{X} \to i\underline{a}\mathrm{G} \times_{i\underline{a}\mathrm{X}} \mathrm{X} \simeq \mathrm{G},$$

which is a map over X since the isomorphisms identify pr_2 with f and h, respectively. Then indeed $\underline{a}(iq \times_{i\underline{a}\mathrm{X}} \mathrm{X}) = q$. On the other hand, given a map $q' : f \to h$, we need to show that the square

$$\begin{tikzcd} \mathrm{F} \arrow[r, "\sim"] \arrow[d, "q'"] & i\underline{a}\mathrm{F} \times_{i\underline{a}\mathrm{X}} \mathrm{X} \arrow[d, "i\underline{a}q'"] \\ \mathrm{H} \arrow[r, "\sim"] & i\underline{a}\mathrm{H} \times_{i\underline{a}\mathrm{X}} \mathrm{X} \end{tikzcd}$$

commutes. These maps correspond via the projections $i\underline{a}\mathrm{H} \times_{i\underline{a}\mathrm{X}} \mathrm{X} \rightrightarrows i\underline{a}\mathrm{H}, \mathrm{X}$ to maps to $i\underline{\mathrm{H}}$ and X. The maps to X agree, as $f = hq'$, and the maps to $i\underline{a}\mathrm{H}$ are $i\underline{a}q' \circ \eta_{\mathrm{F}}$ and $\eta_{\mathrm{H}} \circ q'$. These agree by naturality of the unit. Thus $\tilde{e}_{\mathrm{X}}$ is fully faithful.

Here's another way to show that $\tilde{e}_{\mathrm{X}}$ is an equivalence: we have a second square

$$\begin{tikzcd} \widehat{\mathrm{C}/\mathrm{X}} \arrow[r, "e_{\mathrm{X}}"] \arrow[d] & \hat{\mathrm{C}}/\mathrm{X} \arrow[d] \\ \widetilde{\mathrm{C}/\mathrm{X}} \arrow[r, "\tilde{e}_{\mathrm{X}}"] & \tilde{\mathrm{C}}/\underline{a}\mathrm{X}, \end{tikzcd} \tag{‡}$$

which we know commutes up to natural isomorphism since $\tilde{e}_{\mathrm{X}}$ is naturally isomorphic to $\underline{a}e_{\mathrm{X}}i$, and we have a natural isomorphism $j_{\mathrm{X}!}\underline{a} \simeq \underline{a}\hat{j}_{\mathrm{X}!}$ (1.3 3).

Putting the squares (†, ‡) together two different ways, we know that the top rows compose to give natural autoequivalences of $\hat{\mathrm{C}}/\mathrm{X}$ and $\widehat{\mathrm{C}/\mathrm{X}}$. The same will follow for the bottom rows if we can isolate subcategories of $\widehat{\mathrm{C}/\mathrm{X}}$ and $\hat{\mathrm{C}}/\mathrm{X}$ so that the restriction of sheafification to these subcategories are full and essentially surjective functors to $\widetilde{\mathrm{C}/\mathrm{X}}$ and $\tilde{\mathrm{C}}/\underline{a}\mathrm{X}$, respectively.

For $\widehat{\mathrm{C}/\mathrm{X}}$, we just pick the full subcategory $\widetilde{\mathrm{C}/\mathrm{X}}$. For $\hat{\mathrm{C}}/\mathrm{X}$, we pick the essential image of the functor $\tilde{\mathrm{C}}/\underline{a}\mathrm{X} \to \hat{\mathrm{C}}/\mathrm{X}$ that sends $\mathrm{F} \xrightarrow{\delta} \underline{a}\mathrm{X}$ to $i\mathrm{F} \times_{i\underline{a}\mathrm{X}} \mathrm{X} \xrightarrow{\mathrm{pr}_2} \mathrm{X}$ and on morphisms is defined by the adjunctive bijection of dotted arrows

$$\begin{tikzcd} iF \times_{i\underline{a}X} X \arrow[r, dotted] \arrow[d, "pr_2"] & iG \arrow[d] \\ X \arrow[r, "\eta"] & i\underline{a}X \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} F \arrow[rr, dotted] \arrow[dr, "\delta"] & & G \arrow[dl] \\ & \underline{a}X & \end{tikzcd}$$

provided by $\underline{a}(iF \times_{i\underline{a}X} X \xrightarrow{pr_2} X) \simeq (F \xrightarrow{\delta} \underline{a}X)$. This functor is fully faithful, and the restriction of $\underline{a}$ to its essential image (which is just the full subcategory of $\hat{C}/X$ generated by the $iF \times_{i\underline{a}X} X$) provides a quasi-inverse; therefore in particular the restriction of $\underline{a}$ to this subcategory is full and essentially surjective.

Now we're in the following general situation: we have a functor $a : C \to C'$, endofunctors $r : C \to C$, $r' : C' \to C'$ with the property that $r'a$ is naturally isomorphic to ar, and a full subcategory D of C so that $a|_D$ is full and essentially surjective. Then $r' \simeq \mathrm{id}_{C'}$ if $r \simeq \mathrm{id}_C$, since given objects aF and aG of C' with $F, G \in \mathrm{ob}\, D$, every map between them is of the form af for $f : F \to G$ in D, so the natural isomorphisms

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\wr"] & G \arrow[d, "\wr"] \\ rF \arrow[r, "rf"] & rG \end{tikzcd} \quad \text{give rise to natural isomorphisms} \quad \begin{tikzcd} aF \arrow[r, "af"] \arrow[d, "\wr"] & aG \arrow[d, "\wr"] \\ arF \arrow[r, "arf"] \arrow[d, "\wr"] & arG \arrow[d, "\wr"] \\ r'aF \arrow[r, "r'af"] & r'aG. \end{tikzcd}$$

5.5. 1) The commutativity of the last square follows from the fact that $(F \xrightarrow{\delta} X)$ represents a sheaf $\mathcal{F}$ on C/X (via e_X) iff $(F \xrightarrow{\delta} X) \simeq (F \times_{i\underline{a}X} X \xrightarrow{pr_2} X)$, and $\tilde{e}_X(\mathcal{F}) = \underline{a}(F \xrightarrow{\delta} X)$. Therefore $i\tilde{e}_X(\mathcal{F}) \times_{iaX} X \simeq (F \xrightarrow{\delta} X)$.

2) With the notation of (I 5.12), the map g is induced by $j_{[m]!}$, so $\underline{a}_X Y$ is the sheafification of the presheaf on C/X specified, via e_X, by $Y \xrightarrow{m} X$. The definition of the map f uses the commutativity of (‡) up to natural isomorphism, since the equivalence $\tilde{e}_X$ sends $\underline{a}_X Y$ to $(\underline{a}Y \xrightarrow{am} \underline{a}X) \in \mathrm{ob}\, \tilde{C}/\underline{a}X$. The arrow $\tilde{e}_m$ is just $e_{m!}$ where e_m is defined in (I 5.12). The commutativity of the triangle of (I 5.12) up to natural isomorphism

means that we can draw an arrow induced by $j_{m!}$ so that the triangle in the diagram

$$\begin{tikzcd}
\tilde{C}/\underline{a}X/\underline{a}Y \arrow[dd, equal] & (\widetilde{C/X})/\underline{a}_X Y \arrow[l, "f"'] & \\
& & \widetilde{C/X/Y} \arrow[ul, "g"] \arrow[dl, "\tilde{e}_m"] \\
\tilde{C}/\underline{a}Y & \widetilde{C/Y} \arrow[l, "\tilde{e}_Y"] \arrow[uu] &
\end{tikzcd}$$

commutes up to natural isomorphism. The commutativity of the square up to natural isomorphism follows from $j_Y = j_X \circ j_m$.

3) The commutativity of the right square follows from the last sentence of (5.4).

IV. Topos. ‘We have seen various exactness properties of categories of the form $\tilde{\mathscr{C}}$ = category of sheaves of sets on $\mathscr{C}$, where $\mathscr{C}$ is a small site, properties that one can express by saying that in many respects, these categories (that we call *topos*) inherit familiar properties of the category (Ens) of (small) sets. On the other hand, experience has taught us that there is reason to consider diverse situations in Mathematics *above all as technical means to construct corresponding categories of sheaves* (of sets), i.e. *the corresponding ‘topos.’* It appears that all the truly important notions related to a site (for example, its cohomological invariants, various other ‘topological’ invariants, such as its homotopy invariants recently studied by M. Artin and B. Mazur, and the notions studied in the book of J. Giraud related to non-commutative cohomology) are in fact directly expressed in terms of the associated topos. From this point of view, it’s appropriate to view two sites as being essentially equivalent when their associated topos are equivalent categories, and to consider that the data of a site (at least in the case, especially important in practice, where its topology is finer than the canonical topology) is contained in that of a topos $\mathscr{E}$ (namely the associated topos, consisting of sheaves of sets on the site), and of a generating family of elements of $\mathscr{E}$. This point of view is analogous to the one that associates a group to a system of generators and relations, and which takes the structure of this group as the object of interest, rather than the system of generators and relations that gave rise to it (which are considered auxiliary data in this situation). Moreover the ‘comparison lemma’ provides numerous examples of pairs of sites $\mathscr{C}, \mathscr{C}'$, non-isomorphic, and not even equivalent as categories, which give rise to equivalent topos, so that there is reason to consider $\mathscr{C}$ and $\mathscr{C}'$ as essentially equivalent.'

1.2.1. The proof of the corollary using the comparison lemma rests on the fact that a family of objects $\mathfrak{X}$ of a topos E with the canonical topology form a generating family iff every object of E can be covered for the canonical topology by arrows with their sources in objects in $\mathfrak{X}$. This is because covering families in E are just epimorphic families, and the different notions (I 7.1) of generating subcategories coincide (II 4.9); in particular, a subcategory is generating iff it's generating by epimorphisms.

1.2.3. Let R denote the sieve of H which is generated by the $u : X_i \to H$, $(u, i) \in I(H)$. Indeed by (II 5.1) it suffices to show that $H' := \underline{a}R \to X$ is an epimorphism, where $\underline{a}$ is sheafification for the canonical topology on E. As $\underline{a}$ is left-exact, $H' \hookrightarrow H$, so $\mathrm{Hom}(X_i, H') \hookrightarrow \mathrm{Hom}(X_i, H)$ is injective for each i. Let $u : X_i \to H$. Then $(u, i) \in I(H)$, so u factors through $R \hookrightarrow H$. As $R \hookrightarrow H$ coincides with $R \to i\underline{a}R \hookrightarrow H$, u factors through $H' \hookrightarrow H$, and the map of Homs is indeed a bijection for each i. The definition of generating subcategory then tells that $H' \hookrightarrow H$ is an isomorphism.

1.2.4. For the first point, we know that $\coprod_{i\in I} G_i$ is representable in $\tilde{E}$ and also (conflating the G_i with the objects of E representing them) in E, by b). We would like to know that J_E of the latter is the former. For this, we use (II 4.6 2). By abuse of notation, let $s_i : G_i \to \coprod_{i\in I} G_i$ be the morphisms in E (where the latter coproduct is in E). As coproducts in E are disjoint, conditions b) and c) of (i) are verified. To see condition a), that the s_i are covering for the canonical topology on E, it suffices to know that they form a universally effectively epimorphic family (I 10.3). This is obvious since given any object G' of E,

$$\mathrm{Hom}_E(\coprod_{i\in I} G_i, G') \to \prod_{i\in I} \mathrm{Hom}_E(G_i, G')$$

is a bijection, and this remains true after any base change as coproducts in E are universal.

'le diagramme $K \rightrightarrows G \to H$ est exact' $\rightsquigarrow$ this is assertion b) in the proof of (II 4.3 2).

'K est le carré fibré de G au-dessus de H' $\rightsquigarrow$ the commutation of colimits with finite limits in $\tilde{E}$ implies that $K \rightrightarrows G$ can be identified with $G \times_H G \rightrightarrows G$. Writing K and G for the objects of E that represent them, this implies that $\mathrm{K} \rightrightarrows \mathrm{G}$ is an equivalence relation in E, since for every $\mathrm{A} \in \operatorname{ob} \mathrm{E}$, the image of the map $\mathrm{K(A)} \to \mathrm{G(A)} \times \mathrm{G(A)}$ consists of those sections (a, a') that are sent by $\mathrm{G(A)} \to \mathrm{H(A)}$ to the same section.

The image G of the family $(u : \mathrm{X}_\alpha \to \mathrm{H}, (u, \alpha) \in \mathrm{I(H)})$ is the sheafification of the presheaf image. It is the universal factorization of the arrows u; i.e. for every $(u, \alpha) \in \mathrm{I(H)}$, u factors through $\mathrm{G} \hookrightarrow \mathrm{H}$, and G is universal for this property. In particular,

$$\mathrm{Hom}(\mathrm{X}_\alpha, \mathrm{G}) \to \mathrm{Hom}(\mathrm{X}_\alpha, \mathrm{H})$$

is a bijection for all $\alpha \in \mathrm{A}$. As the $(\mathrm{X}_\alpha)_{\alpha \in \mathrm{A}}$ form a family of generators, $\mathrm{G} \xrightarrow{\sim} \mathrm{H}$.

From the description of G as the sheafification of the presheaf image $\mathrm{R} \hookrightarrow \mathrm{H}$, the family $(u : \mathrm{X}_\alpha \to \mathrm{G}, (u, \alpha) \in \mathrm{I(H)})$ is seen to be epimorphic, since for any presheaf F,

$$\mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \mathrm{F}) \to \prod_{(u,\alpha) \in \mathrm{I(H)}} \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{X}_\alpha, \mathrm{F})$$

is injective, but when F is moreover a sheaf, $\mathrm{Hom}_{\tilde{\mathrm{E}}}(\mathrm{G}, \mathrm{F}) \to \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \mathrm{F})$ is bijective.

'$\mathrm{X}_\alpha \times_{\mathrm{H}} \mathrm{X}_\beta$ est représentable' $\rightsquigarrow$ limits in $\tilde{\mathrm{E}}$ can be computed in $\hat{\mathrm{E}}$, hence can be computed on sections. Given a diagram in E, its limit, if representable, represents the limit of the diagram in $\hat{\mathrm{E}}$.

1.4. The reason this proposition exists is that the Yoneda embedding h does not preserve colimits in general.

'permet de considérer X comme une limite inductive. . . ' $\rightsquigarrow$ as a family of maps $(\mathrm{X}_i \to \mathrm{X})_{i \in \mathrm{I}}$ is covering for the canonical topology on E iff they are universally strictly epimorphic;[12] i.e.

$$\mathrm{Hom}(\mathrm{X}, \mathrm{Y}) \to \prod_{i \in \mathrm{I}} \mathrm{Hom}(\mathrm{X}_i, \mathrm{Y}) \rightrightarrows \prod_{(i,j) \in \mathrm{I} \times \mathrm{I}} \mathrm{Hom}(\mathrm{X}_i \times_{\mathrm{X}} \mathrm{X}_j, \mathrm{Y})$$

is exact for all $\mathrm{Y} \in \operatorname{ob} \mathrm{E}$. This is the same as X being the colimit of the stated diagram (which is disconnected as long as $|\mathrm{I}| > 1$). We conclude by (II 2.4).

[12]Recall (II 4.3) that in fact universal strict epimorphism, universal effective epimorphism, and epimorphism are all the same notion in E.

1.5. If the $\mathcal{U}$-presheaf $\mathrm{Hom}(f(-), \mathrm{Y})$ is a sheaf for all $\mathrm{Y} \in \mathrm{ob}\,\mathrm{E}'$, it defines a functor $\mathrm{E}' \to \tilde{\mathrm{E}}$, which, composed with a quasi-inverse to $\mathrm{J_E} : \mathrm{E} \to \tilde{\mathrm{E}}$, yields a functor $g : \mathrm{E}' \to \mathrm{E}$ with the property that

$$\mathrm{Hom}(f(\mathrm{X}), \mathrm{Y}) \simeq \mathrm{Hom}(\mathrm{X}, g(\mathrm{Y}))$$

for all $\mathrm{X} \in \mathrm{ob}\,\mathrm{E}$, $\mathrm{Y} \in \mathrm{ob}\,\mathrm{E}'$, and conversely.

2.2. If $\emptyset$ is the empty topological space, then $\mathrm{Ouv}(\emptyset)$ is equivalent to the punctual category. The topology on this category is defined by two cribles of $*$: $*$ itself and $\emptyset$.

2.3. This example is discussed further at the end of the appendix to Mac Lane & Moerdijk, *Sheaves in Geometry and Logic*. The crucial point is that the functor that forgets the G-action creates all finite limits and all colimits. It can be helpful to describe these G-equivariant sheaves in terms of the espace étalé associated to a sheaf (Hartshorne II exercise 1.13). A brief review: given a presheaf $\mathcal{F}$ on a topological space X, given an open set $\mathrm{U} \subset \mathrm{X}$ and $s \in \mathcal{F}(\mathrm{U})$, s determines a map $\mathrm{U} \to \mathrm{Sp\acute{e}}(\mathrm{X}) := \coprod_{\mathrm{P}\in\mathrm{X}} \mathcal{F}_{\mathrm{P}}$, and we give $\mathrm{Sp\acute{e}}(\mathrm{X})$ the final topology for all such sections s. The projection $\pi : \mathrm{Sp\acute{e}}(\mathrm{X}) \to \mathrm{X}$ sends a section of $\mathcal{F}_{\mathrm{P}}$ to P, and the sheafification of $\mathcal{F}$ has over U the set of all continuous sections of π over U. By definition, $\mathrm{Y} \subset \mathrm{Sp\acute{e}}(\mathrm{X})$ is open if its preimage under all pairs (s, U) where $\mathrm{U} \subset \mathrm{X}$ is open and $s \in \mathcal{F}(\mathrm{U})$ is open; this preimage consists of those points P of U so that $(\mathrm{P}, s_{\mathrm{P}}) \in \mathrm{Y}$. As any two sections of $\mathcal{F}$ representing the same element of a stalk coincide on an open set, given (s, U), the subset $\{(\mathrm{P}, s_{\mathrm{P}}) : \mathrm{P} \in \mathrm{U}\}$ is open in $\mathrm{Sp\acute{e}}(\mathrm{X})$, and such subsets form a basis for the topology on $\mathrm{Sp\acute{e}}(\mathrm{X})$. In conclusion, a continuous section of π over $\mathrm{U} \subset \mathrm{X}$ is nothing other than an assignment $\mathrm{P} \mapsto s_{\mathrm{P}} \in \mathcal{F}_{\mathrm{P}}$ for every point $\mathrm{P} \in \mathrm{U}$, so that for every P there is a neighborhood $\mathrm{P} \in \mathrm{V_P} \subset \mathrm{U}$ and a section $t \in \mathcal{F}(\mathrm{V_P})$ so that $t_{\mathrm{Q}} = s_{\mathrm{Q}}$ for all $\mathrm{Q} \in \mathrm{V_P}$.

2.7.2. We check that the description of a continuous G-action on a set E is the same as the description given in (2.7.1). Given a continuous G-action on E, write $\mathrm{G} = \varprojlim_{i\in\mathrm{I}} \mathrm{G}_i$ with G_i finite, let $\mathrm{N}_i := \ker(\mathrm{G} \to \mathrm{G}_i)$, and let E_i denote the subset of elements of E fixed by N_i. Then $\mathrm{E} = \bigcup_{i\in\mathrm{I}} \mathrm{E}_i$ because the stabilizer of any given point $x \in \mathrm{E}$ is open, and the N_i form a fundamental system of neighborhoods of $e \in \mathrm{G}$. The action of G on E_i factors through G_i and we have a group action in the sense of (2.7.1).

On the other hand, given a G-action in the sense of (2.7.1), with the G_i finite, if $x \in E$, then $x \in E_i$ for some i, and G acts on x via $G \longrightarrow G_i$. If $x \in E_j$, $j \geq i$, then the action of G_j on G factors through $G_j \longrightarrow G_i$. The resulting action of G on E is continuous, since the stabilizer of x in G is the preimage of the stabilizer of x in G_i.

4.3. The functor defined in terms of T is *left* adjoint, not right adjoint, to the functor $X \mapsto \mathrm{Hom}(T, X)$ from E to (Ens).

4.7.4. To show necessity, the functor f given by the restriction of $F_!$ to C is continuous by the criterion of (III 1.2 iv), with f^s given by $F_!$. Therefore $\tilde{f}^* = F^*$ by the uniqueness of right adjoint up to canonical isomorphism. To see f is cocontinuous, given $G \in \mathrm{ob}\, \tilde{C}'$, $H \in \mathrm{ob}\, \tilde{C}$,

$$\mathrm{Hom}_{\hat{C}'}(i'G, \hat{f}_*H) \simeq \mathrm{Hom}_{\hat{C}}(\hat{f}^* i'G, H) = \mathrm{Hom}_{\tilde{C}}(F^*G, H) \simeq \mathrm{Hom}_{\tilde{C}'}(G, F_*H).$$

As $E' \simeq \tilde{C}'$, C' is a subcategory of E', and $\hat{f}_*H$ is a presheaf on C', $\hat{f}_*H$ is determined by the restriction of the functor $\mathrm{Hom}_{\hat{C}'}(-, \hat{f}_*H)$ to C', where it coincides with $\hat{f}_*H(-)$. In conclusion, we find that $\hat{f}_*H \simeq F_*H$ as presheaves on C'; as the latter is a sheaf, f is cocontinuous by (III 2.2).

4.9.2. The necessity of this condition when the topology on C' is coarser than the canonical one is seen from (III 1.3 3).

4.9.4. Keep in mind, by (1.6), the continuous functors from $\tilde{C}$ to E are the ones with right adjoints. Given such a continuous G, $G\epsilon$ is continuous from C to E, as $\epsilon : C \longrightarrow \tilde{C}$ is continous: given a sheaf F on C, $\hat{\epsilon}^*F$ is the presheaf on C whose sections over $X \in \mathrm{ob}\, C$ are given by

$$\mathrm{Hom}_{\hat{C}}(X, \hat{\epsilon}^*F) = \mathrm{Hom}_{\tilde{C}}(\epsilon(X), F) = \mathrm{Hom}_{\hat{C}}(X, iF) = F(X);$$

i.e. $\hat{\epsilon}^*F = F$. The map $g \mapsto g^s$ takes continuous functors $C \longrightarrow E$ to continuous functors $\tilde{C} \longrightarrow E$ since g^s has right adjoint $g_s = g^*$. We have $g = \underline{a}g = g^s\epsilon$, where the second isomorphism is immediate from (III 1.3 3). On the other hand, a continuous G commutes with inductive limits (1.6) and prolongs $G\epsilon$ in the sense that

$$\begin{tikzcd} C \arrow[r, "G\epsilon"] \arrow[d, "\epsilon"] & E \arrow[d, equal] \\ \tilde{C} \arrow[r, "G"] & E \end{tikzcd}$$

commutes up to canonical isomorphism. Therefore by the last remark of (III 1.2), G and $(G\epsilon)^s$ are naturally isomorphic.

4.9.6. Indeed, the functor F^* commutes with inductive limits and extends f[13] (1.2.1).

6.8. On the subject of the category $E_{/\varphi_p}$; i.e. the full subcategory of $\widehat{(E^\circ)}_{/\varphi_p}$ formed of arrows whose sources are objects of E: the point is that φ_p is a covariant functor $E \to \mathrm{Ens}$; i.e. a presheaf on E°. Objects of E are considered elements of $\widehat{(E^\circ)}$ via $X \mapsto \mathrm{Hom}_E(X, -)$. To give an object of $E_{/\varphi_p}$ is to give an object X of E together with a map $\mathrm{Hom}_E(X, -) \to \varphi_p$. By (I 1.4), this is the same as a section of $\varphi_p(X)$; i.e. an element of the fiber X_p. A morphism $(X, u) \to (Y, v)$ in $E_{/\varphi_p}$ is a morphism in E° over φ_p; i.e. a map $Y \to X$ in E that takes $v \in Y_p$ to $u \in X_p$. It's now clear that the category labeled $\mathbf{Vois}(p)$ is actually the *opposite* of the category $E_{/\varphi_p}$: a morphism $(Y, v) \to (X, u)$ in $\mathbf{Vois}(p)$ is a map $Y \to X$ in E taking v to u. This is confirmed by the claim that there is a canonical (covariant) functor $\mathbf{Vois}(p) \to E$: this is simply the functor source. Considering φ_p and objects of E as presheaves on E°, (I 3.4) gives that

$$F_p = \varphi_p(F) = \varinjlim_{E/\varphi_p} \mathrm{Hom}_E(X, F) = \varinjlim_{\mathbf{Vois}(p)^\circ} F(X).$$

Keeping in mind that $\mathbf{Vois}(p) = (E_{/\varphi_p})^\circ$ rather than the reverse which is claimed, the proof of the statement 'C is generating implies that there is a fundamental system of neighborhoods of any point of E with their sources in C' uses (6.5 a).

6.8.2. This section exists to conform to the classical intuition; it's necessary to pass to $\hat{C}$ because objects of C can be regarded as presheaves, but are not necessarily sheaves.

[13]N.B.: The topologies on C and C′ are subcanonical as they are full subcategories (with the induced topology) of a site with a subcanonical topology. Given a fully faithful functor $u : D \to D'$ between sites with D′ subcanonical, the induced topology on D is subcanonical: given $X \in \mathrm{ob}\, D$, $u^*(u(X)) \simeq X$ as presheaves.

8.5.1. Let's show that if $f : \mathrm{F} \to \mathrm{G}$ is a morphism in E, $(\mathrm{X}_i)_{i\in\mathrm{I}}$ cover the final object of E, and $f_i := j_i^* f : \mathrm{F}_i \to \mathrm{G}_i$ is a monomorphism (resp. epimorphism) for all $i \in \mathrm{I}$ ($\mathrm{F}_i = j_i^*\mathrm{F} = \mathrm{F} \times \mathrm{X}_i$, etc.), then $\mathrm{F} \to \mathrm{G}$ is a monomorphism (resp. epimorphism).

Suppose f is not a monomorphism; then there is a $\mathrm{U} \in \mathrm{E}$ so that $\mathrm{F}(\mathrm{U}) \to \mathrm{G}(\mathrm{U})$ is not a monomorphism; as $\mathrm{F}(\mathrm{U}) = \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \mathrm{F})$ for every crible R of U in E (and likewise for G), we have that $\mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \mathrm{F}) \to \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \mathrm{G})$ is not injective for the crible R generated by $(\mathrm{U}_i)_{i\in\mathrm{I}}$, which is covering. Therefore there is some i and some $\mathrm{V} \to \mathrm{U}_i$ ($\mathrm{V} \in \mathrm{E}$) so that $\mathrm{F}(\mathrm{V}) \to \mathrm{G}(\mathrm{V})$ is not injective. As $\mathrm{Hom}_{\mathrm{E}/\mathrm{X}_i}(\mathrm{V}, \mathrm{F}_i) = \mathrm{F}(\mathrm{V})$, $j_i^* f$ is not a monomorphism.

Suppose f is not an epimorphism; this means that there is a $\mathrm{U} \in \mathrm{E}$ so that $\underline{a} f(\mathrm{F})(\mathrm{U}) \to \mathrm{G}(\mathrm{U})$ is not surjective, so $\mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \underline{a} f(\mathrm{F})) \to \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, \mathrm{G})$ is not surjective for R as in the previous paragraph, so $\underline{a} f(\mathrm{F})(\mathrm{V}) \to \mathrm{G}(\mathrm{V})$ is not surjective for some V as in the previous paragraph (since $\underline{a} f(\mathrm{F}) \subset \mathrm{G}$); i.e.

$$\varinjlim_{\mathrm{R}\in\mathrm{J}(\mathrm{V})} \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}, f(\mathrm{F})) \to \mathrm{G}(\mathrm{V})$$

is not surjective. As $f_i(j_i^*\mathrm{F}) = j_i^* f(\mathrm{F})$ and $j_{i!}$ sets up a bijection $\mathrm{J}(\mathrm{V} \to \mathrm{U}_i) \to \mathrm{J}(\mathrm{V})$ by (III 5.2 1),

$$\begin{array}{ccc}
\varinjlim_{\mathrm{R}\in\mathrm{J}(\mathrm{V}\to\mathrm{X}_i)} \mathrm{Hom}_{\widehat{(\mathrm{E}/\mathrm{X}_i)}}(\mathrm{R}, f_i(\mathrm{F}_i)) & \xrightarrow{\sim} & \varinjlim_{j_{i!}\mathrm{R}\in\mathrm{J}(\mathrm{V})} \mathrm{Hom}_{\hat{\mathrm{E}}}(j_{i!}\mathrm{R}, f(\mathrm{F})) \\
\| & & \| \\
\underline{a} f_i(\mathrm{F}_i)(\mathrm{V}) & & \underline{a} f(\mathrm{F})(\mathrm{V}) \\
\downarrow & & \downarrow \\
j_i^*\mathrm{G}(\mathrm{V}) & = & \mathrm{G}(\mathrm{V})
\end{array}$$

commutes, where we have written V for $\mathrm{V} \to \mathrm{X}_i$ (determined by the map $\mathrm{V} \to \mathrm{U}_i$) and used $\mathrm{Hom}_{\mathrm{E}/\mathrm{X}_i}(\mathrm{V}, \mathrm{G}_i) = \mathrm{Hom}_{\mathrm{E}}(\mathrm{V}, \mathrm{G})$. We conclude that f_i is not an epimorphism.

A more direct argument goes as follows: suppose the f_i are epimorphisms and let $f(\mathrm{F}) \subset \mathrm{G}$ denote the presheaf image. We want to show that $\underline{a} f(\mathrm{F}) = \mathrm{G}$. Fix $s \in \mathrm{G}(\mathrm{U})$; then $g_i : \mathrm{X}_i \times \mathrm{U} \to \mathrm{U}$ cover U and for each i there is a covering sieve $\mathrm{R}_i \in \mathrm{J}(\mathrm{X}_i \times \mathrm{U})$ so that $s \in \mathrm{Hom}_{\hat{\mathrm{E}}}(\mathrm{R}_i, f(\mathrm{F})|_{\mathrm{X}_i\times\mathrm{U}})$. In other words, for every $\mathrm{Y} \to \mathrm{R}_i$ (Y in E), the restriction of s to $G(Y)$ lies in the presheaf image of f. Let R be the full subcategory of E/U with objects consisting of $\{g_i \circ \varphi : \varphi \in R_i\}$. Then R is a sieve of U and it is covering by T 2). Moreover, for every $Y \to R$ (Y in E), $f(F)(Y)$ contains the restriction of s. Therefore $s \in \mathrm{Hom}_{\hat{E}}(R, f(F))$ and $s \in \underline{a}f(F)(U)$.

9.3.4. Let's use $\mathfrak{U}$, $\mathfrak{F}$ for the open subtopos and the complementary closed subtopos; i.e. full subcategory of E consisting of objects whose restriction to $\mathfrak{U}$ is a final object of $\mathfrak{U}$. To verify that $\mathfrak{F}$ is indeed a topos, we need a bijection of adjunction

$$\mathrm{Hom}(X_{\complement U}, Y) \simeq \mathrm{Hom}(X, Y)$$

whenever Y belongs to $\mathfrak{F}$. It comes about as follows. Since $U \times Y \to Y$ is an isomorphism, the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[rr] & & Y \\
U \times X \arrow[u] \arrow[rr] \arrow[dr] & & U \times Y \arrow[u] \arrow[dl] \\
& U &
\end{tikzcd}$$

shows that we get a map $X_{\complement U} \to Y$ by sending U to the inverse of the isomorphism $U \times Y \to U$, followed by the projection $U \times Y \to Y$. This establishes the bijection.

VI. Conditions de finitude. Topos et sites fibrés. Applications aux questions de passage à la limite.

1.2. Recall the description (`00WK`) of sections of the sheafification.

1.4. 'ce qui implique évidemment que la famille finie correspondante des $Y_j \to X_i$ est aussi couvrante' $\rightsquigarrow$ form the fiber product of $Y_j \coprod X_i' \to X_i \coprod X_i' \simeq X$ with $X_i \to X$.

14. *The pro-étale topology for schemes*

These notes are part of my effort to understand *The pro-étale topology for schemes* by Bhatt & Scholze (*Astérisque* **369**, 2015, p. 99–201). The authors make use of homotopy-limits and -colimits in derived categories of Grothendieck abelian categories (usually modules over a ring in a topos). In general, the computation of these uses the model structure on the abelian category of chain complexes. But in almost all cases, these homotopy-limits are sequential, and fit into familiar distinguished triangles. To make sense of the paper, therefore, the reader needs to know nothing of ∞-categories or model categories.

1.2. 'Any map between weakly étale X-schemes is itself weakly étale' $\rightsquigarrow$ I like the proof of [**F**, 1.2].

1.3. 'The property of being weakly étale is clearly étale local on the source and target' $\rightsquigarrow$ on the target, this is true since flatness is fpqc local on the target. On the source, this is true by `092K`, for example, and we see that the property of being weakly étale is actually fpqc local on the source and target.

'For a ring A, the sites defined by weakly étale A-algebras and by ind-étale A-algebras are equivalent' $\rightsquigarrow$ surely what is meant is the following. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the sites both of whose underlying categories is the opposite category of that of weakly étale (resp. ind-étale) A-algebras, with the fpqc topology. The inclusion $\mathscr{C}_2 \hookrightarrow \mathscr{C}_1$ is fully faithful, preserves fiber products, and the topology on $\mathscr{C}_2$ is the induced topology with respect to this inclusion [**SGAA**, Exp. III, 1.6, 3.1].[14] Therefore by the comparison lemma [**SGAA**, Exp. III, 4.1], the topos $\widetilde{\mathscr{C}_1}$ and $\widetilde{\mathscr{C}_2}$ are equivalent.

2.1. See `08YF` for background on spectral spaces. To finish the proof of `09XX`, note that as X is closed in the product with the constructible topology, X is compact for that topology (`0901`). The inclusion $X \subset \prod_{i\in I} W$ factors through

$$X \to \lim_{J\subset I \text{ finite}} X_J \to \lim_{J\subset I \text{ finite}} \prod_{j\in J} W = \prod_{i\in I} W,$$

[14]The inclusion is continuous and cocontinuous for these topologies [**SGAA**, Exp. III 3.4].

hence is injective. As the maps $X \to X_J$ are surjective, $X \to \lim_{J\subset I \text{ finite}} X_J$ is dense for the constructible (i.e. discrete) topology on X_J. As X is compact for the constructible topology and $\lim_{J\subset I \text{ finite}} X_J$ is Hausdorff if we equip X_J with the constructible topology, $X \to \lim_{J\subset I \text{ finite}} X_J$ is bijective (`08YD`). Now equipping X, $\prod_{i\in I} W$, X_J, $\prod_{j\in J} W$ with their usual topologies, the square

$$\begin{tikzcd}
\prod_{i\in I} W \arrow[r, equal] & \lim_{J\subset I \text{ finite}} \prod_{j\in J} W \\
X \arrow[u, hook] \arrow[r] & \lim X_J \arrow[u, hook]
\end{tikzcd}$$

commutes, where the top arrow is a homeomorphism, both vertical arrows are continuous injections, the topology on X is induced from that on $\prod_{i\in I} W$, and the bottom arrow is a continuous bijection. It follows that the bottom arrow is a homeomorphism.

We have that every spectral space is a directed inverse limit of finite T_0 spaces, but to check that $\mathcal{S} = \mathrm{Pro}(\mathcal{S}_f)$ it's necessary to check that the morphisms agree. The essential point is that if X_i is a directed inverse system of finite T_0 spaces and Y is a finite T_0 space, then any map $f : X = \lim_i X_i \to Y$ factors through some X_i. Let $p_i : X \to X_i$ denote the projections. Since the system is directed, there exists an index i so that for every open set $U \subset Y$, $f^{-1}(U) = p_i^{-1}(V)$ for some open $V \subset X_i$. This means that $p_i(f^{-1}(U)) = V \cap p_i(X)$ is open in $p_i(X)$. As every point of Y is locally closed, we get a partition of $p_i(X)$ with its induced topology into locally closed subsets in bijection with points of Y, determining a map g of sets $p_i(X) \to Y$. The map g is continuous, since $g^{-1}(U) = V \cap p_i(X)$. By the Mittag-Leffler condition and the fact that the inverse limit of finite nonempty sets is nonempty, there is some $j \geq i$ so that the image of X_j in X_i is $p_i(X)$. Then the map f factors as $X \to X_j \to p_i(X) \xrightarrow{g} Y$.

2.1.2. A profinite space X is spectral because, writing it as a cofiltered limit $\lim X_i$ of finite discrete spaces with projections $p_i : X \to X_i$, if U_i is open in X_i, $p_i^{-1}(U)$ is a profinite space, hence quasi-compact, so that also we see that an irreducible profinite space is a point (`005F`). To see that a profinite space X is w-local note that given an open cover $\{U_\alpha\}$, since X is quasi-compact, we may assume the cover is finite, and we can shrink each of the finitely many U_α so that each is also quasi-compact. Then we can write $U_\alpha = p_i^{-1}(V_\alpha)$ with $V_\alpha \subset X_i$, and since the limit is cofiltered, one i works for all of the finitely many indices α (`0A2P`). There's a partition of $p_i(X) = \bigsqcup_\alpha T_\alpha$ so that $T_\alpha \subset V_\alpha$ for each α, and the continuous maps $p_i^{-1}(T_\alpha) \hookrightarrow U_\alpha$ give rise to a section $X = \bigsqcup_\alpha p_i^{-1}(T_\alpha) \hookrightarrow \bigsqcup_\alpha U_\alpha$.

2.1.4. That $X \mapsto \pi_0(X)$ is the stated left adjoint follows from the second statement of `08ZL`, which is an easy consequence of the fact that the image of a connected space under a continuous map is connected. To check $\pi_0(X) \to \lim_i \pi_0(X_i)$ is a homeomorphism, by `08YE` it suffices to check that it's a bijection. To see that it's surjective, pick a point $x = (x_i)$ of $\lim_i \pi_0(X_i)$ and let $Y_i \subset X_i$ be the connected component corresponding to x. Then $\lim_i Y_i$ is a nonempty subset of X by `086J`. To see that it's injective, suppose we have two distinct connected components S_1, S_2 of X. Then $S_1 = \bigcap U_\alpha$ where $U_\alpha \supset S_1$ is clopen (`005F`). As a finite intersection of clopens U_α is a U_α and $S_1 \cap S_2 = \emptyset$, $U_\alpha \cap S_2 = \emptyset$ for some α as S_2 is quasi-compact (`005D`). Suppose S_1 and S_2 are sent to the same point $x = (x_i)$ of $\lim_i \pi_0(X_i)$, and let Y_i be as before. As $S_1 \cup S_2 \subset \lim_i Y_i$, $U_\alpha \cap \lim_i Y_i$ is nonempty clopen, so $\lim_i Y_i$ is not connected, contradicting the

Lemma. — *A cofiltered limit of connected finite spaces is connected.*

Proof. — By the Mittag-Leffler condition, if X_i is a directed system of connected finite T_0 spaces, for each i there is a $j \geq i$ so that for all $k \geq j$, the image of X_k in X_i coincides with that of X_j, which is connected as X_j is. So we can replace X_i with the image of X_j for all i and assume that the transition maps are surjective. Next, recall that by `086J`, $p_i : X = \lim_i X_i \to X_i$ is surjective for all i. If U and V are complementary nonempty clopens in X, they are quasi-compact since X is. Therefore since the system is directed there is an i so that $U = p_i^{-1}(A)$ and $V = p_i^{-1}(B)$ for opens A, B in X_i. The opens A and B are disjoint since p_i is surjective and U and V are disjoint. Also A and B partition X_i for the same reason. □

'X^c is profinite when X is w-local' $\rightsquigarrow$ X^c is then closed and w-local by (2.1.3); now use `0905`.

'As X^c is profinite, there is a map s lifting the inclusion' $\rightsquigarrow$ Recall that a profinite space is w-local (2.1.2).

However, I find the rest of the proof much harder to understand than that of `0968`. Note that (1) of `0968` implies that all open covers of X split, and that the last paragraph of the proof of `0968` shows that the condition that every connected component of X has a unique closed point is equivalent to the condition that every point of X specializes to a unique closed point. So `0968` implies that if $X^c \subset X$ is closed, and every connected component of X has a unique closed point, then X is w-local.

2.1.6. The subspace of X comprising all points that specialize to a point of Z is the intersection of all open sets containing Z.

2.1.8. A good reference is D.C. Isaksen, *Calculating limits and colimits in pro-categories*. Cofiltered limits in any pro-category are representable (4.1 of op. cit.). Moreover, $\mathcal{S}$ admits finite (in fact, arbitrary) products, and the forgetful functor commutes with both. By the dual of Kashiwara-Schapira *Categories and Sheaves* 3.2.9, and the fact that equalizers are representable if binary products and pullbacks are, it is enough to show the lemma for pullbacks in $\mathcal{S}$. More generally, $\mathrm{Pro}(\mathcal{C})$ has finite limits over loopless diagrams if $\mathcal{C}$ has finite limits (3.2 of Isaksen), so it will be enough to show that $\mathcal{C}$ has finite limits. It clearly has finite products, and therefore has finite limits if it has fiber products.

For the second proof, recall `0905`. It's easy to check the triangle identities for (α, a): the unit is the identity, while the counit takes a spectral space X to the continuous map from X with the constructible topology to X with its given topology that is the identity on points. For (β, b), the map $\mathrm{Pro}(\mathrm{Set}_f) \to \mathrm{Set}$ is the forgetful functor (there is no topology on Set. Then if A is a set with the discrete topology and B is a profinite space,

$$\mathrm{Hom}_{\mathrm{Set}}(A, b(B)) = \mathrm{Hom}_{\mathrm{Top}}(A, B) = \mathrm{Hom}_{\mathrm{Top}'}(\beta(A), B),$$

where Top′ is the category of compact Hausdorff spaces.

2.1.9. To recall the corresponding facts for $\mathcal{S}$, cofiltered limits are taken care of by `0A2Z`; for fibered products, one would like to check that the fibered product $X \to Z \leftarrow Y$ of spectral spaces is a spectral space. Writing $Z = \lim_i Z_i$ as a cofiltered limit of finite T_0 spaces, $X \times_Z Y = \lim_i X \times_{Z_i} Y$, and it suffices to show that $X \times_{Z_i} Y$

is a spectral space. Each point z of Z_i is locally closed, so $X \times_z Y$ is a constructible subset of $X \times Y$, and $X \times_{Z_i} Y = \bigcup_{z \in Z_i} X \times_z Y$ is also. As $X \times Y$ is spectral (`0907`) and a constructible subset of a spectral space is spectral (`0902`), $X \times_{Z_i} Y$ is spectral.

2.1.10. One can check that the stated *left* adjoint is indeed a left adjoint by the description of morphisms in $\mathrm{Pro}(\mathcal{S}_f)$.

2.1.11. Let X be a spectral space, say $\lim_i X_i$ of finite T_0 spaces X_i.

(*) The set of points of X generalizing $Y \subset X$ coincides with $\lim_i S_i$, where $S_i \subset X_i$ is the set of points generalizing $p_i(Y)$.

We have two descriptions of X^Z, the first coming from (2.1.10) and the second coming from (2.1.11). By (*), a point of X^Z from (2.1.10) is the data of a specialization $x_1 \rightsquigarrow x_0$ of points of X. The set X^Z from (2.1.11) has the same description. Now we show that the topologies are cofinal. Let $q_i : \lim_i X_i^Z \to X_i^Z$ denote the projections. Let $y \rightsquigarrow x$ be a point of X^Z. A basis of neighborhoods of the point $y \rightsquigarrow x$ for the topology of $\lim_i X_i^Z$ is given by open sets of the form $q_i^{-1}(U_x)$, where $U_x \subset X_{p_i(x)} \subset X_i$ is open and $y \in q_i^{-1}(U_x)$. Concretely, $q_i^{-1}(U_x)$ consists of those points $y' \rightsquigarrow x'$ with $p_i(x') = p_i(x)$ and $p_i(y') \in U_x$. As $\{x\} \subset X_i$ is locally closed, $p_i^{-1}(x)$ is a constructible subset of X and (*) describes its space of generalizations $\widetilde{p_i^{-1}(x)}$ in X. If $\mathrm{pr}_j : \widetilde{p_i^{-1}(x)} \to S_j$ are the projections (where S_j is as in (*) for $Y = p_i^{-1}(x)$), then $p_i(y) \in U_x \subset S_i$ and $q_i^{-1}(U_x)$ coincides with the inverse image in

$$\lim_{\{X_j \hookrightarrow X\}} \sqcup_j \widetilde{X_j}$$

of $\mathrm{pr}_i^{-1}(U_x) \subset \widetilde{p_i^{-1}(x)}$ coming from the partition of X into $p_i^{-1}(x)$ and $p_i^{-1}(X - x)$.

On the other hand, a basis for

$$\lim_{\{X_j \hookrightarrow X\}} \sqcup_j \widetilde{X_j}$$

comes from picking a constructible $Y \subset X$ and an open set U of the generalization $\widetilde{Y}$. Call the resulting open set $V \subset X^Z$; V corresponds to those $y' \rightsquigarrow x'$ with $y' \in U$ and $x' \in Y$. Let $y \in U$; then $y \rightsquigarrow x \in Y$. Let S_i be as in (*), so $\widetilde{Y} = \lim_i S_i$. Shrinking U about

y so that it's quasi-compact, it coincides with $\mathrm{pr}_i^{-1}(\mathrm{U}_i)$ for some open $\mathrm{U}_i \subset \mathrm{S}_i$, where $\mathrm{pr}_i : \widetilde{\mathrm{Y}} \to \mathrm{S}_i$. The constructible set Y is determined by the data of two quasi-compact opens Y_1 and Y_2 in X; say $\mathrm{Y} = \mathrm{Y}_1 \cap \mathrm{Y}_2^c$. Possibly changing the index i of U_i, we may suppose $\mathrm{Y}_j = p_i^{-1}(\mathrm{T}_j)$ for T_j open in X_i and $j = 1, 2$, so that $\mathrm{Y} = p_i^{-1}(\mathrm{T}_1 \cap \mathrm{T}_2^c)$. Then $\mathrm{pr}_i(x) \in \mathrm{S}_i \cap \mathrm{T}_1 \cap \mathrm{T}_2^c$, the intersection of U_i with $(\mathrm{X}_i)_{\mathrm{pr}_i(x)}$ is an open set containing $\mathrm{pr}_i(y)$, and $q_i^{-1}(\mathrm{U}_i \cap (\mathrm{X}_i)_{\mathrm{pr}_i(x)})$ is a subset of V containing the point $y \rightsquigarrow x$ of X^{Z}. (It is the set of $y' \rightsquigarrow x'$ with $p_i(x') = p_i(x)$ and $p_i(y') \in \mathrm{U}_i \cap (\mathrm{X}_i)_{\mathrm{pr}_i(x)}$; by the above description, $x' \in \mathrm{Y}$ and $y' \in \mathrm{U}$.) This shows that the two topologies on X^{Z} coincide.

A point $x_1 \rightsquigarrow x_0$ of

$$\mathrm{X}^{\mathrm{Z}} = \lim_{\{\mathrm{X}_j \hookrightarrow \mathrm{X}\}} \sqcup_j \widetilde{\mathrm{X}_j}$$

specializes to another point $x_1' \rightsquigarrow x_0'$ iff $x_0 = x_0'$ and $x_1 \rightsquigarrow x_1'$. Therefore

$$(\mathrm{X}^{\mathrm{Z}})^c = \lim_{\{\mathrm{X}_j \hookrightarrow \mathrm{X}\}} \sqcup_j \mathrm{X}_j;$$

these correspond to the points $\{x \rightsquigarrow x : x \in \mathrm{X}\}$ of X^{Z}.

'The cofinality of affine stratifications inside all constructible stratifications' $\rightsquigarrow$ Suppose given $\mathrm{U}, \mathrm{V} \subset \mathrm{X}$ quasi-compact open; we wish to refine the constructible set $\mathrm{U} \cap \mathrm{V}^c$ by affine constructible sets. Cover U by finitely many affine opens $\{\mathrm{U}_i\}$. Then $\mathrm{U}_1 \cap \mathrm{V}^c$ is affine and $\mathrm{V} \cup \mathrm{U}_1$ is quasi-compact, then $\mathrm{U}_2 \cap (\mathrm{V} \cup \mathrm{U}_1)^c$ is affine and constructible. We proceed inductively until we use the finitely many U_i, at which point we've stratified $\mathrm{U} \cap \mathrm{V}^c$ by affine constructible subsets.

Now if given a finite union of constructible subsets C_i, we do $\mathrm{C}_1 = \mathrm{U} \cap \mathrm{V}^c$, then partition all $\mathrm{C}_i, i > 1$ into $\mathrm{C}_i \cap \mathrm{V}$ and $\mathrm{C}_i \cap \mathrm{V}^c \cap \mathrm{U}^c$, which are constructible, and proceed inductively.

We want to conclude from this fact that we can give

$$\mathrm{X}^{\mathrm{Z}} = \lim_{\{\mathrm{X}_j \hookrightarrow \mathrm{X}\}} \sqcup_j \widetilde{\mathrm{X}_j}$$

the structure of affine scheme. This is a consequence of `096V`.

2.1.12. What does it mean that f is 'a cofiltered limit of Zariski localizations'? I think it means that $W = \lim_i W_i$ is a cofiltered limit in $\mathcal{S}$ and f coincides with $W \to W_i \to V$ for some Zariski localization $W_i \to V$; moreover whenever there's an arrow $j \to i$, the composition $W_j \to W_i \to V$ is a Zariski localization. As soon as this condition is satisfied, one can assume that the conclusion actually holds for all i, by reindexing the limit.

It's immediate that the base change of a Zariski localization is again one, and it follows that the base change of a pro-(Zariksi localization) is again one since the pullback of a limit is the limit of pullbacks ('limits commute with limits').

2.1.13. With the above understanding of what a pro-(Zariski localization) is, any map $P \to Q$ from a profinite set to a finite discrete set is a pro-(Zariski localization), since if $P = \lim_i P_i$ with P_i finite and discrete, the map $P \to Q$ is determined by a map $P_i \to Q$ for some i,[15] and any map of finite discrete sets is a Zariski localization. This immediately implies that whenever there's an arrow $j \to i$, the composition $P_j \to P_i \to Q$ is a Zariski localization.

Applying this to understand the proof of the lemma, we find that $S_i \to T$ is the base change of a pro-(Zariski localization), hence is one, and the map $S = \lim_i S_i \to T$ 'is a cofiltered limit of the maps $S_i \to T$'; i.e. coincides with $S \to S_i \to T$ for any i. (Finding $S = \lim_i S_i$ amounts to saying that to give a map to S and to T which agrees on T_i for every i is the same as giving a map to S. The map $S = \lim S_i \to S_i$ comes from the map $\mathrm{id} : S \to S$ and $f : S \to T$. Indeed, the map $S \to T$ coincides with $\lim_i S_i \to S_i \to T$ for all i.)

It remains only to show that this cofiltered limit of pro-(Zariski localizations) is a pro-(Zariski localization).[16] It's more convenient to work here with directed sets, so we may assume that $S = \lim_j S'_j$ and $T = \lim_i T_i$ are limits over directed sets $\mathcal{J}, \mathcal{I}$, respectively. As we've seen, for each i, there's an index j so that the map $S \to T_i$ factors through S'_j (call such j good wrt i), and $S \times_{T_i} T$ is the limit of the $S'_j \times_{T_i} T$ over the directed subset $\mathcal{J}_i \subset \mathcal{J}$ of those j good wrt i. Endowing the cartesian product $\mathcal{J} \times \mathcal{I}$

[15]The preimage of a point is a quasi-compact open.

[16]One could just as well imitate the proof of `0BSJ` here.

with the order that says $(j,i) \leq (j',i')$ if $j \leq j$ and $i \leq i'$, the subset

$$\mathcal{M} = \{(j,i) \in \mathcal{J} \times \mathcal{I} : j \in \mathcal{J}_i\} \subset \mathcal{J} \times \mathcal{I}$$

is directed. Since the diagram

$$\begin{tikzcd} S'_{j'} \times_{T_{i'}} T \arrow[r] \arrow[d] & S'_{j'} \times_{T_i} T \arrow[d] \\ S'_j \times_{T_{i'}} T \arrow[r] & S'_j \times_{T_i} T \end{tikzcd}$$

commutes whenever j is good wrt i' and $j \leq j'$, $i \leq i'$, there is a functor from $\mathcal{M}$ to the category with objects $S'_j \times_{T_i} T$ with j good wrt i and maps coming from $\mathcal{I}$ and $\mathcal{J}$. The direct limit over this functor is S and exhibits S as a pro-(Zariski localization), since the map $S \to T$ factors through any of the $S'_j \times_{T_i} T$.

2.2.4. Use `0977` to finish the proof that $A \mapsto A^Z$ is left adjoint to the inclusion of the category of w-local rings and maps inside all rings.

2.2.6. Use the 'precise' version of Lemma 2.2.8 for $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and the map $\mathrm{id} : \pi_0(Y) \to \pi_0(Y)$ to get a map $Y \to \pi_0(Y) \times_{\pi_0(X)} X = \operatorname{Spec} C$.

2.2.7. 'Any ind-étale algebra over an absolutely flat ring is also absolutely flat' $\rightsquigarrow$ combine `097N` & `092I`.

'$A \to A_1$ is faithfully flat ind-étale' $\rightsquigarrow$ that any finite product of the $\operatorname{Spec} B_j$ is faithfully flat and étale follows from the stability of these properties under composition and base change. The stability of 'faithfully flat' under the colimit is `090N`.

'$\overline{A}$ is absolutely flat' $\rightsquigarrow$ `097K` + `0BSJ` imply that $A \to \overline{A}$ is ind-étale over an absolutely flat ring, hence absolutely flat (it's faithfully flat by another application of `090N`).

'Any faithfully flat étale $\overline{A}$-algebra has a section' $\rightsquigarrow$ if $\overline{A} \to B$ is étale and surjective, it's obtained as the base change to $\overline{A}$ of a finitely presented map $A_n \to B'$ for some n

(`05N9`); in other words, the square

$$\begin{tikzcd} \mathrm{Spec}\,B \arrow[r] \arrow[d] & \mathrm{Spec}\,B' \arrow[d] \\ \mathrm{Spec}\,\overline{A} \arrow[r] & \mathrm{Spec}\,A_n \end{tikzcd}$$

is cartesian. As $A_n \to \overline{A}$ is faithfully flat, $A_n \to B$ is étale by fpqc descent (`02VN`), and surjective since $\mathrm{Spec}\,B \to \mathrm{Spec}\,\overline{A}$ is, so $\mathrm{Spec}\,B \to \mathrm{Spec}\,A_n$ is surjective. The existence of a section $B \to \overline{A}$ comes from the dotted section in the diagram

$$\begin{tikzcd} \mathrm{Spec}\,B \arrow[r] \arrow[d] & \mathrm{Spec}\,B' \times_{\mathrm{Spec}\,A_n} \mathrm{Spec}\,A_{n+1} \arrow[r] \arrow[d] & \mathrm{Spec}\,B' \arrow[d] \\ \mathrm{Spec}\,\overline{A} \arrow[r] & \mathrm{Spec}\,A_{n+1} \arrow[u, dotted, bend right] \arrow[r] & \mathrm{Spec}\,A_n \end{tikzcd}$$

in which both squares are cartesian.

2.2.8. Recall (`078L`) that given a small filtered category $\mathcal{I}$ with set of objects I and an $\mathcal{I}$-diagram F of commutative rings,

$$\mathrm{Spec}(\mathrm{colim}_{i\in I}\,F(i)) = \lim_{i\in I} \mathrm{Spec}(F(i)),$$

where the limit is taken over the $\mathcal{I}^{\mathrm{op}}$-diagram of affine schemes or spectral spaces.

Given a map $T \to \pi_0(\mathrm{Spec}(A))$ of profinite sets, let $T = \lim_j T'_j$ and $\pi_0(\mathrm{Spec}(A)) = \lim_i S_i$ be profinite presentations. For each i, there is a full cofiltered subcategory of the T'_j consisting of those j so that the map $T \to S_i$ factors through T'_j. Call such j good wrt i.

When j is good wrt i, the space $T'_j \times_{S_i} \mathrm{Spec}\,A$ is $\mathrm{Spec}\,B_{ij}$ for the ring B_{ij} obtained as follows. The points of S_i correspond to disjoint clopens in Spec A corresponding to a decomposition $A = \prod_{x\in S_i} A_x$. The x-fiber in T'_j, notated $T'_j \times_{S_i} x$, is a finite discrete space; let $B_x = \prod_{|T'_j \times_{S_i} x|} A_x$. Then $B_{ij} = \prod_{x\in S_i} B_x$ has $\mathrm{Spec}\,B_{ij} = T'_j \times_{S_i} \mathrm{Spec}\,A$.

Suppose j is good wrt i' and we have arrows $j' \to j, i' \to i$. Then

$$\begin{array}{ccc} T'_{j'} \times_{S_{i'}} \operatorname{Spec} A & \longrightarrow & T'_{j'} \times_{S_i} \operatorname{Spec} A \\ \downarrow & & \downarrow \\ T'_j \times_{S_{i'}} \operatorname{Spec} A & \longrightarrow & T'_j \times_{S_i} \operatorname{Spec} A \end{array}$$

is a commutative diagram of affine schemes. Suppose $i' \to i$ and write $T = \lim_j T'_j$ with j good wrt i' for all j. The commutativity of the above diagram gives maps $\lim_j \operatorname{Spec} B_{i'j} \to \lim_j \operatorname{Spec} B_{ij}$ corresponding to the maps $T'_j \times_{S_{i'}} \operatorname{Spec} A \to T'_j \times_{S_i} \operatorname{Spec} A$, and corresponding to maps $\operatorname{colim}_j B_{i'j} \leftarrow \operatorname{colim}_j B_{ij}$. Call the (filtered) colimit of these maps B; then Spec B coincides as a spectral space with $\lim_i T \times_{S_i} \operatorname{Spec} A = T \times_{\pi_0(\operatorname{Spec} A)} \operatorname{Spec} A$. The map $T \times_{\pi_0(\operatorname{Spec} A)} \operatorname{Spec} A \to \pi_0(\operatorname{Spec} A)$ induces a map $\pi_0(T \times_{\pi_0(\operatorname{Spec} A)} \operatorname{Spec} A) \to \pi_0(\operatorname{Spec} A)$ (08ZL). The fiber of this map over a point $x \in \pi_0(\operatorname{Spec} A)$ is $\pi_0(T \times_{\pi_0(X)} Z) = \pi_0(T_x \times Z)$, where $T_x := T \times_{\pi_0(X)} x$ is the x-fiber and $Z \subset \operatorname{Spec} A$ is the closed subset corresponding to x.[17] As Z is connected, the clopens of $T \times_{\pi_0(\operatorname{Spec} A)} Z$ are in bijection with clopens of the x-fiber of T.[18] As the connected component of a spectral space containing a given point is the intersection of the clopens containing that point (005F), it follows that $\pi_0(T \times_{\pi_0(\operatorname{Spec} A)} Z) = \pi_0(T \times_{\pi_0(\operatorname{Spec} A)} x) = T \times_{\pi_0(\operatorname{Spec} A)} x$.[19] As the map $\pi_0(T \times_{\pi_0(\operatorname{Spec} A)} \operatorname{Spec} A) \to T$ is over $\pi_0(\operatorname{Spec} A)$ and the fibers coincide, the map is a bijection, hence a homeomorphism. (Note that in BS's proof, $T_i = T \times_{S_i} \pi_0(\operatorname{Spec} A)$.)

Once $T \times_{\pi_0(X)} X$ has been given the structure of affine X-scheme, it's easy to check that this construction is right adjoint to $\pi_0(Y)$ on maps of profinite $\pi_0(X)$-sets. It amounts to the observation that the map of X-schemes $Y \to T'_j \times_{S_i} X$ is determined by the map $\pi_0(Y) \to T$ in the following way: the discrete finite set S_i corresponds to a finite clopen

[17] Given a map of topological spaces $A \to B$, the map $A \to \pi_0(B)$ factors as $A \to \pi_0(A) \to \pi_0(B)$, so the preimage in $\pi_0(A)$ of a point $b \in \pi_0(B)$ is the image in $\pi_0(A)$ of the b-fiber A_b of A. Since A_b is a union of connected components of A, the map $\pi_0(A_b) \to \pi_0(A)$ is injective, so this image is $\simeq \pi_0(A_b)$.

[18] Let $Q \subset T_x \times Z$ be clopen, then by intersecting with closed subsets of the form $a \times Z$, $a \in T_x$, we see Q is of the form $S \times Z$ for $S \subset T_x$. Taking a closed point $z \in Z$, $S = Q \cap (T_x \times z)$ is clopen in $T_x \times z \simeq T_x$.

[19] Indeed, given a quasi-compact space X so that every connected component of X containing $x \in X$ is the intersection of clopens containing x, given $X = \coprod_i X_i$, replacing X_i by a point we get a finite discrete space. Doing this for each disjoint union decomposition of X, we get a cofiltered category C of finite discrete spaces and continuous maps, and $\pi_0(X) = \lim C$.

partition of X and Y, and for each $x \in S_i$, the map $\pi_0(Y) \times_{S_i} x \to T \times_{S_i} x \to T'_j \times_{S_i} x$ decomposes the clopen subset $\pi_0(Y) \times_{S_i} x \subset \pi_0(Y)$ into finitely many clopen subsets corresponding to the finitely many points $p \in T'_j$ that are sent to x. Then these clopens $Y_p \subset Y$ get sent to the finitely many copies of $X \times_{\pi_0(X)} x$[20] corresponding to these finitely many p, under the unique map $Y_p \to X \times_{\pi_0(X)} x$ over X.

To check that the map $S \times_{\pi_0} X \to X$ is a pro-(Zariski localization) and pro-finite, follow the last paragraph of the note to 2.1.13.

2.2.10. Observe the colimit is filtered: given maps of A-algebras $A' \to B_0$ and $A'' \to B_0$, both factor through the map $A' \otimes_A A'' \to B_0$. Given two maps $A' \rightrightarrows A''$ making a commutative triangle with B_0, we let A''' be the coequalizer of these two maps; by the universal property it has a unique map to B_0, and $A \to A'''$ is étale (c.f., e.g. `021F`).

We check the adjunction in detail. The counit is given as follows. Starting with $B_0 \in \mathrm{Ind}(B_{\text{ét}})$, we give a map $\mathrm{Hens}_A(B_0) \otimes_A B \to B_0$ corresponding to a map $\mathrm{Hens}_A(B_0) \to B_0$ making the square

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hens}_A(B_0) \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_0 \end{array}$$

commute. This is the map which sends each diagram $A \to A' \xrightarrow{c} B_0$ to the map $c : A' \to B_0$, determining a map $\mathrm{colim}\, A' \to B_0$ that makes the above square commute. In the purported adjunction

$$\mathrm{Hom}_B(A_0 \otimes_A B, B_0) \underset{b}{\overset{a}{\rightleftarrows}} \mathrm{Hom}_A(A_0, \mathrm{Hens}_A(B_0)),$$

the map a is clear: if $A_0 = \mathrm{colim}\, A_i$ with $A \to A_i$ étale, a map $A_0 \otimes_A B \to B_0$ is determined by maps $A_i \to B_0$ of A-algebras compatible with the maps $A_i \to A_{i'}$. This diagram of maps of A-algebras is a sub-diagram of the one defining $\mathrm{Hens}_A(B_0)$ and the canonical embedding of diagrams determines a map $A_0 \to \mathrm{Hens}_A(B_0)$.

[20] A clopen subset of X.

This description makes it clear that $b \circ a = \mathrm{id}$: given a compatible family of maps $\mathrm{A}_i \to \mathrm{B}_0$ of A-algebras,

$$\begin{tikzcd} \mathrm{A}_i \arrow[r] \arrow[d] & \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0) \arrow[r] & \mathrm{B}_0 \arrow[dll, "\mathrm{id}"] \\ \mathrm{B}_0 & & \end{tikzcd}$$

commutes and coincides with $b \circ a(\mathrm{A}_i \to \mathrm{B}_0)$.

To check $a \circ b = \mathrm{id}$: given a compatible family of maps $\mathrm{A}_i \to \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0)$, b is computed by $- \otimes_{\mathrm{A}} \mathrm{B}$ followed by the counit: $\mathrm{A}_i \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0) \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{B}_0$, which determines a map $\mathrm{A}_i \to \mathrm{B}_0$ of A-algebras. As $\mathrm{A} \to \mathrm{A}_i$ is of finite presentation,

$$\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_i, \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0)) = \operatorname{colim} \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_i, \mathrm{A}'). \tag{00QO}$$

This fact combined with the description of the counit shows that the map $\mathrm{A}_i \to \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0)$ is determined by a commutative triangle

$$\begin{tikzcd} \mathrm{A}' \arrow[dr] & \\ & \mathrm{B}_0 \\ \mathrm{A}_i \arrow[uu] \arrow[ur] & \end{tikzcd}$$

of A-algebras. This of course means that the maps of A-algebras $\mathrm{A}_i \to \mathrm{B}_0$ and $\mathrm{A}_i \to \mathrm{A}' \to \mathrm{B}_0$ determine the same element of $\operatorname{colim} \mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_i, \mathrm{A}')$. In other words, we can represent the map $\mathrm{A}_0 \to \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0)$ by the compatible family of maps $\mathrm{A}_i \to \mathrm{B}_0$ obtained by composing the given maps with the canonical $\mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0) \to \mathrm{B}_0$. After writing the $\mathrm{A}_i \to \mathrm{Hens}_{\mathrm{A}}(\mathrm{B}_0)$ this way, it's clear that $a \circ b = \mathrm{id}$.

2.2.11.

Lemma. — Let $\mathrm{A} \to \mathrm{B}$ *be ind-étale. Then* $\mathrm{B} = \operatorname{colim} \mathrm{A}_i$*, where the colimit is indexed by diagrams* $\mathrm{A} \to \mathrm{A}_i \to \mathrm{B}$ *of* A*-algebras with* $\mathrm{A} \to \mathrm{A}_i$ *étale.*

Proof. — The colimit is filtered; see the note to 2.2.10, so it suffices to check surjectivity and injectivity of sets. Write $\mathrm{B} = \operatorname{colim}_j \mathrm{B}_j$ as a filtered colimit of étale A-algebras, and let $\mathrm{A}' := \operatorname{colim} \mathrm{A}_i$. Surjectivity: let $f \in \mathrm{B}$; then $f \in \mathrm{B}_j$ for some j. As $\mathrm{A} \to \mathrm{B}_j$ is étale, $B_j \to B$ figures among the $A_i \to B$. Injectivity: suppose $f, g \in A'$; we may take suppose f, g are represented by elements of some étale A-algebra A_i equipped with a map of A-algebras $A_i \to B$, which therefore necessarily factors through some B_j (finite presentation). Suppose f, g go to the same element of B. This just means that there is some $B_j \to B_{j'}$ so that the images of these elements coincide in $B_{j'}$.

$$\begin{tikzcd} & & B_{j'} \arrow[dr] & \\ A \arrow[r] & A_i \arrow[r] & B_j \arrow[u] \arrow[r] & B \end{tikzcd}$$

This diagram gives a morphism of maps of A-algebras $(A_i \to B) \to (B_{j'} \to B)$ and f, g are carried to the same element of $B_{j'}$, hence represent the same element of A'. □

Corollary. — If $A \to C$ *is an ind-étale morphism of* A*-algebras, then the canonical morphism* $\mathrm{Hens}_A(C) \xrightarrow{\sim} C$ *is an isomorphism.*

To check $\mathrm{Hens}_A(C) = \mathrm{Hens}_{A'}(C)$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] & \\ B \arrow[r] & B \otimes_A A' \arrow[r] & C \end{tikzcd}$$

As $A \to A'$ is ind-étale, $B \to B \otimes_A A'$ is (`097J`), and $B \to C$ is ind-étale by hypothesis, so $B \otimes_A A' \to C$ is ind-étale by `097M`. Given $A_0 \in \mathrm{Ind}(A_{\text{ét}})$,

$$\begin{tikzcd} \mathrm{Hom}_B(A_0 \otimes_A B, C) \arrow[r, equal] \arrow[d, equal] & \mathrm{Hom}_{B\otimes_A A'}(A_0 \otimes_A B \otimes_A A', C) \arrow[d, equal] \\ \mathrm{Hom}_A(A_0, \mathrm{Hens}_A(C)) & \mathrm{Hom}_{A'}(A_0 \otimes_A A', \mathrm{Hens}_{A'}(C)). \end{tikzcd}$$

As

$$\mathrm{Hom}_{A'}(A_0 \otimes_A A', \mathrm{Hens}_{A'}(C)) = \mathrm{Hom}_A(A_0, \mathrm{Hens}_A(\mathrm{Hens}_{A'}(C)))$$

and $\mathrm{Hens}_A(\mathrm{Hens}_{A'}(C)) \xrightarrow{\sim} \mathrm{Hens}_{A'}(C)$ by the previous corollary, $\mathrm{Hens}_A(C) = \mathrm{Hens}_{A'}(C)$.

2.2.12. 'Any étale A/I-algebra lifts to an étale A-algebra' ⇝ `04D1`.

2.2.13. Let $Z \subset \operatorname{Spec} A$ be the closed locus cut out by I. When invoking Lemma 2.1.7 (and later when concluding Z is w-local assuming $\operatorname{Spec} A$ is), it's necessary to check that the map $Z \hookrightarrow \operatorname{Spec} A$ induces a bijection on connected components $\pi_0(Z) \xrightarrow{\sim} \pi_0(\operatorname{Spec} A)$. This is a corollary of the fact that (A, I) is a henselian pair if and only if for every finite A-algebra B, $B \to B/IB$ induces a bijection on idempotents (`09XI`), since if $W_0 := \operatorname{Spec} A_0 \subset \operatorname{Spec} A$ is a connected component, A_0/IA_0 has no nontrivial idempotents, so $W_0 \cap Z = \operatorname{Spec} A_0/IA_0$ is connected, so must be a connected component of Z.

'$\mathrm{Hens}_A(A/I) = A$' $\rightsquigarrow$ this is a corollary of the fact that given any map of A-algebras $\sigma : A' \to A/I$ with $A \to A'$ étale, there's a map $A' \to A$ of A-algebras lifting σ (`09XI`). In other words, we can always complete $A \to A' \to A/I$ to a commutative diagram

$$\begin{tikzcd} & A \arrow[dr] & \\ A \arrow[ur, "\mathrm{id}"] \arrow[r] & A' \arrow[u] \arrow[r] & A/I, \end{tikzcd}$$

which shows that $A \xrightarrow{\mathrm{id}} A \to A/I$ is cofinal in the system of diagrams over which the colimit is taken to compute $\mathrm{Hens}_A(A/I)$, so indeed $\mathrm{Hens}_A(A/I) = A$.

'Any faithfully flat étale A/I-algebra B_0 is the reduction modulo I of a faithfully flat étale A-algebra B' $\rightsquigarrow$ holds without the words 'faithfully flat' by the note to 2.2.12, and with the words 'faithfully flat' since Z contains all the closed points of $\operatorname{Spec} A$. Note that if $\operatorname{Spec} A$ is w-local, then $\operatorname{Spec} A/I$ is by (2.1.3).

2.2.14. It remains only to check the obvious: that if $I \subset A$ is an ideal in a ring, and $A_0 \in \mathrm{Ind}((A/I)_{\text{ét}})$, then $\mathrm{Hens}_A(A_0)$ is henselian along I,[21] since if this is true, $\mathrm{Hens}_{A^Z}(\overline{A^Z/I_{A^Z}})$ is w-strictly local iff its reduction modulo I_{A^Z} is (2.2.13), but its reduction modulo I_{A^Z} coincides with $\overline{A^Z/I_{A^Z}}$ by (2.2.12), which is indeed w-strictly local by construction. To check the claim, it suffices (`09XI`) to show that given $\mathrm{Hens}_A(A_0) \to A'$ étale and a $\mathrm{Hens}_A(A_0)$-algebra map $\sigma : A' \to \mathrm{Hens}_A(A_0) \otimes_A A/I = A_0$ (2.2.12), there's a $\mathrm{Hens}_A(A_0)$-algebra map $A' \to \mathrm{Hens}_A(A_0)$ lifting σ. As $A' \otimes_A A/I = A' \otimes_{\mathrm{Hens}_A(A_0)} A_0$ and, bizarrely, $\mathrm{Hens}_A(A_0) = \mathrm{Hens}_{\mathrm{Hens}_A(A_0)}(A_0)$ (by

[21] Technically along $I\,\mathrm{Hens}_A(A_0)$.

Remark 2.2.11), we're done since

$$\mathrm{Hom}_{A_0}(A' \otimes_A A/I, A_0) \simeq \mathrm{Hom}_{\mathrm{Hens}_A(A_0)}(A', \mathrm{Hens}_A(A_0)).$$

2.3.7. The map f^Z sends I_{A^Z} into I_{B^Z} since it is w-local.

'$\mathrm{Spec}(A')^c = \mathrm{Spec}(A_0)$' ⇝ As we've checked in the note to 2.2.14, A' is henselian along $I_{A^Z}A'$, so $\mathrm{Spec}(A')^c = \mathrm{Spec}(A_0)^c = \mathrm{Spec}(A_0)$ by (2.2.12). (In particular, the Jacobson radical of A' is $I_{A^Z}A'$.)

2.3.8. To conclude that A' is w-strictly local, you can localize A at a closed point so that it becomes a strictly henselian local ring by (2.2.9), and then the corresponding localization of A' is an ind-(Zariski localization) over A. Localizing A' at a closed point, you have a local homomorphism of local rings $A \to A'$ that is an ind-(Zariski localization) with A strictly henselian, so $A = A'$.

'Replacing A with A'' ⇝ `092L`.

2.4.2. 'The section s is a closed immersion since π is separated' ⇝ Given

$$\begin{tikzcd} X \arrow[r, "\pi"'] & Y, \arrow[l, bend right, dotted, "s"'] \end{tikzcd}$$

the square

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "s"] & X \arrow[d, "s\circ\pi"', "\mathrm{id}"] \\ X \arrow[r, "\Delta"] & X \times_Y X \end{tikzcd}$$

is cartesian. (Call the pullback P for a moment. One checks the two maps $P \to X$ must coincide with the same map t, and $t = s \circ \pi \circ t$, so we see that $\pi \circ t : P \to Y$ and $s : Y \rightrightarrows X$ make the diagram commute. In other words, Y is universal, so $P \simeq Y$.)

2.4.6. 'If X is a discrete set, $\beta(X)$ is extremally disconnected' $\rightsquigarrow$ suppose $Y \twoheadrightarrow \beta(X)$ surjects, then we can find a map $X \to Y$ making

$$\begin{tikzcd} & \beta(X) \\ X \arrow[ur, "\eta_X"] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

commute. The adjoint map $\beta(X) \to Y$ is carried via $Y \to \beta(X)$ to the adjoint of η_X which is the identity.

'If X is a compact Hausdorff space, then $\beta(X) \to X$ is a continuous surjection' $\rightsquigarrow$ $\mathrm{id} : X \to X$ is obtained from $\varepsilon_X : \beta(X) \to X$ by applying the forgetful functor to **Set** and precomposing with $\eta_X : X \to \beta(X)$.

'Any extremally disconnected space is a retract of $\beta(X)$ for some X' $\rightsquigarrow$ Let Y be our extremally disconnected space. Then as seen, $\varepsilon_Y : \beta(Y) \twoheadrightarrow Y$ is a continous surjection. The section s is continuous and injective; since $\varepsilon_Y \circ s = \mathrm{id}_Y$, s is a homeomorphism onto its image. In other words, Y is a retract of $\beta(Y)$.

2.4.8. 'Every continuous surjection $T \to \pi_0(\mathrm{Spec}(A))$ of profinite sets has a section' $\rightsquigarrow$ (2.2.8) gives an ind-Zariski localization $A \to B$ with $\mathrm{Spec}\, B \simeq T \times_{\pi_0(\mathrm{Spec}\, A)} \mathrm{Spec}\, A$ over A, so $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is faithfully flat, hence admits a section, which gives a section of $T \to \pi_0(\mathrm{Spec}\, A)$ upon applying π_0. One concludes that $\pi_0(\mathrm{Spec}\, A)$ is extremally disconnected by Gleason's Theorem 1.2, since the category of profinite sets and continuous maps satisfies his conditions (a–c).

Lemma. — A strictly w-local ring A *is henselian along its Jacobson radical* I_A.

Proof. — Let $I := I_A$; the ring A/I is w-strictly local by (2.2.15). We've already checked that $A' := \mathrm{Hens}_A(A/I)$ is henselian along I (c.f. the note to 2.2.14); moreover IA' coincides with the Jacobson radical of A' since $A' \otimes_A A/I = A/I$ (2.2.12), which is absolutely flat. The unit $\eta_A : A \to A'$ reduces mod I to $\mathrm{id} : A/I \to A/I$, so is a w-local map of strictly w-local rings that is a homeomorphism on closed points. Moreover, $A \to A'$ is ind-étale. Then just as in the proof of (2.3.8), for each maximal ideal $\mathfrak{m} \subset A$, the ring $A'/\mathfrak{m}A'$ has a unique maximal ideal and is absolutely flat, so is a field, and $\mathfrak{m}A'$

is a maximal ideal. The morphism $A_{\mathfrak{m}} \to A'_{\mathfrak{m}A'}$ is an ind-étale map of local rings with $A_{\mathfrak{m}}$ strictly henselian, so it's an isomorphism. Since $A'_{\mathfrak{m}A'} = A' \otimes_A A_{\mathfrak{m}}$,[22] $A \to A'$ is an isomorphism. $\square$

The set $s(\operatorname{Spec} A) \subset T$ is closed since, endowed with the subspace topology, it's homeomorphic to Spec A, which is quasi-compact. If $s(\operatorname{Spec} A) = \operatorname{Spec} C'$ with reduced structure, we have a map of absolutely flat rings $h : \operatorname{Spec} C' \to \operatorname{Spec} A$ which is a homeomorphism on spectra with the property that the local ring at a point x of Spec C′ coincides with the local ring of Spec A at $h(x)$. Therefore h is an isomorphism, since as we've just seen, $C'_x = C' \otimes_A A_{h(x)}$.

2.4.9. Of course, a Hausdorff extremally disconnected space is totally disconnected: given distinct points x and y, there are disjoint neighborhoods U and V of each point, and $\overline{U} \cap V = \emptyset$. As $\overline{U}$ is clopen, x and y don't belong to the same connected component.

Lemma 2.2.7 produces the w-strictly ind-étale local faithfully flat A^Z/I_{A^Z}-algebra $\overline{A^Z/I_{A^Z}}$. As this ring is absolutely flat, $X := \operatorname{Spec} \overline{A^Z/I_{A^Z}}$ is homeomorphic to $\pi_0(X)$ and $\beta(X) \to X$ is a surjective map from an extremally disconnected (profinite) space to a profinite space. Lemma 2.2.8 gives us an ind-(Zariski localization) $\overline{A^Z/I_{A^Z}} \to A_0$ which is faithfully flat since the corresponding map on spectra is $\beta(X) \times_{\pi_0(X)} X = \beta(X) \to X$, which is surjective. For the same reason, A_0 is w-local. By (2.2.9), to check that A_0 is w-strictly local, we need to check that its local rings are strictly henselian. The local rings of $\overline{A^Z/I_{A^Z}}$ are spectra of separably closed fields as $\overline{A^Z/I_{A^Z}}$ is w-strictly local and absolutely flat. The x-fiber of the map $\beta(X) \to X$ at a point $x \in X$ can be obtained by localizing $\overline{A^Z/I_{A^Z}}$ at x. By construction, all of the local rings of $\beta(X) \times_X x$ coincide with $k(x)$.[23] This shows that all the local rings of A_0 are also spectra of separably

[22]To check this, it suffices to show that the spectra of the two rings coincide, because if this is true and $a \in A' \setminus \mathfrak{m}$, then there's no proper ideal of $A' \otimes_A A_{\mathfrak{m}}$ containing a. Suppose $\mathfrak{p} \subset A'$ is prime and not contained in $\mathfrak{m}A'$. Since A' is w-local, $\mathfrak{p}$ belongs to a connected component distinct from the one corresponding to $\mathfrak{m}A'$. As $A \to A'$ is a w-local map of w-local rings and a homeomorphism on closed points, the connected component of Spec A′ containing $\mathfrak{p}$ is the preimage of a connected component of Spec A not containing $\mathfrak{m}$. Suppose $J \subset A$ cuts out this latter closed subset. Then the preimage of $\mathfrak{p}$ in A contains J. Pick some $b \in J$, $b \notin \mathfrak{m}$. As the image of b in A' is contained in $\mathfrak{p}$, $1 \in \mathfrak{p}A_{\mathfrak{m}}$.

[23]Suppose $\beta(X) \times_X x = \operatorname{Spec} S$. Lemma 2.2.8 writes S as a filtered colimit of finite products of $k(x)$, in particular as a colimit of reduced rings, so $\beta(X) \times_X x$ is absolutely flat. Let $\beta(X) = \lim \beta_i$ be a profinite presentation with projections $p_i : \beta(X) \to \beta_i$ and let $y \in \beta(X)$. Then y corresponds to an ideal $\mathfrak{m}$ of S in

closed fields, so A_0 is w-strictly local. Since $\pi_0(\beta(X)) = \beta(X)$, (2.4.8) gives that A_0 is w-contractible, so $\mathrm{Hens}_{A^Z}(A_0)$ is w-contractible by (2.4.3) since it is henselian along I_{A^Z} (note to 2.4.8).

Of course, $\operatorname{Spec} A' \to \operatorname{Spec} A^Z$ is surjective since it's open and surjects onto closed points, since $\operatorname{Spec}(A^Z)^c = \operatorname{Spec}(A^Z/I_{A^Z})$ and the reduction modulo I_{A^Z} of our map is $\operatorname{Spec} A_0 \to \operatorname{Spec} A^Z/I_{A^Z}$, which is surjective by construction.

2.4.10. I can't substantiate the assertion about π_0 for silly reasons like $\mathbf{F}_{p^n} \otimes_{\mathbf{F}_p} \mathbf{F}_{p^n}$. Instead I follow the proof of `0985` + `0986` to show that $\pi_0(\operatorname{Spec} B)$ is extremally disconnected. I believe the hypothesis that Spec A be profinite should be added to the statement of `0985`. Let's show that the pullback square there is a pullback square on points. Let $x \in X$. The x-fiber of $X' \times_X Y$ coincides with the fiber product of the x-fibers over x, so we replace X by x. The points don't change if we replace X' by $(X')_{\mathrm{red}}$, so X' is a product of purely inseparable field extensions of $k(x)$. Picking one, we reduce to $X' \to X$ a radicial, faithfully flat map; i.e. a universal homeomorphism. The claim is now obvious.

3.1.1. 'Surjection' = 'epimorphism.'

3.1.2. 'The forgetful functor commutes with connected limits' $\rightsquigarrow$ `002T`.

'and preserves surjections' $\rightsquigarrow$ follows since it preserves connected colimits (`04AR`).

In fact what you need and have is that the forgetful functor creates epimorphisms.

3.1.3. Suppose $X \in \operatorname{ob} \mathcal{X}$ with $X \twoheadrightarrow 1$ (1 being a final object in $\mathcal{X}$). If $F \twoheadrightarrow G$, then $F \times X \twoheadrightarrow G \times X$ since epimorphisms are universal in any topos; conversely if $F \times X \twoheadrightarrow G \times X$, then $F \times X \twoheadrightarrow G$, so $F \twoheadrightarrow G$.

3.1.4. 'The topos of sets is replete' $\rightsquigarrow$ this follows from the Mittag-Leffler condition.

'hence so is the topos of presheaves on a small category' $\rightsquigarrow$ limits of presheaves compute pointwise [**SGAA**, I 3.1].

the following way. A function $f \in S$ coming from the image of $\prod_{\beta_i} k(x)$ in S is in $\mathfrak{m}$ if $f(p_i(y)) = 0$. As $S/\mathfrak{m} = k(x)$, all the local rings of S are isomorphic to $k(x)$.

'As a special case, the classifying topos of a finite group G (which is simply the category of presheaves on B(G)) is replete' $\rightsquigarrow$ here B(G) refers to the one-object category with G as its Hom set. Then indeed the category of G-sets is the category of presheaves on B(G) [**SGAA**, IV 2.6].

3.1.5. When $\overline{k}/k$ is finite, $\widetilde{\mathrm{Spec}(k)_{\text{ét}}}$ is the classifying topos of $\mathrm{Gal}(\overline{k}/k)$, which we know from (3.1.4) to be replete. Let $x := \mathrm{Spec}\, k, \overline{x} := \mathrm{Spec}\, \overline{k}$. The étale sheaf represented by $\overline{x}$ then covers the final object x of $x_{\text{ét}}$, so is an epimorphism in $\widetilde{x_{\text{ét}}}$ [**SGAA**, II 4.4]. For the same reason, when $\overline{k}/k$ is an infinite extension, the maps $\mathrm{Spec}\, k_i \to \mathrm{Spec}\, k_{i-1}$ are epimorphisms. To see that the diagram $\cdots \to \mathrm{Spec}\, k_2 \to \mathrm{Spec}\, k_1 \to \mathrm{Spec}\, k_0$ has limit the initial object of $\widetilde{x_{\text{ét}}}$, by [**SGAA**, II 4.1.1] it suffices to show that there is no cone of this diagram belonging to $x_{\text{ét}}$, which is obvious.

3.1.6. An element f of the limit would set up an injection $\mathrm{T} \hookrightarrow \mathbf{Z}$ with $f(t) = f_{\mathrm{S}}(t)$ for $t \in \mathrm{T}$ and S large enough.

3.1.7. Let's connect the notion of epimorphism of sheaves in $\tilde{\mathrm{C}}$, where here C is the fpqc site of an affine scheme, to the intuition. Given $f : \mathrm{F} \to \mathrm{G}$ in $\tilde{\mathrm{C}}$, f is an epimorphism iff the sheafification of the presheaf image $f(\mathrm{F}) \subset \mathrm{G}$ is G; i.e. if $\underline{a}f(\mathrm{F}) = \mathrm{G}$. As $f(\mathrm{F})$ is a sub-presheaf of a separated presheaf, it's separated. Therefore we only have to apply the plus construction once to sheafify $f(\mathrm{F})$ [**SGAA**, II 3.2], so given $\mathrm{Y} = \mathrm{Spec}\, \mathrm{A} \in \mathrm{ob}\, \mathrm{C}$,

$$\underline{a}f(\mathrm{F})(\mathrm{Y}) = \varinjlim_{\mathrm{R} \in \mathrm{J}(\mathrm{Y})} \mathrm{Hom}_{\hat{\mathrm{C}}}(\mathrm{R}, f(\mathrm{F})).$$

In this case the topology on C is defined by a pretopology, and conjugating `022E` with `040I` we may assume each R in our colimit comes as the presheaf image (in Spec A) of a fpqc ring map $\mathrm{A} \to \mathrm{B}$. Then the condition that f be an epimorphism of sheaves equates to the property that given any section $s \in \mathrm{G}(\mathrm{Spec}\, \mathrm{A})$, there is such an $\mathrm{A} \to \mathrm{B}$ so that s is in $\mathrm{Hom}_{\hat{\mathrm{C}}}(\mathrm{R}, f(\mathrm{F}))$, where $\mathrm{R} \subset \mathrm{Spec}\, \mathrm{A}$ is the presheaf image of Spec B. This is equivalent to the property that the image of s in $\mathrm{G}(\mathrm{Spec}\, \mathrm{B})$ lies in $f(\mathrm{F})(\mathrm{Spec}\, \mathrm{B})$.

3.1.8. That $X_{n+1} \to X_n$ is epimorphic follows since $F_{n+1} \to F_n \times_{G_n} G_{n+1}$ is presumed epimorphic. The compatibility of the maps t_n with n comes about since the diagram

$$\begin{tikzcd}
X_{n+1} \arrow[r] \arrow[d, "t_{n+1}"] & X_n \arrow[d] \arrow[rd, bend left, "t_n"] & \\
F_{n+1} \arrow[r] \arrow[rrd, bend right] & F_n \times_{G_n} G_{n+1} \arrow[rd, "\mathrm{pr}_1"] & \\
& & F_n
\end{tikzcd}$$

commutes. That t_{n+1} lifts s_{n+1} means the diagram

$$\begin{tikzcd}
X_{n+1} \arrow[r] \arrow[d, "t_{n+1}"] & X_n \arrow[r] \arrow[d, "t_n"] & X \arrow[d, "s_n"] \\
F_{n+1} \arrow[r] & F_n \arrow[r] & G_n
\end{tikzcd}$$

commutes. This follows inductively from the case $n = 0$ via the compatibility of t_n with n. It remains to specify the map $X_n \to F_n \times_{G_n} G_{n+1}$; the map $X_n \to F_n$ is t_n, while the map $X_n \to G_{n+1}$ is $X_n \to X \xrightarrow{s_{n+1}} G_{n+1}$. Checking that these maps give the same map G_n amounts to verifying the commutativity of the above square, which we just did.

Forming this construction for $X = \lim G$ and $s = \mathrm{id}$ finds an epimorphism $X' \twoheadrightarrow \lim G$ that factors through $\lim F$.

3.1.9. Applying 3.1.8 in this case, the conditions to verify are that $\prod_{i<n} F_i \to \prod_{i<n} G_i$ and $\prod_{i<n+1} F_i \to \prod_{i<n} F_i \times G_{n+1}$ are epimorphic, provided that $F_i \to G_i$ is for each i. The latter follows as the base change of $F_{n+1} \twoheadrightarrow G_{n+1}$, since epimorphisms are universal in $\mathcal{X}$. The former follows by factoring $\prod_{i<n} G_i \to \prod_{i<n} F_i$ as

$$\prod_{i<n} G_i \to \prod_{i<n-1} G_i \times F_n \to \prod_{i<n-2} G_i \times F_{n-1} \times F_n \to \cdots \to \prod_{i<n} F_i$$

and using the same fact.

The assertion that countable products are exact should be understood in the context of abelian sheaves, since although we've seen that countable products preserve epimorphism, they needn't preserve coproducts in general; i.e. given sets A and B and a functor

$\mathrm{F} : \mathrm{A} \times \mathrm{B} \to \mathcal{X}$, the map

$$\coprod_{a \in \mathrm{A}} \prod_{b \in \mathrm{B}} \mathrm{F}(a, b) \to \prod_{b \in \mathrm{B}} \coprod_{a \in \mathrm{A}} \mathrm{F}(a, b)$$

is almost never an isomorphism, as can be checked when $\mathcal{X}$ is the category of sets and A and B are finite.

3.1.10. '$\prod_n \mathrm{F}_n$ computes the derived product in $\mathrm{D}(\mathcal{X})$' ⇝ `07KC`.

3.2.3. If $\mathrm{F} \to \mathrm{G}$ in $\mathcal{X}$ is an epimorphism, this means that the sheafification of the presheaf image is an epimorphism of presheaves. Recalling the description of sections of the sheafification (`00WK`), for every $s \in \mathrm{G}(\mathrm{Y})$, there is an epimorphic family[24] $\mathrm{Y}_i \to \mathrm{Y}$ so that $s|_{\mathrm{Y}_i}$ is in the image of $\mathrm{F}(\mathrm{Y}_i) \to \mathrm{G}(\mathrm{Y}_i)$ for all i. Letting $\mathrm{Y}' := \coprod_i \mathrm{Y}_i$, $s|_{\mathrm{Y}'}$ is the image of a section $t \in \mathrm{F}(\mathrm{Y}') \to \mathrm{G}(\mathrm{Y}')$. As $\mathrm{Y}' \to \mathrm{Y}$ is an epimorphism and Y is weakly contractible, there exists a section $s : \mathrm{Y} \to \mathrm{Y}'$, and $t|_{\mathrm{Y}}$ is sent to s.

The converse follows from the fact that the weakly contractible objects form a generating subcategory C of $\mathcal{X}$ [**SGAA**, II 4.9], so $\mathcal{X} \to \tilde{\mathrm{C}}$ is an equivalence [**SGAA**, IV 1.2.1]. The repleteness of $\mathcal{X}$ then follows from the fact that the category of sets is replete.

For (2), the notation $j_!$ means $j_{\mathrm{Y}!}$ [**SGAA**, IV 11.3.3], and $\mathrm{Hom}(\mathbf{Z}, j_{\mathrm{Y}}^*(-)) = \mathrm{H}^0(\mathrm{Y}, -)$ [**SGAA**, V 2.2]. As Y is weakly contractible, $\mathrm{H}^0(\mathrm{Y}, -)$ is an exact functor. As Y is quasi-compact, $\mathrm{H}^0(\mathrm{Y}, -)$ commutes with direct sums (`0935`). We conclude by (`094D`) that $j_{\mathrm{Y}!}\mathbf{Z}$ is compact. It remains to show that for every nonzero object K of $\mathrm{D}(\mathcal{X})$, there exists some i and a nonzero map from some $j_{\mathrm{Y}!}\mathbf{Z}$ to $\mathrm{K}[i]$. The fact that K is nonzero means that $\mathrm{H}^i\mathrm{K} \neq 0$ for some i; i.e. there is some Z in $\mathcal{X}$ so that $(\mathrm{H}^i\mathrm{K})(\mathrm{Z}) \neq 0$. Take some covering map $\mathrm{Y}_i \to \mathrm{Z}$ in $\mathcal{X}$ with Y_i weakly contractible coherent. Then $(\mathrm{H}^i\mathrm{K})(\mathrm{Y}_i) \neq 0$ for some i, which determines a map of complexes $j_{\mathrm{Y}!}\mathbf{Z} \to \mathrm{K}[i]$ inducing the specified nonzero map on H^0.

[24] Recall that a topos is given the canonical topology by default, and in any topos every epimorphic family is automatically universal and effective, hence covering for the canonical topology.

3.3.2. To find that $\mathrm{R}\,\mathrm{Hom}(\mathrm{K}, -)$ commutes with $\mathrm{R}\lim$, it's enough to check that $\mathrm{R}\,\mathrm{Hom}(\mathrm{K}, -)$ commutes with countable products, which is true since the product of K-injective complexes is K-injective (`0BK6`), $\mathrm{Hom}_{\mathrm{Ab}(\mathcal{X})}$ commutes with countable products in the second input, and countable products in Ab are exact.

The categories $\mathrm{D}(\mathcal{X}^{\mathbf{N}^\circ})$ and $\mathrm{D}(\mathcal{X})^{\mathbf{N}^\circ}$ are not equivalent; however, there is an obvious functor from the former to the latter which is essentially surjective (`0CQ9`). Moreover, while one can define $\mathrm{R}\lim$ beginning with $\mathrm{D}(\mathcal{X}^{\mathbf{N}^\circ})$ or $\mathrm{D}(\mathcal{X})^{\mathbf{N}^\circ}$ (the text adopts the latter definition), the two notions coincide (`08U5`). Therefore there is no ambiguity when applying $\mathrm{R}\lim$ to objects of $\mathrm{D}(\mathcal{X}^{\mathbf{N}^\circ})$.

About the purported exact triangle *when* $\{\mathrm{G}_n\}$ *is in* $\hat{\mathrm{D}}(\mathrm{X})$: we can replace $\{\mathrm{G}_n\}$ (in $\hat{\mathrm{D}}(\mathrm{X})$) by an inverse system of complexes $\{\mathrm{I}_n\}$ so that each I_n is a bounded below complex of injectives, and the the transition maps $\mathrm{I}_{n+1} \to \mathrm{I}_n$ are surjective, termwise-split morphisms of complexes. In this case, $\mathrm{Hom}(\mathrm{F}_*, \mathrm{I}_n)$ computes $\mathrm{R}\,\mathrm{Hom}(\mathrm{F}_*, \mathrm{G}_n)$ $(* = n, n+1)$, and indeed

$$\prod_n \mathrm{Hom}(\mathrm{F}_n, \mathrm{I}_n) \to \prod_n \mathrm{Hom}(\mathrm{F}_{n+1}, \mathrm{I}_n)$$

surjects. Moreover, $\{\mathrm{I}_n\}$ will be K-injective in $\mathrm{Ch}(\mathrm{Ab}(\mathrm{X}^{\mathbf{N}^\circ}))$. To construct I_n, we follow `070F`. We start with an inverse system of complexes $\{\mathrm{G}_n\}$ in $\hat{\mathrm{D}}(\mathrm{X})$ (so, in particular, the transition maps $\mathrm{G}_{n+1} \to \mathrm{G}_n$ are morphisms of complexes). We take a quasi-isomorphism $\mathrm{G}_0 \to \mathrm{I}_0$ into a complex of injectives zero in degrees < 0. Proceeding inductively, we may assume we have

$$\begin{tikzcd} \mathrm{G}_n \arrow[r] & \mathrm{G}_{n-1} \arrow[d] \\ & \mathrm{I}_{n-1}. \end{tikzcd}$$

We choose a distinguished triangle $\mathrm{C} \to \mathrm{G}_n \to \mathrm{I}_{n-1} \to$ in $\mathrm{K}(\mathrm{Ab}(\mathcal{X}))$ and a quasi-isomorphism $\mathrm{C} \to \mathrm{Q}$ into a bounded-below complex of injectives. We build a distinguished triangle $\mathrm{I}_{n-1}[-1] \to \mathrm{Q} \to \mathrm{I}_n \to$ off the composition $\mathrm{I}_{n-1}[-1] \to \mathrm{C} \to \mathrm{Q}$, and we may pick it so that the rotated distinguished triangle $\mathrm{Q} \to \mathrm{I}_n \to \mathrm{I}_{n-1} \to$ has the property that $0 \to \mathrm{Q}^p \to \mathrm{I}_n^p \to \mathrm{I}_{n-1}^p \to 0$ is split-exact for every p (`0G6C`); this implies that I_n^p is injective for all p and that $\mathrm{I}_n^p \to \mathrm{I}_{n-1}^p$ is a termwise split surjection whose kernel is injective. We get a morphism of distinguished triangles in $\mathrm{K}(\mathrm{Ab}(\mathcal{X}))$

$$\begin{tikzcd}
\mathrm{I}_{n-1}[-1] \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{C} \arrow[r] \arrow[d] & \mathrm{G}_n \arrow[r] \arrow[d] & {} \\
\mathrm{I}_{n-1}[-1] \arrow[r] & \mathrm{Q} \arrow[r] & \mathrm{I}_n \arrow[r] & {}
\end{tikzcd}$$

which is a quasi-isomorphism. The square

$$\begin{tikzcd}
\mathrm{G}_n \arrow[r] \arrow[d] & \mathrm{G}_{n-1} \arrow[d] \\
\mathrm{I}_n \arrow[r] & \mathrm{I}_{n-1}
\end{tikzcd}$$

only commutes up to homotopy, but as $\mathrm{I}_n \to \mathrm{I}_{n-1}$ is a termwise-split surjection, we can modify the map $\mathrm{G}_n \to \mathrm{I}_n$ by a homotopy to make the square commute on the nose.

To see that $\{\mathrm{I}_n\}$ is K-injective in $\mathrm{Ch}(\mathrm{Ab}(\mathrm{X}^{\mathbf{N}^\circ}))$: we must show that every map from an acyclic complex $\{\mathrm{M}_n\}$ to $\{\mathrm{I}_n\}$ is homotopic to 0. Fix a map f, and suppose we have constructed homotopies $h_n^i : \mathrm{M}_n^i \to \mathrm{I}_n^{i-1}$ in all degrees i and $n < m$ making commutative squares

$$\begin{tikzcd}
 & \mathrm{M}_n^i \arrow[dl] \arrow[d] \\
\mathrm{I}_n^{i-1} \arrow[d] & \mathrm{M}_{n-1}^i \arrow[dl] \\
\mathrm{I}_{n-1}^{i-1}. &
\end{tikzcd}$$

Writing $\mathrm{I}_m^i = \mathrm{I}_{m-1}^i \oplus \mathrm{J}_m^i$ with J_m^i injective, the map $\mathrm{M}_m^i \to \mathrm{I}_m^i \to \mathrm{I}_{m-1}^i$ coincides with $\mathrm{M}_m^i \to \mathrm{M}_{m-1}^i \to \mathrm{I}_{m-1}^i$ for all i, so a homotopy $h_m^{i+1} : \mathrm{M}_m^{i+1} \to \mathrm{I}_m^i$ is determined on the I_{m-1}^i factor by $\mathrm{M}_m^{i+1} \to \mathrm{M}_{m-1}^{i+1} \to \mathrm{I}_{m-1}^i$, where the second map is h_{m-1}^{i+1}. To determine what the homotopy h_m^{i+1} should send to the J_m^i factor, assuming h_m^i has been fixed (compatible with h_{m-1}^i), we have a commutative diagram

$$\begin{tikzcd}
\mathrm{M}_m^{i-1} \arrow[r, "d_{\mathrm{M}}^{i-1}"] \arrow[d] & \mathrm{M}_m^i \arrow[r, "d_{\mathrm{M}}^i"] \arrow[d, "f_m^i"] \arrow[dl, "h_m^i"'] & \mathrm{M}_m^{i+1} \arrow[d] \\
\mathrm{I}_m^{i-1} \arrow[r, "d_{\mathrm{I}}^{i-1}"] & \mathrm{I}_{m-1}^i \oplus \mathrm{J}_m^i \arrow[r] & \mathrm{I}_m^{i+1}.
\end{tikzcd}$$

As the map $\mathrm{M}_m^i \to \mathrm{I}_m^i$ given by $f_m^i - d_{\mathrm{I}}^{i-1} h_m^i$ factors through $\operatorname{coker} d_{\mathrm{M}}^{i-1}$ and J_m^i is injective,

$$\begin{tikzcd}
\operatorname{coker} d_{\mathrm{M}}^{i-1} \arrow[r, hook] \arrow[d, "f_m^i - d_{\mathrm{I}}^{i-1} h_m^i"] & \mathrm{M}_m^{i+1} \arrow[d, dotted] \\
\mathrm{I}_{m-1}^i \oplus \mathrm{J}_m^i \arrow[r] & \mathrm{J}_m^i
\end{tikzcd}$$

we get a map $\mathrm{M}_m^{i+1} \to \mathrm{J}_m^i$ which, complemented by the map $\mathrm{M}_m^{i+1} \to \mathrm{M}_{m-1}^{i+1} \to \mathrm{I}_{m-1}^i$ coming from h_{m-1}^{i+1}, determines a homotopy h_m^{i+1}. We have that $f_m^i = d_{\mathrm{I}}^{i-1} h_m^i + h_m^{i+1} d_{\mathrm{M}}^i$ after projection to J_m^i by construction, and after projection to I_{m-1}^i since the commutativity of

$$\begin{tikzcd}
\mathrm{I}_m^{i-1} \arrow[r] \arrow[d] & \mathrm{I}_m^i \arrow[d] \\
\mathrm{I}_{m-1}^{i-1} \arrow[r] & \mathrm{I}_{m-1}^i
\end{tikzcd}$$

shows that $\mathrm{J}_m^{i-1} \hookrightarrow \mathrm{I}_m^{i-1} \to \mathrm{I}_m^i \to \mathrm{I}_{m-1}^i$ is the null map, h_m^i was chosen compatible with h_{m-1}^i, and what h_m^{i+1} sends to I_{m-1}^i is determined by h_{m-1}^{i+1}. For the same reason, h_m^{i+1} is compatible with h_{m-1}^{i+1} in the sense that

$$\begin{tikzcd}
& \mathrm{M}_m^{i+1} \arrow[dl, "h_m^{i+1}"'] \arrow[dd] \\
\mathrm{I}_m^i \arrow[dd] & \\
& \mathrm{M}_{m-1}^{i+1} \arrow[dl, "h_{m-1}^{i+1}"'] \\
\mathrm{I}_{m-1}^i &
\end{tikzcd}$$

commutes. In this way we construct homotopies $h^i : \{\mathrm{M}_n\}^i \to \{\mathrm{I}_n\}^{i-1}$ which are maps in $\mathrm{Ch}(\mathrm{Ab}(\mathfrak{X}^{\mathbf{N}^\circ}))$ so that $f^i = d_{\mathrm{I}} h^i + h^{i+1} d_{\mathrm{M}}$, so indeed $\{\mathrm{I}^n\}$ is K-injective. In conclusion, letting $\mathrm{G} = \{\mathrm{G}_n\}$ and $\mathrm{I} = \{\mathrm{I}_n\}$, we've shown that whatever is $\mathrm{F} \in \operatorname{ob} \mathrm{Ch}(\mathrm{Ab}(\mathfrak{X}^{\mathbf{N}^\circ}))$,

$$0 \to \operatorname{Hom}(\mathrm{F}, \mathrm{I}) \to \prod_n \operatorname{Hom}(\mathrm{F}_n, \mathrm{I}_n) \to \prod_n \operatorname{Hom}(\mathrm{F}_{n+1}, \mathrm{I}_n) \to 0$$

is exact, $\operatorname{Hom}(\mathrm{F}, \mathrm{I})$ computes $\mathrm{R}\operatorname{Hom}(\mathrm{F}, \mathrm{G})$ in $\mathrm{D}(\mathfrak{X}^{\mathbf{N}^\circ})$, and $\operatorname{Hom}(\mathrm{F}_*, \mathrm{I}_n)$ computes $\mathrm{R}\operatorname{Hom}(\mathrm{F}_*, \mathrm{G}_n)$ in $\mathrm{D}(\mathfrak{X}^{\mathbf{N}^\circ})$ ($* = n, n+1$), giving rise to the claimed distinguished triangle.

It's easy to see that when $\mathrm{K} \in \mathrm{ob}\, \mathrm{D}(\mathcal{X})$ and $\{\mathrm{L}_n\} \in \mathrm{ob}\, \hat{\mathrm{D}}(\mathcal{X}^{\mathbf{N}^\circ})$, we have a map of distinguished triangles

$$\begin{array}{ccccccc} \mathrm{R}\,\mathrm{Hom}_{\hat{\mathrm{D}}(\mathcal{X})}(\tau(\mathrm{K}), \{\mathrm{L}_n\}) & \to & \prod_n \mathrm{R}\,\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\tau_{\geq -n}\mathrm{K}, \mathrm{L}_n) & \to & \prod_n \mathrm{R}\,\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\tau_{\geq -n-1}\mathrm{K}, \mathrm{L}_n) & \to \\ \downarrow & & \downarrow & & \downarrow & \\ \mathrm{R}\lim \mathrm{R}\,\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathrm{K}, \mathrm{L}_n) & \longrightarrow & \prod_n \mathrm{R}\,\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathrm{K}, \mathrm{L}_n) & \xrightarrow{\mathrm{id}-t} & \prod_n \mathrm{R}\,\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathrm{K}, \mathrm{L}_n) & \longrightarrow \end{array}$$

where t is induced by the transition map $\mathrm{L}_{n+1} \to \mathrm{L}_n$, and the upper triangle is the one just constructed. It will suffice to show that the leftmost map induces an isomorphism on H^0. This follows from the five lemma and the fact that

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\tau_{\geq *}\mathrm{K}, \mathrm{L}_n[i]) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{X})}(\mathrm{K}, \mathrm{L}_n[i])$$

is an isomorphism for $* = -n, -n-1$ and $i \leq 0$.

3.3.3. To complete the proof, it would be nice to show instead that $\mathrm{H}^{-j}\mathrm{K} \simeq \mathrm{H}^{-j}\mathrm{K}_i$ for $j \in \mathbf{Z}$, $j \leq i$. Indeed $\mathrm{H}^{-j}(\prod_n \mathrm{I}_n) = \prod_{n \geq j} \mathrm{H}^{-j}\mathrm{K}_n$, and $\mathrm{H}^{-j}\mathrm{K}_n = \mathrm{H}^{-j}\mathrm{K}_i$ when $j \leq i$. As the map $\mathrm{H}^{-j}(t - \mathrm{id})$ is surjective for all j with kernel $\lim_n \mathrm{H}^{-j}\mathrm{K}_n = \lim_n \mathrm{H}^{-j}\mathrm{K}_i = \mathrm{H}^{-j}\mathrm{K}_i$ when $j \leq i$, we're done.

3.3.7. (2) I will prove that $\mathrm{D}(\mathcal{X})$ is left-complete under the hypothesis that $\mathcal{X}$ *has enough points* and there exists a set $\mathcal{B}$ of objects of $\mathcal{X}$ with the properties that (i) every object of $\mathcal{X}$ can be covered by elements of $\mathcal{B}$,[25] and (ii) for each $\mathrm{X} \in \mathcal{B}$ and $\mathrm{F} \in \mathrm{ob}\, \mathrm{Ab}(\mathcal{X})$, $\mathrm{H}^q(\mathrm{X}, \mathrm{F}) = 0$ for $q > d$.[26]

Note that we may assume $\mathcal{X} = \tilde{\mathrm{C}}$ where C is a full subcategory of $\mathcal{X}$ containing $\mathcal{B}$ with the property that the limit (in $\mathcal{X}$) of any finite diagram of objects in C is also in C.[27] In particular, this means that the inclusion functor commutes with these finite limits, so that [**SGAA**, III 3.3] gives a description of covering families in C in terms of those in $\mathcal{X}$. By this description, a family of morphisms in C is covering iff it's epimorphic, and the fact that $\mathcal{B}$ is generating in C implies that if two presheaves in $\hat{\mathrm{C}}$ agree on $\mathcal{B}$, their

[25] i.e. there exists a jointly epimorphic family of maps with their sources in $\mathcal{B}$; i.e. $\mathcal{B}$ is generating.
[26] This statement may seem stronger than the hypothesis of the proposition, but it really isn't: if the corresponding statement is true for $\mathrm{H}^i\mathrm{K}$ for all i and all K in $\mathrm{D}(\mathcal{X})$, then it's true for all $\mathrm{F} \in \mathrm{ob}\, \mathrm{Ab}(\mathcal{X})$.
[27] Follow the proof of [**SGAA**, IV 1.2] iii) $\Rightarrow$ i').

sheafifications coincide. (We may not be able to take $\mathrm{C} = \mathcal{X}$ because of set-theoretic issues when discussing presheaves.)

As $\mathcal{B}$ is generating, given a point p of $\mathcal{X}$ and $\mathrm{F} \in \operatorname{ob}\mathcal{X}$,

$$\mathrm{F}_p = \varinjlim_{\mathrm{Vois}(p)^\circ} \mathrm{F}(\mathrm{X}),$$

and if $\mathcal{B}$ continues to refer to the full subcategory generated by the set $\mathcal{B}$, the full subcategory $\mathcal{B}_{/\varphi_p}$ of $\mathrm{Vois}(p) = \mathcal{X}_{/\varphi_p}$ consisting of those neighborhoods (X, u) with $\mathrm{X} \in \mathcal{B}$ is cofinal in $\mathrm{Vois}(p)$ [**SGAA**, IV 6.8]. As $\mathcal{X}$ has enough points, this allows us to use the argument of `0D62`.

Conjugating `071B` and `070M` finds that τ is fully faithful, noting that the condition that $\forall m \exists p(m) : \mathrm{H}^p(\mathrm{U}, \mathcal{H}^{m-p}) = 0$ for $p > p(m)$ is satisfied by $p(m) = d$, when U is one of the opens on which $\mathcal{H}^i := \mathrm{H}^i\mathrm{K}$ has cohomological dimension $\leq d$. To see that τ is essentially surjective, it suffices, given $\{\mathrm{K}_n\} \in \operatorname{ob}\hat{\mathrm{D}}(\mathcal{X})$, to find an isomorphism in $\hat{\mathrm{D}}(\mathcal{X})$ with a system of complexes $\{\mathrm{I}_n\}$ satisfying the conditions of `08BY`, as then $\mathrm{I} = \lim_n \mathrm{I}_n$ would have $\mathrm{H}^p(\mathrm{I})$ equal to the stable value of the system $\mathrm{H}^p(\mathrm{K}_n)$, and $\tau_{\geq -n}\mathrm{I} \to \mathrm{I}_n \simeq \mathrm{K}_n$ would be a quasi-isomorphism.

To find $\{\mathrm{I}_n\}$, we can by the note to 3.3.2 find a quasi-isomorphism $\{\mathrm{K}_n\} \to \{\mathrm{I}_n\}$ in $\mathrm{Ch}(\mathrm{Ab}(\mathcal{X}^{\mathbf{N}^\circ}))$, where each I_n is a bounded below complex of injectives, and the transition maps $\mathrm{I}_{n+1} \to \mathrm{I}_n$ are surjective and termwise-split. Then whatever $\mathrm{U} \in \mathcal{B}$ and $m \in \mathbf{Z}$, the systems of abelian groups $\mathrm{I}_n^{m-2}(\mathrm{U})$ and $\mathrm{I}_n^{m-1}(\mathrm{U})$ satisfy ML, and it will suffice to show that there exists an $n_0 \in \mathbf{Z}$ so that $\mathrm{H}^m(\mathrm{I}_n(\mathrm{U})) = \mathrm{H}^m(\mathrm{I}_{n_0}(\mathrm{U}))$ whenever $n \geq n_0$. As $\mathrm{I}_n(\mathrm{U})$ computes $\mathrm{R}\Gamma(\mathrm{U}, \mathrm{K}_n)$, we have a distinguished triangle

$$\mathrm{R}\Gamma(\mathrm{U}, \mathrm{C}) \to \mathrm{R}\Gamma(\mathrm{U}, \mathrm{K}_n) \to \mathrm{R}\Gamma(\mathrm{U}, \mathrm{K}_{n_0}) \to$$

where C is concentrated in degrees $[-n, -n_0)$. Since each cohomology sheaf of C has cohomological dimension $\leq d$ on U, it suffices to take $n_0 \geq d - m$.

3.4.1. Note that $\mathrm{T}(\mathrm{K}, x) \simeq \mathrm{R}\mathcal{H}om_{\mathrm{R}}(\mathrm{R}_x, \mathrm{K})$ in $\mathrm{D}(\mathcal{X}, \mathrm{R})$ (`0997`), which makes it obvious that $\mathrm{D}_{\mathrm{comp}}(\mathcal{X}, \mathrm{R})$ is a triangulated subcategory of $\mathrm{D}(\mathcal{X}, \mathrm{R})$.

3.4.6. $\rightsquigarrow$ `01EB`. $\mathrm{R}(\mathcal{X}) := \Gamma(\mathcal{X}, \mathrm{R}) = \mathrm{R}(e_{\mathcal{X}})$.

3.4.7. I prefer the proof `0997`. The proof of `0996` (2) relies on the adjunction

$$\mathrm{Hom}_{\mathrm{R}}(\mathrm{N},\mathrm{M}) = \mathrm{Hom}_{\mathrm{S}}(\mathrm{N}, \mathscr{H}\!om_{\mathrm{R}}(\mathrm{S},\mathrm{M}))$$

where $\mathrm{R} \to \mathrm{S}$ is a map of sheaves of rings, N is an R-module, and M is an S-module. Given an element α of the left-hand side and $n \in \mathrm{N}(\mathrm{U})$, we get an element of $\mathrm{Hom}_{\mathrm{R}|_{\mathrm{U}}}(\mathrm{S}|_{\mathrm{U}}, \mathrm{M}|_{\mathrm{U}})$ that sends a local section s of $\mathrm{S}|_{\mathrm{U}}(\mathrm{V})$ to $\alpha(\mathrm{V})(sn)$. On the other hand, an element β of the right-hand side determines a homomorphism of R-modules $\mathrm{N} \to \mathrm{M}$ by composing β with evaluation on the global section 1; i.e. $n \in \mathrm{N}(\mathrm{U})$ goes to the evaluation of $\beta(\mathrm{U})(n)$ on $1 \in \mathrm{S}(\mathrm{U})$.

3.4.9. I followed the exposition of `0995` to get to this result, which works in the greater generality of a ringed site.

Intermezzo: `0995`.

0998. The assignment $\mathrm{U} \mapsto \mathrm{I}(\mathrm{U})$ defines a presheaf, since if $\mathrm{U} \in \mathscr{C}$ and $j_{\mathrm{U}} : \mathscr{C} \to \mathscr{C}/\mathrm{U}$, homotopy limits commute with j_{U}^*. So I is a sub-presheaf of $\mathcal{O}$, and it's in fact a subsheaf, since $\mathrm{T}(\mathrm{K}|_{\mathrm{U}}, f) = 0$ means that all its cohomology sheaves are zero. Given a sheaf on $\mathscr{C}/\mathrm{U}$, if $j_i : \mathrm{V}_i \to \mathrm{U}$ cover U, then the functors $(j_i^*)_{i\in\mathrm{I}}$ are conservative.[28] So if $f_i \in \mathrm{I}(\mathrm{V}_i)$ are the restrictions of a section $f \in \mathcal{O}(\mathrm{U})$, then in fact $f \in \mathrm{I}(\mathrm{U})$.

0999. $\mathrm{D}_{\mathrm{comp}}(\mathcal{O})$ is preserved under products since given $(\mathrm{K}_i)_{i\in\mathrm{I}}$, $\prod_i \mathrm{K}_i$ in $\mathrm{D}(\mathcal{O})$ is computed by taking a quasi-isomorphism $\mathrm{K}_i \xrightarrow{\sim} \mathrm{I}_i$ where I_i is a K-injective complex of $\mathcal{O}$-modules, and then forming the product $\prod_{i\in\mathrm{I}} \mathrm{I}_i$, which is also K-injective (`0BK6`), so

$$\mathrm{R}\mathscr{H}\!om_{\mathcal{O}}\Big(\mathcal{O}_f, \prod_{i\in\mathrm{I}} \mathrm{K}_i\Big) = \mathscr{H}\!om_{\mathcal{O}}\Big(\mathcal{O}_f, \prod_{i\in\mathrm{I}} \mathrm{I}_i\Big) = \prod_{i\in\mathrm{I}} \mathscr{H}\!om_{\mathcal{O}}(\mathcal{O}_f, \mathrm{I}_i) = \prod_{i\in\mathrm{I}} \mathrm{R}\mathscr{H}\!om_{\mathcal{O}}(\mathcal{O}_f, \mathrm{K}_i).$$

$\mathrm{D}_{\mathrm{comp}}(\mathcal{O})$ is stable under homotopy limits in $\mathrm{D}(\mathcal{O})$ since it's stable under products and is triangulated.

[28]This follows from [**SGAA**, IV 8.5.1] and its note since then $\underline{a}\mathrm{V}_i \to \underline{a}\mathrm{U}$ cover the final object $\underline{a}\mathrm{U}$ of $\tilde{\mathrm{C}}/\underline{a}\mathrm{U}$ [**SGAA**, II 4.4, III 5.2 1)], and the functors $\widetilde{\mathrm{C}/\mathrm{U}} \to \tilde{\mathrm{C}}/\underline{a}\mathrm{U}$, $\widetilde{\mathrm{C}/\mathrm{V}_i} \to \tilde{\mathrm{C}}/\underline{a}\mathrm{V}_i$, and $\widetilde{\mathrm{C}/\mathrm{U}/\mathrm{V}_i} \to \widetilde{\mathrm{C}/\mathrm{V}_i}$ are equivalences [**SGAA**, III 5.4, 5.5 2)].

099B. If Q denotes the extended alternating Čech complex (0G6F)

$$\mathcal{O} \to \prod_{i_0} \mathcal{O}_{f_{i_0}} \to \prod_{i_0<i_1} \mathcal{O}_{f_{i_0} f_{i_1}} \to \cdots \to \mathcal{O}_{f_1 \cdots f_r},$$

then the map $K \to K^{\wedge}$ is induced by the map to the stupid truncation $Q \to \sigma_{\leq 0} Q$ (0118), which fits into the exact sequence

$$0 \to \sigma_{>0} Q \to Q \to \sigma_{\leq 0} Q \to 0.$$

If K is derived complete, it suffices to show that $R\mathcal{H}om_{\mathcal{O}}(\sigma_{>0} Q, K) = 0$. By repeated appeal to distinguished triangles coming from the exact sequences of stupid truncations and the fact that finite products and finite coproducts coincide in an additive category, this fact reduces to the stated one that $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K) = 0$ for all $f = f_{i_0} \cdots f_{i_p}$, $p \geq 0$.

The same argument gives that

$$R\mathcal{H}om_{\mathcal{O}}(K^{\wedge}, E) \to R\mathcal{H}om_{\mathcal{O}}(K, E)$$

is an isomorphism if $E \in \operatorname{ob} D_{\mathrm{comp}}(\mathcal{O})$. Indeed, $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is in $D(\mathcal{O}_{f_i})$ for some i, so

$$R\mathcal{H}om_{\mathcal{O}}(R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K), E) = 0$$

by (0997).

0A0E. To check that the Koszul complex is self-dual up to a shift: let E be a free R-module of rank n with basis $e_1, e_2, \ldots, e_n$, and $f : E \to R$ be a map of R-modules. The Koszul complex $K(E, f)$ looks like

$$\wedge^n E \to \wedge^{n-1} E \to \cdots \to E \to R$$

with R in degree 0. Putting $\wedge^n E$ instead in degree 0 and forming $\mathrm{Hom}_R(-, R)$, we get a complex whose terms are also zero outside $[-n, 0]$. Fixing an isomorphism $\wedge^n E \simeq R$ yields an identification $\mathrm{Hom}_R(\wedge^p E, R) \simeq \wedge^{n-p} E$, so that $e_{i_1} \wedge \cdots \wedge e_{i_p}$ is identified with the dual of $e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}$, where $\{1, \ldots, n\} = \{i_1, \ldots, i_p\} \coprod \{j_1, \ldots, j_{n-p}\}$. Let $(e_{j_1} \wedge \cdots \wedge e_{j_{n-p}})^{\vee}$ denote the basis element of $\mathrm{Hom}(\wedge^{n-p} E, R)$ that sends $e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}$

to 1. The differentials agree up to sign:

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum (-1)^{k+1} f(e_{i_k}) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p}$$

$$d((e_{j_1} \wedge \cdots \wedge e_{j_{n-p}})^{\vee}) = \sum_{j \notin \{j_1, \ldots, j_{n-p}\}} (-1)^t f(e_j)(e_{j_1} \wedge \cdots \wedge e_{j_t} \wedge e_j \wedge e_{j_{t+1}} \wedge \cdots \wedge e_{j_{n-p}})^{\vee},$$

where here and everwhere $1 \leq i_1 < \cdots < i_k < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_t < j < j_{t+1} < \cdots < j_{n-p} \leq n$. It's best to just think of removing e_{i_k} from $e_{i_1} \wedge \cdots \wedge e_{i_p}$ and adding it to $e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}$. Removing it gives $(-1)^{k+1} f(e_{i_k})$, adding it gives $(-1)^t f(e_{i_k})$, where $j_t < i_k < j_{t+1}$. The point is that $k + t = i_k$ since $\{1, \ldots, n\} = \{i_1, \ldots, i_p\} \coprod \{j_1, \ldots, j_{n-p}\}$. Therefore if we send $e_{i_1} \wedge \cdots \wedge e_{i_p}$ to $(-1)^{i_1 + \cdots + i_p + p}(e_{j_1} \wedge \cdots \wedge j_{n-p})^{\vee}$, we obtain an isomorphism of complexes $\mathrm{K} \to \mathrm{Hom}_{\mathrm{R}}(\mathrm{K}(\mathrm{E}, f)[-n], \mathrm{R})$.

This can be checked in the case $n = 2$: our morphism gives

$$\begin{tikzcd}
e_1 \wedge e_2 \arrow[r, "{(-1,1)}"] \arrow[d, "-1"] & e_1 \oplus e_2 \arrow[r, "{(1,1)}"] \arrow[d, shift right=3, "1"] \arrow[d, shift left=3, "-1"] & 1 \arrow[d, "1"] \\
1 \arrow[r, "{(1,1)}"] & e_2 \oplus e_1 \arrow[r, "{(1,-1)}"] & e_1 \wedge e_2,
\end{tikzcd}$$

between the Koszul complex (associated to $f(e_1) = 1 = f(e_2)$) in the top row and its dual in the bottom.

The reference to `0A0A` uses the coincidence of sequential colimits and homotopy colimits (`093W`), true since filtered colimits are exact in a topos.

099D. 'The category of the $(\mathrm{A}_0, \mathrm{I}_0) \to (\mathrm{A}, \mathrm{I})$ is filtered' $\rightsquigarrow$ Suppose given $f : (\mathrm{A}_0, \mathrm{I}_0) \to (\mathrm{A}, \mathrm{I})$ and $f' : (\mathrm{A}'_0, \mathrm{I}'_0) \to (\mathrm{A}, \mathrm{I})$. Suppose $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_m$) generate A_0 (resp. A'_0) as a $\mathbf{Z}$-algebra, and $a_1, \ldots, a_p$ (resp. $b_1, \ldots, b_q$) generate I_0 (resp. I'_0) as an ideal of A_0 (resp. A'_0). Let B_0 denote the subring of A generated as a $\mathbf{Z}$-algebra by the images of the x_i and y_i, and let J_0 denote the ideal of B_0 generated by the images of the a_i and b_i. The ideal J_0 has the property that $\mathrm{AJ}_0 = \mathrm{I}$, but it may be that J_0 is not generated as an ideal of B_0 by the images of the a_i only (or by the images of the b_i only). We know, however, that for each $i \in [1, p]$, a_i can be written as a linear combination of the b_j with coefficients in A, and vice versa for $j \in [1, q]$. Adding finitely many generators to B_0, therefore, $g : (\mathrm{B}_0, \mathrm{B}_0\mathrm{J}_0) \to (\mathrm{A}, \mathrm{I})$ has the property that f and f' factor though g and $\mathrm{B}_0 f(\mathrm{I}_0) = \mathrm{J}_0 = \mathrm{B}_0 f'(\mathrm{I}'_0)$.

Next, given a commutative diagram

$$\begin{tikzcd} & (\mathrm{A}, \mathrm{I}) & \\ (\mathrm{A}_0, \mathrm{I}_0) \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] & & (\mathrm{A}'_0, \mathrm{I}'_0), \arrow[ul] \end{tikzcd}$$

let B_0 be the subring of A generated as a $\mathbf{Z}$-algebra by the images of the finitely many generators of A'_0, and let J_0 be the ideal of B_0 generated by the image of I'_0 in B_0. Then $(\mathrm{B}_0, \mathrm{J}_0)$ is in $\mathscr{A}_0$ and the diagram

$$\begin{tikzcd} (\mathrm{A}, \mathrm{I}) & (\mathrm{B}_0, \mathrm{J}_0) \arrow[l] \\ (\mathrm{A}_0, \mathrm{I}_0) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & (\mathrm{A}'_0, \mathrm{I}'_0) \arrow[u] \arrow[ul] \end{tikzcd}$$

commutes.

'Since the topology on $\mathscr{C}_0$ is chaotic, the value $\mathscr{I}^{\bullet}(\mathrm{U})$ is a resolution of A by injective A-modules' $\rightsquigarrow$ it is a resolution since 'sections over U' is an exact functor. This functor preserves injectives because it has an exact left adjoint; namely, the functor which assigns to an A-module M the presheaf on $\mathscr{C}_0$ with sections over U given by M and sections over U′ given by 0 ($\mathrm{U}' \neq \mathrm{U}$). This is a presheaf of $\mathbb{O}$-modules.

The existence of $\mathscr{K}^{\bullet}$ making the given diagram commute *up to homotopy* follows from the fact that the quasi-isomorphisms in $\mathrm{K}(\mathbb{O})$ form a multiplicative system.

In verifying properties (a–d) in general, knowing them for $\mathscr{A}_0$, (a) and (b) follow directly from the construction, and (c) is an immediate consequence of (d). To show (d), pick $(\mathrm{A}'_0, \mathrm{I}'_0)$ in $\mathscr{A}_0$ and a map $(\mathrm{A}'_0, \mathrm{I}'_0) \to (\mathrm{A}, \mathrm{I})$ so that if $\mathrm{I}'_0 = (f'_1, \dots, f'_r)$, the image of f'_i in A is f_i. Then

$$\mathrm{K}(\mathrm{A}'_0, \mathrm{I}'_0) \simeq \Big(\mathrm{A}'_0 \to \prod_{i_0} (\mathrm{A}'_0)_{f'_{i_0}} \to \prod_{i_0 < i_1} (\mathrm{A}'_0)_{f'_{i_0} f'_{i_1}} \to \cdots \to (\mathrm{A}'_0)_{f'_1 \cdots f'_r}\Big)$$

in $D(A'_0)$ and for any map $(A'_0, I'_0) \to (A_0, I_0)$ in $\mathscr{A}_0$, $K(A'_0, I'_0) \otimes^{\mathbf{L}}_{A'_0} A_0 \xrightarrow{\sim} K(A_0, I_0)$ is an isomorphism in $D(A_0)$. As the alternating Čech complex is a K-flat complex of A'_0-modules, tensor product commutes with colimits, and filtered colimits are exact,

$$\begin{aligned}
K(A, I) &= \operatorname*{colim}_{(A_0,I_0)\to(A,I)} K(A_0, I_0) \\
&\simeq \operatorname*{colim}_{(A_0,I_0)\to(A,I)} \Big(A'_0 \to \prod_{i_0} (A'_0)_{f'_{i_0}} \to \prod_{i_0<i_1} (A'_0)_{f'_{i_0} f'_{i_1}} \to \cdots \to (A'_0)_{f'_1 \dots f'_r}\Big) \otimes_{A'_0} A_0 \\
&= \Big(A'_0 \to \prod_{i_0} (A'_0)_{f'_{i_0}} \to \prod_{i_0<i_1} (A'_0)_{f'_{i_0} f'_{i_1}} \to \cdots \to (A'_0)_{f'_1 \dots f'_r}\Big) \otimes_{A'_0} \operatorname*{colim}_{(A_0,I_0)\to(A,I)} A_0 \\
&= A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0} f_{i_1}} \to \cdots \to A_{f_1 \dots f_r},
\end{aligned}$$

where the $\simeq$ is in $D(A)$. These maps are clearly compatible with the maps to A.

0955. The formal argument that gives (1) $\Rightarrow$ (2): let $i : D(I^\infty\text{-torsion}) \to D_{I^\infty\text{-torsion}}(A)$ be the functor 0A6N. To show that i is an equivalence, it suffices to show that $\eta : K \to R\Gamma_I\, i(K)$ is an isomorphism in $D(I^\infty\text{-torsion})$. Since the functors H^i are conservative and factor through i, it suffices to show that $i\eta$ is an isomorphism. But

$$iK \xrightarrow{i\eta} i\, R\Gamma_I\, iK \xrightarrow{\varepsilon i} iK$$

composes to the identity and ε is an isomorphism by hypothesis.

As for (3), if $j : D_{I^\infty\text{-torsion}}(A) \hookrightarrow D(A)$, then a right adjoint of ji is $R\Gamma_I$, but another is $R\Gamma_I|_{D_{I^\infty\text{-torsion}}(A)}\, R\Gamma_Z$. Therefore these two functors are isomorphic, and the transformation of functors $R\Gamma_I(K) \xrightarrow{\sim} R\Gamma_Z(K)$ is more properly written

$$jiR\Gamma_I \simeq ji\, R\Gamma_I|_{D_{I^\infty\text{-torsion}}(A)}\, R\Gamma_Z \xrightarrow{\varepsilon} jR\Gamma_Z$$

where $\varepsilon : i\, R\Gamma_I|_{D_{I^\infty\text{-torsion}}(A)} \to \mathrm{id}$ is an isomorphism.

099E. Recalling the definition of a finite type sheaf of ideals (03DL), the comparison lemma [**SGAA**, III 4.1] gives that $\widetilde{\mathscr{C}} \xrightarrow{\sim} \widetilde{\mathscr{C}}'$.

If I and I′ are both finitely generated ideals of $\mathscr{I}(U)$ that generate $\mathscr{I}|_U$, then

$$\mathscr{I}_U = \mathrm{im}(\mathscr{O}_U \otimes I \to \mathscr{O}_U) = \mathrm{im}(\mathscr{O}_U \otimes I'\mathscr{O}_U).$$

Letting $\mathscr{M}, \mathscr{M}'$ denote the presheaves $V \mapsto I\mathscr{O}(V)$ (resp. $V \mapsto I'\mathscr{O}(V)$) on $\mathscr{C}/U$,

$$\operatorname*{colim}_{R\in J(U)} \mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R, \mathscr{I}) = \mathscr{I}(U) = \operatorname*{colim}_{R\in J(U)} \mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R, \mathscr{M}) = \operatorname*{colim}_{R\in J(U)} \mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R, \mathscr{M}')$$

As the category $J(U)$ of covering sieves of U is cofiltered, we can find a $R' \in J(U)$ so that the image of $\mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R', \mathscr{M})$ in $\mathscr{I}(U)$ contains I', and so that the image of $\mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R', \mathscr{M}')$ in $\mathscr{I}(U)$ contains I. In other words, the image of

$$\mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R', \mathscr{M}) \to \mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R', \mathscr{I})$$

contains the image of I' via $\mathscr{I}(U) \to \mathrm{Hom}_{\widehat{\mathscr{C}/U}}(R', \mathscr{I})$, and likewise for $\mathscr{M}'$ and I. For any $U_j \to R'$ (U_j in $\mathscr{C}/U$), then, $I' \subset \mathscr{M}(U_j)$ and $I \subset \mathscr{M}'(U_j)$, so $\mathscr{M}(U_j) = \mathscr{M}'(U_j)$.

099F. To show that $E^\wedge$ is derived complete, it suffices to show that the sheaf of ideals $\mathscr{T} \subset \mathscr{O}$ of elements $f \in \mathscr{O}(U)$ such that $T(E^\wedge|_U, f) = 0$ contains $\mathscr{I}$ (0998). This is clear since for any U in $\mathscr{C}$, there is a covering $\{U_j \to U\}$ so that $\mathscr{I}|_{U_j}$ is generated by finitely many global sections, so $\mathscr{I}|_{U_j} \subset \mathscr{T}|_{U_j}$ by 099B. For the same reason, $E \to E^\wedge$ is an isomorphism since if $u : \mathscr{C}' \subset \mathscr{C}$, $u^*E \xrightarrow{\sim} u^*E^\wedge$ is, and $u^* : \widetilde{\mathscr{C}} \xrightarrow{\sim} \widetilde{\mathscr{C}}'$ (in the notation of 099E).

3.4.12. It's useful to recall the coincidence of the right derived functor of lim and the notion of homotopy limit for inverse sequential systems in a topos (0940). It's easy to check directly that given a topos $\mathscr{X}$, the category $\mathscr{X}^{\mathbf{N}^\circ}$ of functors $\mathbf{N}^\circ \to \mathscr{X}$[29] is a topos. It is simple to check since everything is computed pointwise: initial and final objects are given by pointwise initial and final objects; pushouts, coequalizers, fiber products and direct sums are computed pointwise; a map is a monomorphism, epimorphism, or (universal) effective epimorphism iff it is pointwise. If $\mathscr{X} = \tilde{C}$, then $\mathscr{X}^{\mathbf{N}^\circ}$ is given as sheaves on the site with objects $C \times \mathbf{N}$ with morphisms and topology as in (0940). The point is that if $U_i \to U$ is a covering family in C generating a sieve $R \hookrightarrow U$, then the sieve of $C \times \mathbf{N}$ generated by the family $(U_i, n) \to (U, n)$ is the sieve with objects (V, m) with V in R and $m \leq n$. This description makes it easy to verify the axioms T 1–3) of a topology, and by [**SGAA**, I 3.5, I 2.12], it's easy to see that sheaves on $C \times \mathbf{N}$ are inverse systems of sheaves on C; i.e. objects of $\mathscr{X}^{\mathbf{N}^\circ}$.

[29] $\ldots \to 3 \to 2 \to 1 \to 0$.

If $-_n : \mathcal{X}^{\mathbf{N}^\circ} \to \mathcal{X}$ denotes the functor that picks out the nth object, then it commutes with fiber products and sends final object to final object, so by [**SGAA**, I 10.9] if $\mathrm{F} \rightrightarrows \mathrm{G}$ is an equivalence relation on G in $\mathcal{X}^{\mathbf{N}^\circ}$, then $\mathrm{F}_n \rightrightarrows \mathrm{G}_n$ is an equivalence relation on G_n in $\mathcal{X}$, so is effective universal. As this is true for all n, and pushouts in $\mathcal{X}^{\mathbf{N}^\circ}$ are computed pointwise, there is a morphism π (determined by the π_n) so that the square

$$\begin{tikzcd} \mathrm{F} \arrow[r] \arrow[d] & \mathrm{G} \arrow[d, "\pi"] \\ \mathrm{G} \arrow[r, "\pi"] & \mathrm{H} \end{tikzcd}$$

is cartesian and co-cartesian. As π_n is a universal effective epimorphism for all n, π is the same. Last, for every object E in a $\mathcal{U}$-small generating family for $\mathcal{X}$ and for each $n \in \mathbf{N}$, there is an object of $\mathcal{X}^{\mathbf{N}^\circ}$ with E in the mth spot for $m \leq n$ (with transition maps the identity) and the initial object in the rest. This family of objects is a $\mathcal{U}$-small generating family for $\mathcal{X}^{\mathbf{N}^\circ}$.

The functor lim gives g_* in a morphism of topos $g : \mathcal{X}^{\mathbf{N}^\circ} \to \mathcal{X}$, with g^* the functor that takes an object of $\mathcal{X}$ to the constant inverse system. For each $n \in \mathbf{N}$ we also have a morphism of topos $i_n : \mathcal{X} \to \mathcal{X}^{\mathbf{N}^\circ}$ with i_n^* picking out the nth term and i_{n*} taking an object E of $\mathcal{X}$ to the system which is constant with value E in terms $\geq n$ and consists of the final object in terms $< n$. The functor i_n^* also has a left adjoint $i_{n!}$ that takes E to the system described in the previous paragraph: constant with value E in terms $\leq n$ and the initial object in terms $> n$. The sequence $(i_{n!}, i_n^*, i_{n*})$ of adjoint functors determines morphisms of topos $(i_{n!}, i_n^*) : \mathcal{X}^{\mathbf{N}^\circ} \to \mathcal{X}$ and $(i_n^*, i_{n*}) : \mathcal{X} \to \mathcal{X}^{\mathbf{N}^\circ}$.

There are transformations of functors $i_{n+1}^* \to i_n^*$, so given $\mathrm{F} \in \mathrm{D}(\mathcal{X}^{\mathbf{N}^\circ})$, we get an inverse system (F_n) in $\mathrm{D}(\mathcal{X})$, and we can take the homotopy limit of this system. On the other hand, we can form the right derived functor of g_*, R lim. It turns out that $\mathrm{R}g_*$ computes the homotopy limit (`0940`) of the system. If R is a ring in $\mathcal{X}$, given an exact triangle

$$(\mathrm{K}_n) \to (\mathrm{L}_n) \to (\mathrm{M}_n) \to$$

in $\mathrm{D}(\mathcal{X}^{\mathbf{N}^\circ}, \mathrm{R})$ and an object J in $\mathrm{D}(\mathcal{X}, \mathrm{R})$, then the triangle

$$\mathrm{R}\lim((\mathrm{K}_n) \otimes^{\mathbf{L}}_{\mathrm{R}} g^*\mathrm{J}) \to \mathrm{R}\lim((\mathrm{L}_n) \otimes^{\mathbf{L}}_{\mathrm{R}} g^*\mathrm{J}) \to \mathrm{R}\lim((\mathrm{M}_n) \otimes^{\mathbf{L}}_{\mathrm{R}} g^*\mathrm{J}) \to$$

(with $\otimes^{\mathbf{L}}_{\mathrm{R}} := \otimes^{\mathbf{L}}_{g^*\mathrm{R}}$ computed in $\mathrm{D}(\mathfrak{X}^{\mathbf{N}^\circ}, g^*\mathrm{R})$), is distinguished. Applying this to the distinguished triangle

$$((x_1^n, \ldots, x_r^n)) \to (\mathbf{Z}[x_1, \ldots, x_r]) \to (\mathbf{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n)) \to$$

in $\mathrm{D}(\mathfrak{X}^{\mathbf{N}^\circ}, \mathbf{Z}[x_1, \ldots, x_r])$, we get a distinguished triangle

$$\mathrm{R}\lim(g^*\mathrm{J} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]} ((x_1^n, \ldots, x_r^n))) \to \mathrm{J} \to \mathrm{R}\lim(g^*\mathrm{J} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]} (\mathbf{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n))) \to .$$

As the two senses of $\mathrm{R}\lim$ coincide, and

$$i_n^*(g^*\mathrm{J}\otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]}(\mathbf{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n))) = \mathrm{J}\otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]}\mathbf{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n),$$

$$\mathrm{R}\lim(g^*\mathrm{J}\otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]}(\mathbf{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n))) \simeq \mathrm{R}\lim(\mathrm{J}\otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\ldots,x_r]}\mathbf{Z}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n)),$$

where the $\mathrm{R}\lim$ on the left side is $\mathrm{R}g_*$ and on the right side is the homotopy limit.

One way of defining $\mathrm{R}\lim(\mathrm{K} \xrightarrow{x^n} \mathrm{K})$ goes like this: if $x \in \mathrm{I}(\mathfrak{X})$, the inverse systems

$$\mathrm{K}_x := \cdots \to \mathrm{K} \xrightarrow{x} \mathrm{K} \xrightarrow{x} \mathrm{K} \quad \text{and}$$

$$\mathrm{K}_{\mathrm{id}} := \cdots \to \mathrm{K} \xrightarrow{\mathrm{id}} \mathrm{K} \xrightarrow{\mathrm{id}} \mathrm{K}$$

define objects of $\mathrm{Comp}(\mathfrak{X}^{\mathbf{N}^\circ}, \mathrm{R})$, and $\mathrm{K} \xrightarrow{x^n} \mathrm{K}$ defines a morphism $\mathrm{K}_x \to \mathrm{K}_{\mathrm{id}}$ in $\mathrm{Comp}(\mathfrak{X}^{\mathbf{N}^\circ}, \mathrm{R})$. If C denotes the cone of this morphism in $\mathrm{D}(\mathfrak{X}^{\mathbf{N}^\circ}, \mathrm{R})$, then the triangle

$$\mathrm{T}(\mathrm{K}, x) \to \mathrm{K} \to \mathrm{R}\lim(\mathrm{C}) \to$$

is distinguished, and can say $\mathrm{R}\lim(\mathrm{K} \xrightarrow{x^n} \mathrm{K}) := \mathrm{R}\lim(\mathrm{C})$.

The reference to Lemmas 3.4.6 and 3.4.7 is equivalent to `0998`.

(4) $\Rightarrow$ (3): Let B := Replacing $\mathfrak{X}$ by $\mathfrak{X}/\mathrm{U}_i$, R by $\mathrm{R}|_{\mathrm{U}_i}$, K by $\mathrm{K}|_{\mathrm{U}_i}$, etc., we want to see that $\mathrm{R}\mathscr{H}om_{\mathrm{R}}(\mathrm{R}_{x_i}, \mathrm{K})$ is acyclic. Since homotopy limits commute with localization and shifts, it will suffice to show that

$$\mathrm{H}^0\mathrm{R}\Gamma(\mathfrak{X}, \mathrm{R}\mathscr{H}om_{\mathrm{R}}(\mathrm{R}_{x_i}, \mathrm{K})) = \mathrm{Hom}_{\mathrm{D}(\mathrm{R})}(\mathrm{R}_{x_i}, \mathrm{K})$$

vanishes. The exact sequence

$$\begin{aligned} &0 \to \mathrm{R}^1 \lim \mathrm{Hom}_{\mathrm{D(R)}}(\mathrm{R}_{x_i}, \mathrm{K}[-1] \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathbf{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n)) \\ &\to \mathrm{Hom}_{\mathrm{D(R)}}(\mathrm{R}_{x_i}, \mathrm{K}) \to \lim \mathrm{Hom}_{\mathrm{D(R)}}(\mathrm{R}_{x_i}, \mathrm{K} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathbf{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n)) \to 0 \end{aligned} \tag{0919}$$

means that it's enough to show that

$$\mathrm{Hom}_{\mathrm{D(R)}}(\mathrm{R}_{x_i}, \mathrm{K} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathbf{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n)) = 0$$

for any K in D(R). Let $\mathrm{L} := \mathrm{K} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathbf{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n)$. Any such morphism

$$\mathrm{R}_{x_i} \leftarrow \mathrm{A} \to \mathrm{L}$$

in D(R) is equivalent to the morphism

$$\mathrm{R}_{x_i} \leftarrow \mathrm{A} \xleftarrow{x_i^n} \mathrm{A} \xrightarrow{x_i^n} \mathrm{A} \to \mathrm{L} = \mathrm{R}_{x_i} \xleftarrow{x_i^n} \mathrm{R}_{x_i} \leftarrow \mathrm{A} \to \mathrm{L} \xrightarrow{x_i^n} \mathrm{L},$$

which is zero.

(3) $\Rightarrow$ (4) This is equivalent to `0A0E`, since the vanishing of $\mathrm{T}(\mathrm{K}, x_i)$ implies (`0998`) that K is derived complete since the x_i generate I, so $\mathrm{K} \to \mathrm{K}^\wedge$ is an isomorphism. The Koszul complex $\mathrm{K}'_n = \mathrm{K}(\mathbf{Z}[x_1,\dots,x_r], x_1^n,\dots,x_r^n)$ is a free resolution of $\mathbf{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n)$, and the Koszul complex $\mathrm{K}_n = \mathrm{K}(\mathrm{R}, x_1^n,\dots,x_r^n)$ is obtained as $\mathrm{K}'_n \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathrm{R}$. In conclusion,

$$\begin{aligned} \mathrm{K} \xrightarrow{\sim} \mathrm{K}^\wedge &= \mathrm{R}\lim \mathrm{K} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{K}_n = \mathrm{R}\lim(\mathrm{K} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathrm{K}'_n) \\ &= \mathrm{R}\lim(\mathrm{K} \otimes^{\mathbf{L}}_{\mathbf{Z}[x_1,\dots,x_r]} \mathbf{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n)). \end{aligned}$$

3.4.13. This is the first result in the section that uses that $\mathcal{X}$ is replete,[30] and all the results that follow this one in §3.4 require repleteness. The business about commuting homotopy-limits is essentially `0A07`. Namely, as homotopy-limits commute with localization, suppose given an inverse system (F_n) of objects of $\mathcal{X}$ and a global section $x \in \mathrm{I}(\mathcal{X})$. We can replace the system (F_n) by a system with transition maps that

[30]Excepting 3.4.2 and 3.4.4, which also require repleteness.

are morphisms of complexes, and form the lattice

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathrm{F}_{n+1} \arrow[r, "x"] \arrow[d] & \mathrm{F}_{n+1} \arrow[r, "x"] \arrow[d] & \mathrm{F}_{n+1} \arrow[d] \\
\cdots \arrow[r] & \mathrm{F}_{n} \arrow[r, "x"] \arrow[d] & \mathrm{F}_{n} \arrow[r, "x"] \arrow[d] & \mathrm{F}_{n} \arrow[d] \\
 & \vdots & \vdots & \vdots
\end{tikzcd}$$

which is an object F of $\mathrm{D}((\mathcal{X}^{\mathbf{N}^\circ})^{\mathbf{N}^\circ})$. There are two ways of viewing an object of $\mathrm{D}((\mathcal{X}^{\mathbf{N}^\circ})^{\mathbf{N}^\circ})$ as an inverse system of inverse systems of complexes, corresponding to two different morphisms of topos $g_1, g_2 : \mathrm{D}((\mathcal{X}^{\mathbf{N}^\circ})^{\mathbf{N}^\circ}) \to \mathrm{D}(\mathcal{X}^{\mathbf{N}^\circ})$. The square

$$\begin{tikzcd}
(\mathcal{X}^{\mathbf{N}^\circ})^{\mathbf{N}^\circ} \arrow[r, "g_1"] \arrow[d, "g_2"] & \mathcal{X}^{\mathbf{N}^\circ} \arrow[d, "g"] \\
\mathcal{X}^{\mathbf{N}^\circ} \arrow[r, "g"] & \mathcal{X}
\end{tikzcd}$$

commutes, as can be seen since $g_2^* g^* = g_1^* g^*$. As sequential limits in $\mathcal{X}^{\mathbf{N}^\circ}$ are computed pointwise, one of $g_{1*}g_*$ or $g_{2*}g_*$ equals $\mathrm{T}(\mathrm{R}\lim \mathrm{F}_n, x)$ and the other equals $\mathrm{R}\lim(\mathrm{T}(\mathrm{F}_n, x))$, so if the F_n are derived I-complete, then $\mathrm{R}\lim \mathrm{F}_n$ is too.

0A07. Given a morphism of ringed topos $f : (\mathcal{X}, \mathrm{R}) \to (\mathcal{Y}, \mathrm{S})$, we get a morphism of ringed topos $f \times 1 : (\mathcal{X}^{\mathbf{N}^\circ}, \mathrm{R}) \to (\mathcal{Y}^{\mathbf{N}^\circ}, \mathrm{S})$, where $\mathrm{R} = g^*\mathrm{R}$, etc. by applying f^*, f_* termwise. The resulting functor $(f \times 1)^*$ is exact since finite limits in $\tilde{\mathrm{C}}'^{\mathbf{N}^\circ}$ are computed termwise. It's then clear that the square

$$\begin{tikzcd}
\mathcal{X}^{\mathbf{N}^\circ} \arrow[r, "g"] \arrow[d, "f\times 1"] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{Y}^{\mathbf{N}^\circ} \arrow[r, "g"] & \mathcal{Y}
\end{tikzcd}$$

commutes since $(f \times 1)^* g^* = g^* f^*$.

3.4.13. If $\mathrm{I}^n\mathrm{M} = 0$, then any local section $x \in \mathrm{I}(\mathrm{U})$ for U in $\mathcal{X}$ acts nilpotently on $\mathrm{M}|_{\mathrm{U}}$, so $\mathrm{T}(\mathrm{M}|\mathrm{U}, x) = \mathrm{R}\mathscr{H}om_{\mathrm{R}|_{\mathrm{U}}}((\mathrm{R}|_{\mathrm{U}})_x, \mathrm{M}|_{\mathrm{U}}) = 0$ since

$$\mathrm{H}^i\mathrm{R}\Gamma(\mathrm{V}, \mathrm{R}\mathscr{H}om_{\mathrm{R}|_{\mathrm{U}}}((\mathrm{R}|_{\mathrm{U}})_x, \mathrm{M}|_{\mathrm{U}})) = \mathrm{Hom}_{\mathrm{D}(\mathrm{R}|_{\mathrm{V}})}((\mathrm{R}|_{\mathrm{V}})_x, \mathrm{M}|_{\mathrm{V}}[i]) = 0$$

for all V in $\mathcal{X}/\mathrm{U}$ and $i \in \mathbf{Z}$, by the same argument as in (4) $\Rightarrow$ (3) in the note to 3.4.12; i.e. any such morphism

$$(\mathrm{R}|_{\mathrm{V}})_x \leftarrow \mathrm{A} \rightarrow \mathrm{M}|_{\mathrm{V}}[i]$$

in $\mathrm{D}(\mathrm{R}|_{\mathrm{V}})$ is equivalent to the morphism

$$(\mathrm{R}|_{\mathrm{V}})_x \leftarrow \mathrm{A} \xleftarrow{x^n} \mathrm{A} \xrightarrow{x^n} \mathrm{A} \rightarrow \mathrm{M}|_{\mathrm{V}}[i] = (\mathrm{R}|_{\mathrm{V}})_x \xleftarrow{x^n} (\mathrm{R}|_{\mathrm{V}})_x \leftarrow \mathrm{A} \rightarrow \mathrm{M}|_{\mathrm{V}}[i] \xrightarrow{x^n} \mathrm{M}|_{\mathrm{V}}[i],$$

which is zero.

3.4.14. It's clear from the proof that the claim is that the full subcategory consisting of derived I-complete R-modules is a *weak* Serre abelian subcategory of R-Mod. The given distinguished triangle is just the distinguished triangle associated to $\tau_{<0} \rightarrow \mathrm{id} \rightarrow \tau_{\geq 0} \rightarrow$ applied to the complex $\mathrm{L} := \mathrm{M} \rightarrow \mathrm{N}$, with N in degree 0. Applying $\mathrm{T}(-, x)$ to the distinguished triangle $\mathrm{M} = \sigma_{<0}\mathrm{L} \rightarrow \mathrm{L} \rightarrow \sigma_{\geq 0}\mathrm{L} = \mathrm{N} \rightarrow$ shows that $\mathrm{T}(\mathrm{L}, x) = 0$.

3.4.2. The existence of this exact sequence follows from the same proof as for abelian groups (`07KY`), since products in $\mathcal{X}$ are exact, so the right derived functor of lim on inverse systems F_n in $\mathrm{Ab}(\mathcal{X})$, which is always a homotopy-limit (`0941`), is represented by the complex $\prod \mathrm{F}_n \rightarrow \prod \mathrm{F}_n$, with the usual map given by the difference between the identity and the transition (`07KC` & 3.1.11). Of course, the topologies on M given by $\mathrm{I}^n\mathrm{M}$ and $(x_1^n, \ldots, x_r^n)\mathrm{M}$ are cofinal; i.e. the pro-objects $(\mathrm{M}/\mathrm{I}^n\mathrm{M})$ and $(\mathrm{M}/(x_1^n, \ldots, x_r^n)\mathrm{M})$ are isomorphic.

`099M`. In the course of the proof (`0921`) that the pro-objects (K_n) and (Λ/I^n) are isomorphic, we saw that the kernel of the map $(\mathrm{K}_n) \rightarrow (\mathrm{H}^0\mathrm{K}_n) = (\Lambda/(f_1^n, \ldots, f_r^n))$ is pro-zero in $\mathrm{D}(\Lambda)$ using the criterion of (`0G3B`). The same criterion in turn implies that the inverse system of constant presheaves with value this kernel is pro-zero, and, since sheafification is exact, the inverse system of constant sheaves is pro-zero. Similarly, we see that $(\Lambda/(f_1^n, \ldots, f_r^n)) \rightarrow (\Lambda/\mathrm{I}^n)$ is an isomorphism of pro-objects.

Given that the pro-systems $(\underline{\mathrm{K}_n})$ and $(\underline{\Lambda/\mathrm{I}^n})$ are isomorphic, we wish to conclude that $\mathrm{R}\lim(\mathrm{K}\otimes^{\mathbf{L}}_{\Lambda}\underline{\mathrm{K}_n}) \to \mathrm{R}\lim(\mathrm{K}\otimes^{\mathbf{L}}_{\Lambda}\underline{\Lambda/\mathrm{I}^n})$ is an isomorphism. The pro-objects $\mathrm{K}\otimes^{\mathbf{L}}_{\Lambda}\underline{\mathrm{K}_n}$ and $\mathrm{K}\otimes^{\mathbf{L}}_{\Lambda}\underline{\Lambda/\mathrm{I}^n}$ are isomorphic, so it amounts to showing the following

Lemma. — If $(\mathrm{A}_n) \to (\mathrm{B}_n)$ *is a morphism in* $\mathscr{D}^{\mathbf{N}^\circ}$ *which is an isomorphism of pro-objects, where* $\mathscr{D}$ *is a triangulated category with countable products, then* $\mathrm{R}\lim\mathrm{A}_n \to \mathrm{R}\lim\mathrm{B}_n$ *is an isomorphism in* $\mathscr{D}$.

Proof. — Given any $\mathrm{X}\in\mathrm{ob}\,\mathscr{D}$ and $i\in\mathbf{Z}$, the map $(\mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{A}_n[i])) \to (\mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{B}_n[i]))$ is an isomorphism of pro-systems, so we have a map of exact sequences (`08TB`)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{R}^1\lim\mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{A}_n[-1]) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{R}\lim\mathrm{A}_n) & \longrightarrow & \lim\mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{A}_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{R}^1\lim\mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{B}_n[-1]) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{R}\lim\mathrm{B}_n) & \longrightarrow & \lim\mathrm{Hom}_{\mathscr{D}}(\mathrm{X},\mathrm{B}_n) & \longrightarrow & 0 \end{array}$$

in which the outer arrows are isomorphisms by `091B`. □

`099I`. To see that $\mathrm{E}'\simeq\mathrm{M}'$ it suffices to check that r of the given map, composed with the inverse of the isomorphism $\mathrm{E}\to r(\mathrm{E}')$, is an isomorphism. The diagram

$$\begin{tikzcd} r\mathrm{M}' = r\mathrm{R}\mathscr{H}\!om_{\mathscr{O}'}(\mathscr{O}',\mathrm{M}') \arrow[r] \arrow[dr, equal] & r\mathrm{R}\mathscr{H}\!om_{\mathscr{O}}(r(\mathscr{O}'),r(\mathrm{M}')) \arrow[r] \arrow[d] & \mathrm{R}\mathscr{H}\!om_{\mathscr{O}}(r\mathrm{K}',r\mathrm{M}') = r\mathrm{E}' \arrow[d] \\ & \mathrm{R}\mathscr{H}\!om_{\mathscr{O}}(\mathscr{O},r\mathrm{M}') \arrow[r] & \mathrm{R}\mathscr{H}\!om_{\mathscr{O}}(\mathrm{K},r\mathrm{M}') = \mathrm{E} \end{tikzcd}$$

commutes. The equality follows from the isomorphism in $\mathrm{D}(\mathscr{O})$

$$r\mathrm{R}\mathscr{H}\!om_{\mathscr{O}'}(\mathrm{K}\otimes^{\mathbf{L}}_{\mathscr{O}}\mathscr{O}',\mathrm{L}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}\!om_{\mathscr{O}}(\mathrm{K},r\mathrm{L})$$

for $\mathrm{K}\in\mathrm{ob}\,\mathrm{D}(\mathscr{O}), \mathrm{L}\in\mathrm{ob}\,\mathrm{D}(\mathscr{O}')$. The map is constructed as in (`0E1W`). To check that it's an isomorphism, it suffices to show that for each U in $\mathscr{C}$,

$$\mathrm{R}\,\mathrm{Hom}_{\mathscr{O}'|_{\mathrm{U}}}(\mathrm{K}|_{\mathrm{U}}\otimes^{\mathbf{L}}_{\mathscr{O}|_{\mathrm{U}}}\mathscr{O}'|_{\mathrm{U}},\mathrm{L}|_{\mathrm{U}}) \xrightarrow{\sim} \mathrm{R}\,\mathrm{Hom}_{\mathscr{O}|_{\mathrm{U}}}(\mathrm{K}|_{\mathrm{U}},r\mathrm{L}|_{\mathrm{U}})$$

is an isomorphism, since if this is true, the cone of the map on $\mathrm{R}\mathscr{H}\!om$ would have vanishing $\mathrm{H}^p(\mathrm{U},-)$ for all U in $\mathscr{C}$ and $p\in\mathbf{Z}$, so this cone would be quasi-isomorphic to a K-injective complex that is in fact acyclic as a complex of presheaves, hence *a fortiori* as a complex of sheaves. This map on R Hom is an isomorphism since base change is left adjoint to restriction as functors between the abelian categories of sheaves, by the same argument as in (`05DQ`); then use (`09T5`). We're done, since the map

$$\mathrm{R}\mathscr{H}om_{\mathbb{O}}(\mathbb{O}, r\mathrm{M}') \to \mathrm{R}\mathscr{H}om_{\mathbb{O}}(\mathrm{K}, r\mathrm{M}')$$

is the isomorphism $\mathrm{E} \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{O}}(\mathrm{K}, \mathrm{E})$.

`0A0G`. Note that the ideal $\mathcal{I}'$ is the image of the map $f^*\mathcal{I} = f^{-1}\mathcal{I} \otimes_{f^{-1}\mathbb{O}} \mathbb{O}' \xrightarrow{f^\sharp} \mathbb{O}'$, and see (`03DO`).

`099N`. Let's first show that if Λ is a set, abelian group, ring, etc. and $f : \mathfrak{X} \to \mathcal{Y}$ is a morphism of topos, then $f^*\underline{\Lambda} = \underline{\Lambda}$. We may assume $\mathfrak{X} = \tilde{\mathrm{C}}, \mathcal{Y} = \tilde{\mathrm{D}}$, C and D have all finite limits, and f is induced by a continuous left-exact functor $u : \mathrm{D} \to \mathrm{C}$ [**SGAA**, IV 1.2, 4.9.4]. Then $f^*\underline{a} = \underline{a}u_!$ [**SGAA**, III 1.3], so it suffices to show that $u_!$ of the constant presheaf with value Λ is the same. This follows from the adjunction

$$\mathrm{Hom}_{\hat{\mathrm{C}}}(u_!\Lambda, \mathrm{F}) = \mathrm{Hom}_{\hat{\mathrm{D}}}(\Lambda, u^*\mathrm{F}) = \mathrm{Hom}(\Lambda, u^*\mathrm{F}(e_{\mathrm{D}})) = \mathrm{Hom}(\Lambda, \mathrm{F}(e_{\mathrm{C}}))$$

for any presheaf F on C, where $e_{\mathrm{C}}, e_{\mathrm{D}}$ are final objects in C and D.

So if Λ is a ring, f induces a canonical morphism of ringed topos $(\mathfrak{X}, \underline{\Lambda}) \to (\mathcal{Y}, \underline{\Lambda})$ with $f^\sharp$ given by $f^{-1}\underline{\Lambda} = \underline{\Lambda} \xrightarrow{\mathrm{id}} \underline{\Lambda}$ so that f^* is exact and $\mathrm{L}f^* = f^*$ (`06YV`). Therefore for any Λ-module M, $f^*\underline{\mathrm{M}} = \underline{\mathrm{M}}$.

There a more direct way of doing the same thing, which is to see that the constant sheaves are the pullbacks of sets under the unique morphism of topos to the final topos [**SGAA**, IV 4.3].

3.5.4. In the statement of this lemma and its proof, it is asserted that certain maps between objects of $\mathrm{D}(\mathfrak{X}, \mathrm{R})$ are pro-isomorphisms when what is meant is that they induce pro-isomorphisms on cohomology. For inverse systems of complexes with fixed finite amplitude these notions are the same (`0G3D`), but in general I don't think they are.

So when it's asserted that $(\mathrm{R}/\mathfrak{m}^n \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}) \to (\mathrm{R}/\mathfrak{m})$ is a pro-isomorphism, it's actually asserted that the map on cohomologies is a pro-isomorphism, which does follow from Artin-Rees (`0911`). The stronger statement is actually true (the map is actually a pro-isomorphism), but the proof is not so simple (`0G9M`).

We are talking about the spectral sequence

$$E_2^{pq} = H^p(K_n \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} H^q(R/\mathfrak{m}^n \otimes^{\mathbf{L}}_R R/\mathfrak{m})) \Rightarrow H^{p+q}(K_n \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} R/\mathfrak{m}^n \otimes^{\mathbf{L}}_R R/\mathfrak{m}).$$

The point is that the pro-system $H^q(R/\mathfrak{m}^n \otimes^{\mathbf{L}}_R R/\mathfrak{m})$ is essentially constant with value $R/\mathfrak{m}$ if $q = 0$ and 0 otherwise. Since only finitely many (p, q) contribute to a fixed $p + q$, the inverse systems

$$(H^i(K_n \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} R/\mathfrak{m}^n \otimes^{\mathbf{L}}_R R/\mathfrak{m})) \quad \text{and} \quad (H^i(K_n \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} R/\mathfrak{m}))$$

are pro-isomorphic for each i. Since countable products in $\mathfrak{X}$ are exact, for each inverse system (K_n) in $D(\mathfrak{X}, R)$we have exact sequences (`0CQE`)

$$0 \to R^1 \lim_n H^{i-1}(K_n) \to H^i(R\lim K_n) \to \lim_n H^i(K_n) \to 0,$$

which is enough to conclude that

$$R\lim(K_n \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} R/\mathfrak{m}^n \otimes^{\mathbf{L}}_R R/\mathfrak{m}) \xrightarrow{\sim} R\lim(K_n \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} R/\mathfrak{m}).$$

(The pro-systems $(R/\mathfrak{m}^n)$ and $(R/(f_1^n, \ldots, f_r^n))$ are always isomorphic; regularity is not required. It doesn't even matter, since all you need is that $R/\mathfrak{m}$ is R-perfect.)

3.5.5. To assume $k = 1$, it suffices to show that the functor $- \otimes^{\mathbf{L}}_{R/\mathfrak{m}^k} R/\mathfrak{m}$ on $D(\mathfrak{X}, R/\mathfrak{m}^k)$ is conservative. If this is true, then the result for unbounded complexes follows straightaway, since $R/\mathfrak{m}$ is R-perfect when $\mathfrak{m}$ is regular, and the commutation of $- \otimes^{\mathbf{L}}_R P$ with $R\lim$ when P is R-perfect follows from the fact that $- \otimes^{\mathbf{L}}_R P$ commutes with countable products in $D(\mathfrak{X}, R)$ (indeed all limits), since it is right adjoint to $- \otimes^{\mathbf{L}}_R R\mathcal{H}om(K, R)$ (`08JJ`).

To see that $- \otimes^{\mathbf{L}}_{R/\mathfrak{m}^k} R/\mathfrak{m}$ is conservative, it suffices to show that if an object A of $D(\mathfrak{X}, R/\mathfrak{m}^k)$ has $A \otimes^{\mathbf{L}}_{R/\mathfrak{m}^k} R/\mathfrak{m}^{k-1} = 0$, then $A = 0$. If $A \otimes^{\mathbf{L}}_{R/\mathfrak{m}^k} R/\mathfrak{m}^{k-1} = 0$, then

$$A = A \otimes^{\mathbf{L}}_{R/\mathfrak{m}^k} \mathfrak{m}^{k-1} = A \otimes^{\mathbf{L}}_{R/\mathfrak{m}^k} R/\mathfrak{m}^{k-1} \otimes^{\mathbf{L}}_{R/\mathfrak{m}^{k-1}} \mathfrak{m}^{k-1} = 0.$$

The conclusion about $\mathrm{Cone}(c)$ follows from Verdier's exercise (`05R0`).

3.5.7. 'π_* preserves completeness' $\rightsquigarrow$ (3.5.5) combined with $\hat{R} \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^k = R/\mathfrak{m}^k$ gives $K_k = (R\lim K_n) \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^k = (R\lim K_n) \otimes^{\mathbf{L}}_{\hat{R}} \hat{R} \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^k = (R\lim K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^k$, so

$$R\lim_k((\pi_* K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^k) = R\lim_k K_k = \pi_* K_n.$$

The fact that $K_k = (\pi_* K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^k$ implies the full faithfulness of π_*.

A little more explicitly, note by its definition in terms of an exact triangle, $R\lim K_n$ is a cone of the inverse system $(K_n)_n$. By adjunction, the morphisms $R\lim K_n \to K_n$ factor canonically via $R\lim K_n \to (R\lim K_n)\otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n$. Similarly, the morphism $(R\lim K_n)\otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^{n+1} \to K_n$ factors canonically via $(R\lim K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^{n+1} \to (R\lim K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n$. So the diagram

$$\begin{tikzcd}
R\lim K_n \arrow[r] \arrow[dr] & (R\lim K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^{n+1} \arrow[r] \arrow[d] & K_{n+1} \arrow[d] \\
& (R\lim K_n) \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^{n} \arrow[r] & K_n
\end{tikzcd}$$

commutes in $D(\mathfrak{X}, \hat{R})$, and we conclude the desired isomorphism on $R\lim$ by the lemma in the note to `099M`.

4.1.1. The definition provided of fpqc covering is not correct; a covering $X \to Y$ of schemes if a fpqc covering if it is faithfully flat and every quasi-compact open of Y is the image of a quasi-compact open of X. A covering $X_i \to Y$ is a fpqc covering if $\coprod X_i \to Y$ is; i.e. for every quasi-compact open $U \subset Y$, there is a finite subset of indices $i_1, \ldots, i_n$ and quasi-compact opens $U_{i_j} \subset X_{i_j}$ so that U is the image of the U_{i_j}.

4.1.4. $\rightsquigarrow$ (`092Q`).

4.1.8. $\rightsquigarrow$ (`092N`).

4.1.9. Perhaps this is true when only X is assumed coherent (qcqs), but the standard reference would prefer that Y be assumed coherent (`01ZC`).

4.1.10. Claim: the coherent objects in $\mathrm{Spec}(k)_{\text{proét}}$ are the affine ones. If $X \to \mathrm{Spec}(k)$ is weakly étale with X coherent, then X is spectral and every point is closed (`094Q` and the fact that every weakly étale k-algebra is ind-étale, hence has profinite spectrum), so the underlying topological space of X is profinite (`0905`). Each of the finitely many open affines that cover X is clopen for the constructible topology on X, but the topology on X is the constructible topology, so each is open and closed in X; i.e. X is the disjoint union of finitely many affines, hence is affine.

Define a category $\mathscr{W}$ of weakly étale affine k-schemes as follows. Objects of $\mathscr{W}$ are weakly étale morphisms $X \to Y$ in Aff/k, the category of affine schemes over $\mathrm{Spec}\, k$. A morphism $(X \to Y) \to (X' \to Y')$ is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'. \end{array}$$

The functor p that sends $X \to Y$ to Y makes $\mathscr{W}$ into a category fibered over Aff/k. The fiber over $Y \in \mathrm{ob}\,\mathrm{Aff}/k$ is the full subcategory of $Y_{\text{proét}}$ consisting of affine Y-schemes. If we give Aff/k the fpqc topology, then

Proposition. — The functor

$$p : \mathscr{W} \to (\mathrm{Aff}/k)_{\text{fppf}}$$

defines a stack over $(\mathrm{Aff}/k)_{\text{fppf}}$.

Proof. — Fpqc descent for weakly étale morphisms of affine k-algebras is enabled by (`0245`) and (`02L2`), the latter of which says that the property of being flat is fpqc local on the base. Let's verify that this implies that the property of being weakly étale is, too. By (`094U`), the property of being weakly étale is stable under base change, and by (`094S`) it is Zariski local on the base. By (`02KP`) it suffices to show that if $S \to S'$ is a flat surjective morphism of affine schemes and $f : X \to S$ is any map, then f is weakly étale if $f' : X' := X \times_S S' \to S'$ is. Since the property of being flat is fpqc local on the base (`02L2`), f is flat. It remains to show that $X \to X \times_S X$ is flat. By hypothesis $X' \to X' \times_{S'} X' = X \times_S X \times_S S'$ is flat, so since flatness is fpqc local on the base, $X \to X \times_S X$ is flat. □

To finish, we recall the result of descent along a torsor (c.f., e.g., Vistoli's notes, §4.4).

Theorem. — Let $X \to Y$ *be a* G*-torsor in a subcanonical site* $\mathscr{C}$ *which has all finite limits, and* $\mathscr{F} \to \mathscr{C}$ *be a stack. There exists a canonical equivalence of categories between* $\mathscr{F}(Y)$ *and the category of* G*-equivariant objects* $\mathscr{F}^{G}(X)$.

This theorem, applied to our stack $\mathscr{W}$ and $\underline{G}$-torsor $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$, gives the result that the coherent objects in $\operatorname{Spec}(k)_{\text{proét}}$ are the profinite continuous G-sets.

In a few more words, a G-equivariant object in $\mathscr{W}(\operatorname{Spec} \overline{k})$ is an object $\operatorname{Spec} R \to \operatorname{Spec} \overline{k}$ of $\mathscr{W}(\operatorname{Spec} \overline{k})$ (i.e. an ind-étale $\overline{k}$-algebra) together with an action $\underline{G} \times \operatorname{Spec} R \to \operatorname{Spec} R$ making a commutative diagram

$$\begin{array}{ccc} \underline{G} \times \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \underline{G} \times \operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} \overline{k}. \end{array}$$

First suppose $\overline{k} \to R$ is finite étale. Then as topological spaces, $\underline{G}$ is simply G, $\operatorname{Spec} R$ is a discrete set, and the morphism of schemes $\underline{G} \times \operatorname{Spec} R \to \operatorname{Spec} R$ induces a continuous map of topological spaces. As every ind-étale $\overline{k}$-algebra is a filtered colimit of finite étale ones, this recovers the familiar description of the category of $\underline{G}$-equivariant objects of $\mathscr{W}(\operatorname{Spec} \overline{k})$ as the pro-objects in the category of finite discrete G-sets.

'Evaluation on $\underline{S}$ is exact precisely when S is extremally disconnected; note that this functor is not a topos-theoretic point as it does not commute with finite coproducts' $\rightsquigarrow$ here 'exact' is used in the sense of abelian sheaves (see the discussion at the end of `0991`). On the business of $F \mapsto F(\underline{S})$ not commuting with coproducts when S is disconnected: generally, in any site C, given any sheaf F in $\tilde{C}$, $\operatorname{Hom}_{\tilde{C}}(\coprod_i \epsilon_{X_i}, F) = \prod_i F(X_i)$, (where $\epsilon_{X_i} = \underline{a} h_{X_i}$). If C has coproducts which are disjoint and so that $(X_{i'} \to \coprod_i X_i)_{i'}$ is covering, then also $F(\coprod_i X_i) = \prod_i F(X_i)$, so in this case $\epsilon_{\coprod_i X_i} = \coprod_i \epsilon_{X_i}$. In the case of $C = X_{\text{proét}}$, let h_* be the sheaf represented by $\{*\}$, the final object in $\{*\}_{\text{proét}}$ $(* := \operatorname{Spec} \overline{k})$. Then $(h_* \coprod h_*)(\underline{S}) = h_{* \coprod *}(\underline{S}) = \operatorname{Hom}(S, * \coprod *)$, which does not coincide with the two-element set $\operatorname{Hom}(S, *) \coprod \operatorname{Hom}(S, *) = * \coprod *$ when S is disconnected.

4.1.12. First note that Y is affine, since the map $Y \to X$ is affine, since above small enough affine opens U of X it looks like $U \otimes \underline{S} \to U$. A pro-étale map between two affine schemes is defined to be Spec of an ind-étale map, so the Y_i are also affine. The zero section of $Y \to X$ defines sections of the f_i, and the image of X under any one of these sections is a clopen subset of Y_i (`024U`), whence the statement about the zero section as an intersection of clopens.

'Any clopen subset $U \subset Y$ defines a clopen subset $U_p \subset S$ that is stable under T' $\rightsquigarrow$ this amounts to choosing a geometric point $\overline{p}$ of X at p and using the map of schemes $\overline{p} \otimes \underline{S} \to Y$. The resulting clopen subset is stable under T since if $x \in U_p$, both x and $T(x)$ belong to U_q, hence both x and $T(x)$ belong to U_p, all by the construction of Y.

4.1.13. This example is not true as written, and needs an additional hypothesis on X, for example, that X is noetherian (because then every open subset of X would be quasi-compact). A counterexample is $X = \operatorname{Spec} \mathbf{C}[x_1, x_2, \ldots]$, the spectrum of the polynomial ring in countably infinitely many indeterminates. This ring is Jacobson (!) (`00FU`), and if you let 0 denote the closed point $(x_1, x_2, \ldots)$, then $X \smallsetminus \{0\}$ is not quasi-compact (`01K8`).[31] Since the topological space of X is Jacobson, the only open set of X containing all closed points of X except for 0 is $X \smallsetminus \{0\}$. Therefore there is no quasi-compact open of $X \smallsetminus \{0\}$ that contains all closed points of $X \smallsetminus \{0\}$, and the map

$$\operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,0}) \coprod (X \smallsetminus \{0\}) \to X$$

is not a fpqc covering.

4.2.2. We can suppose that V is a cofiltered limit over a diagram with final object V_0, and by (`01ZC`) the morphism $U \to V_0$ determined by h factors through $U \to U_i$ for some i; changing the presentation for U we may assume $U_i = U_0$. Then the square

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d] & V \arrow[d] \\ U_0 \arrow[r] & V_0 \end{tikzcd}$$

[31]The increasing series of ideals $(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \cdots$ correspond to closed sets whose intersection is $\{0\}$, so together constitute an open cover of $X - \{0\}$ with no finite subcover.

commutes, making U pro-(affine étale) over V_0 and the morphism h a map over V_0, so we can use (`097M`).

4.2.3. It is easy to see that $X^{\mathrm{aff}}_{\mathrm{proét}}$ has all finite connected limits and all cofiltered limits for the reason that a cofiltered limit of cofiltered limits of affines in $X_{\text{ét}}$ can be written as a single cofiltered limit of the same,[32] and pullbacks and equalizers stay within the category of affine schemes and within the category $X_{\text{ét}}$. (The equalizer E of two maps $U \rightrightarrows V$ of affine schemes is Spec of the coequalizer of the corresponding map of rings, and E also sits in a cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r, "\Delta"] & V \times_X V, \end{tikzcd}$$

so if $U \to X$ and $V \to X$ are both étale, $E \to V$ is as well.) It follows that $X^{\mathrm{aff}}_{\mathrm{proét}}$ has all connected limits since any connected limit can be written as a cofiltered limit of finite connected limits. First, note that a category that has all pullbacks and equalizers has all finite connected limits (nLab). Then, every limit can be written as a cofiltered limit of finite limits, and if the limit is connected, these finite limits can be taken to be connected. I follow this math.SE answer. Suppose C is a category with finite limits, and let $D : I \to C$ be a diagram. Let J be the set of pairs (o, m), where o is a finite set of objects in I and m a finite set of morphisms in I. We order J by $(o, m) \leq (o', m')$ if $o \subset o'$ and $m \subset m'$. With this order, J is a directed set. For each $j = (o, m) \in J$, let L_j be the limit of the diagram formed by j (*not* the one generated by it, which may not be finite).[33] The L_j form a diagram indexed by J, and the limit of this diagram is the same as a limit of D. Moreover, if I is a connected category, and C has only finite connected limits, then the $j \in J$ corresponding to finite connected diagrams are cofinal in J, so $\lim D$ is also the directed limit of the L_j corresponding to finite connected limits.

[32] Following, e.g., (`0BSJ`), using (`01ZC`).

[33] Explicitly, consider the category category F_j with objects $o \times \{0, 1\}$ and non-identity morphisms given by $(A, 0) \to (B, 1)$ for each non-identity morphism $A \to B$ in m. There's an obvious functor $F_j \to I$, and L_j is the limit of the composition $F_j \to I \xrightarrow{D} C$.

For finite products, suppose U and V are in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ and suppose U and V are limits of affine schemes in $X_{\acute{e}t}$ over directed sets I and J; then $U \times V$ is the limit of the same over the directed set $I \times J$ and if C_{ij} is defined by the cartesian diagram

$$\begin{tikzcd} C_{ij} \arrow[r] \arrow[d] & U_i \times V_j \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

then C_{ij} is étale over X, is affine if Δ is, and $\lim_{I\times J} C_{ij} \simeq U \times_X V$, which is the product of $U \to X$ and $V \to X$ in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$.

4.2.4. That is not what it means for $\tilde{X}_{\mathrm{pro\acute{e}t}}$ to be generated by $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$; $\tilde{X}_{\mathrm{pro\acute{e}t}}$ is generated by $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ if every *sheaf* on $X_{\mathrm{pro\acute{e}t}}$ admits such a surjection. By [**SGAA**, II 4.10], it suffices to show that $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ is topologically generating in $X_{\mathrm{pro\acute{e}t}}$; i.e. every object $X_{\mathrm{pro\acute{e}t}}$ admits a covering family of maps in $X_{\mathrm{pro\acute{e}t}}$ with their sources in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$. Cover X and Y by open affines X_i, Y_{ij} respectively so that Y_{ij} maps inside X_i. By (2.3.4), each Y_{ij} admits a faithfully flat map $U_{ij} \to Y_{ij}$ with U_{ij} in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$. Then the family $U_{ij} \to Y$ is a fpqc covering, and the source of each map is in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$.

4.2.5. $\tilde{X}_{\mathrm{pro\acute{e}t}} = \tilde{X}^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}} \rightsquigarrow$ [**SGAA**, III 4.1]. Note that the topology on $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ is again fpqc by [**SGAA**, III 3.3]. Also note that this equivalence does not require X to be affine.

4.2.6. I don't think this lemma holds, for the simple reason that given Y in $X_{\mathrm{pro\acute{e}t}}$, Y need not admit a Zariski covering consisting of affines in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$. I think the lemma holds if the condition (1) is replaced by the condition

1. For any affine $U = \operatorname{Spec} A$ in $X_{\mathrm{pro\acute{e}t}}$ and faithfully flat ind-étale map $A \to B$ with $V := \operatorname{Spec} B$ in $X^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$, the sequence $F(U) \to F(V) \rightrightarrows F(V \times_U V)$ is exact.

Then one can model a proof on (`022H`).

Remember, a ring A is w-contractible if every faithfully flat ind-étale $A \to B$ admits a section. In general a covering in $X_{\mathrm{pro\acute{e}t}}$ ($X = \operatorname{Spec} A$) admits a refinement by a finite, jointly surjective covering $(U_i \to X)$ with $U_i = \operatorname{Spec} B_i$, $A \to B_i$ ind-étale. This covering corresponds to a crible $R \subset X$ and $\coprod_i U_i \to X$ does not factor through this crible unless one of the $U_i \to X$ is surjective. In other words, id : $X \to X$ doesn't belong to R unless one of the U_i is surjective.

What does happen is that the section $X \to \coprod_i U_i$ determines a finite decomposition of X into disjoint clopens $X = \coprod X_i$, and the maps $X_i \to U_i \to X$ belong to R. So every $(U_i \to X)$ can be refined by such a covering $(X_i \to X)$ with X_i discrete clopens.

4.2.7. $\rightsquigarrow$ [**SGAA**, III 5.4].

4.2.8. I don't see how it follows immediately from (2.4.9), since (2.4.9) is tested on objects of the site, while (3.2.1) is tested on objects of the resulting topos. Instead I proceed following (`090J` & `0F4N`), which means (3.2.3) should be checked for a site with enough weakly contractible coherent objects.

Proposition. — If $\mathscr{C}$ is a site with enough weakly contractible quasi-compact objects, then $\widetilde{\mathscr{C}}$ is replete and $D(\widetilde{\mathscr{C}}) := D(\widetilde{\mathscr{C}}, \mathbf{Z})$ *is compactly generated.*

The proof goes the same way as (3.2.3) and its note. For 'replete,' $F \twoheadrightarrow G \Rightarrow F(Y) \twoheadrightarrow G(Y)$ for Y weakly contractible is by definition (`090L`), and the comparison lemma [**SGAA**, III 4.1] gives the reverse direction. For 'compactly generated,' the proof is essentially the same.

As any affine U in $X_{\text{proét}}$ is quasi-compact, the proposition holds (`0990`) for $X_{\text{proét}}$.

4.2.10. Let $Y = \coprod_{a\in A} X$; then for every S in $X_{\text{proét}}$, $\mathrm{Hom}_X(S, Y)$ coincides with continuous maps $\pi_0(S) \to A$, where A is topologized discretely. Therefore the sheaf on $X_{\text{proét}}$ represented by Y is pulled back from the sheaf on $\pi_0(X)_{\text{proét}} = \{*\}_{\text{proét}}$ (4.2.13) represented by $\coprod_{a\in A}\{*\}$, and it suffices to show that this sheaf is the constant sheaf on $\{*\}_{\text{proét}}$ with value A. This sheaf is nothing other than the sheaf G of locally constant functions $S \to A$, and the discussion in the note to (4.2.11) (with Λ replaced by A) shows that the datum of a covering family $\{g_i : S_i \to S\}$ in the sense of (4.1.10) and *constant* functions $f_i : S_i \to A$ belonging to the equalizer $\prod_i G(S_i) \rightrightarrows \prod_{i,j} G(S_i \times_S S_j)$ is equivalent to the datum of a locally constant function $S \to A$. This shows that the sheaf G is indeed the sheafification of the constant presheaf on $\{*\}_{\text{proét}}$ with value A.

Returning to X, since $\underline{\mathrm{A}}$ is represented by Y and $\mathrm{Y} \to \mathrm{X}$ is locally of finite presentation, the formula of (4.1.9) applies.

4.2.11. Given a covering family $\{g_i : \mathrm{S}_i \to \mathrm{S}\}$, the property that functions $f_i : \mathrm{S}_i \to \Lambda$ lie in the equalizer of $\prod_i \mathrm{F}(\mathrm{S}_i) \rightrightarrows \prod_{i,j} \mathrm{F}(\mathrm{S}_i \times_{\mathrm{S}} \mathrm{S}_j)$ is equivalent to the condition that for any two indices i, j and point $s \in g_i(\mathrm{S}_i) \cap g_j(\mathrm{S}_j)$, the restrictions of f_i and f_j to the s-fibers $g_i^{-1}(s)$ and $g_j^{-1}(s)$, respectively, are constant with the same value. The same is true with F replaced by G. So there is no problem defining a function $f : \mathrm{S} \to \Lambda$: as every $s \in \mathrm{S}$ is in the image of some g_i, just set $f(s)$ to equal the value of f_i on $g_i^{-1}(s)$. To check that this is locally constant on S in the case the f_i are, of course this property is the same as being a continuous function $\mathrm{S} \to \Lambda$ where Λ is discretely topologized, and in that case each S_i decomposes into finitely many clopen subspaces according to the function f_i. Regrouping, we may assume that each S_i is sent by f_i to one value depending only on i. There is still a finite subset J so that $\coprod_{j \in \mathrm{J}} g_j$ is surjective, and, regrouping again, we may assume that if $j \neq j'$ then $f_j(\mathrm{S}_j) \neq f_{j'}(\mathrm{S}_{j'})$. As the $g_j(\mathrm{S}_j)$ are closed and $\mathrm{S} = \coprod_{j \in \mathrm{J}} g_j(\mathrm{S}_j)$, we're done.

4.2.12. Regarding the last sentence, we've already checked in the note to (4.2.10) that the constant sheaf on $\{*\}_{\text{proét}}$ with value T is the sheaf that sends a profinite set S to continuous maps $\mathrm{S} \to \mathrm{T}$, where T is discretely topologized. Recalling (`005F` & `0900`) as well as the note to (4.2.5), which shows that a sheaf on $\mathrm{X}_{\text{proét}}$ is known once it is known on coherent objects, we see that when T is discretely topologized, the sheaf $\mathscr{F}_{\mathrm{T}}$ is the pullback of the constant sheaf with value T on $\{*\}_{\text{proét}}$ via the morphism of topos $\tilde{\mathrm{X}}_{\text{proét}} \to \widetilde{\pi_0(\mathrm{X})}_{\text{proét}}$ (4.2.13).

As for the proof, we use the criteria of (4.2.6) ameliorated by its note. Continuous maps $\mathrm{U} \to \mathrm{T}$ clearly form a Zariski sheaf in U, so we only need to show that given a faithfully flat ind-étale homomorphism of rings $\mathrm{A} \to \mathrm{B}$ corresponding to a map $p : \mathrm{V} := \operatorname{Spec} \mathrm{B} \to \mathrm{U} := \operatorname{Spec} \mathrm{A}$, then $\mathscr{F}_{\mathrm{T}}(\mathrm{U}) \to \mathscr{F}_{\mathrm{T}}(\mathrm{V}) \rightrightarrows \mathscr{F}_{\mathrm{T}}(\mathrm{V} \times_{\mathrm{U}} \mathrm{V})$ is exact. Note that if a function $f \in \mathscr{F}_{\mathrm{T}}(\mathrm{V})$ is in the equalizer, then f is constant along U-fibers, since given any two points of V over u, there is a point of $\mathrm{V} \times_{\mathrm{U}} \mathrm{V}$ mapping to both [**EGA**, I 3.4.7]. Therefore for such an f we can unambiguously define a function $g : \mathrm{U} \to \mathrm{T}$ and we have to check that it's continuous. If $W \subset T$ is open, then $p^{-1}g^{-1}(W) = f^{-1}(W)$, so we've reduced to the claim proved in the lemma.

The reference for the last sentence of the proof is (08ZL).

'Any faithfully flat étale $\overline{A}$-algebra has a section' $\rightsquigarrow$ if $\overline{A} \to B$ is étale and surjective, it's obtained as the base change to $\overline{A}$ of a finitely presented map $A_n \to B'$ for some n (05N9); in other words, the square

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} B' \arrow[d] \\ \operatorname{Spec} \overline{A} \arrow[r] & \operatorname{Spec} A_n \end{tikzcd}$$

is cartesian. As $A_n \to \overline{A}$ is faithfully flat, $A_n \to B$ is étale by fpqc descent (02VN), and surjective since $\operatorname{Spec} B \to \operatorname{Spec} \overline{A}$ is, so $\operatorname{Spec} B \to \operatorname{Spec} A_n$ is surjective. The existence of a section $B \to \overline{A}$ comes from the dotted section in the diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} B' \times_{\operatorname{Spec} A_n} \operatorname{Spec} A_{n+1} \arrow[r] \arrow[d] & \operatorname{Spec} B' \arrow[d] \\ \operatorname{Spec} \overline{A} \arrow[r] & \operatorname{Spec} A_{n+1} \arrow[r] \arrow[u, dotted, bend right] & \operatorname{Spec} A_n \end{tikzcd}$$

in which both squares are cartesian.

4.2.13. In what follows, we construct $\pi^{-1} : \pi_0(X)_{\text{proét}} := \{*\}_{\text{proét}}/\pi_0(X) \to X_{\text{proét}}$; in fact, π^{-1} is a functor from profinite sets over $\pi_0(X)$ to affine X-schemes which is right adjoint to the functor $Y \mapsto \pi_0(Y)$ on affine X-schemes, and $\pi^{-1}(S \to \pi_0(X)) \to X$ is an affine morphism that is pro-finite (hence integral) and a pro-(Zariski localization) over an affine Zariski cover of X, hence weakly étale.

To construct π^{-1}: let $T \to \pi_0(X)$ be a profinite set over $\pi_0(X)$. For each $V \subset U \subset X$ with V and U affine, we have with the notation of the note to (2.2.8) affine U-schemes $T \times_{\pi_0(X)} V := \lim_i \lim_j T'_j \times_{S_i} V$ and $T \times_{\pi_0(X)} U := \lim_i \lim_j T'_j \times_{S_i} U$,[34] and this defines a functor from the directed set of affine opens in X (and morphisms over X) to affine

[34]Note that $T \times_{\pi_0(X)} V$ does indeed have underlying topological space given by this fiber product, and likewise for U.

X-schemes. Of course, the induced map $T \times_{\pi_0(X)} V \to (T \times_{\pi_0(X)} U) \times_X V$ is an isomorphism, as limits commute with limits. This defines (01LG) an affine X-scheme, call it $\boxed{T \times_{\pi_0(X)} X,}$ and a functor $\pi^{-1} : \pi_0(X)_{\text{proét}} \to X_{\text{proét}}$, where $\pi_0(X)_{\text{proét}}$ is just the localized site $\{*\}_{\text{proét}}/\pi_0(X)$. To show that π^{-1} preserves limits, it suffices to look over U, and show that $T \times_{\pi_0(X)} U \simeq (T \times_{\pi_0(X)} \pi_0(U)) \times_{\pi_0(U)} U$, since the latter functor commutes with limits by (2.2.8). Evidently these schemes are homeomorphic with the same π_0, so to do this it suffices to show that for every affine U-scheme Y,

$$\mathrm{Hom}_{\pi_0(U)}(\pi_0(Y), T \times_{\pi_0(X)} \pi_0(U)) \simeq \mathrm{Hom}_U(Y, T \times_{\pi_0(X)} U).$$

We need to show that maps $Y \to T'_j \times_{S_i} U$ over U are determined by a choice of map $\pi_0(Y) \to T \times_{\pi_0(X)} \pi_0(U)$. Let $t \in T'_j$ map to a point $s \in S_i$. The points s and t determine a clopen of $T \times_{\pi_0(X)} \pi_0(U)$ and hence a clopen $W \subset Y$. Then W must be sent to $t \times_{S_i} U$ via the map determined by the structural arrow $Y \to U$.

The affine X-scheme $T \times_{\pi_0(X)} X$ has underlying topological space given by this fiber product, it is coherent since X is, and we have a map $\pi_0(T \times_{\pi_0(X)} X) \to T$ over $\pi_0(X)$. Following the argument in the note to (2.2.8), the fiber over $x \in \pi_0(X)$ of $\pi_0(T \times_{\pi_0(X)} X) \to T$ is $\pi_0(T \times_{\pi_0(X)} Z) = \pi_0(T_x \times Z) \to T_x := T \times_{\pi_0(X)} x$, where $Z \subset X$ is the connected component corresponding to x, and $\pi_0(T_x \times Z) = T_x$. Therefore $\pi_0(T \times_{\pi_0(X)} X) \to T$ is a bijection and a homeomorphism.

It follows that $T \mapsto T \times_{\pi_0(X)} X$ is a fully faithful right adjoint to the functor from affine X-schemes to profinite $\pi_0(X)$-sets given by π_0. Given an affine X-scheme Y and a morphism $\pi_0(Y) \to T$ over $\pi_0(X)$, we have to show that there is a unique morphism $Y \to T \times_{\pi_0(X)} X$ over X that coincides with the given map upon applying π_0. By uniqueness and glueing, it suffices to show this over a given open affine $U \subset X$, so we have to show that there is a unique morphism

$$Y_U := Y \times_X U \longrightarrow T \times_{\pi_0(X)} U \simeq (T \times_{\pi_0(X)} \pi_0(U)) \times_{\pi_0(U)} U$$

over U agreeing with the map of $\pi_0(U)$-sets

$$\pi_0(Y \times_X U) \to \pi_0(Y) \times_{\pi_0(X)} \pi_0(U) \to T \times_{\pi_0(X)} \pi_0(U) = \pi_0(T \times_{\pi_0(X)} U)$$

determined by the map of $\pi_0(X)$-sets $\pi_0(Y) \to T$. This is true by the adjunction in the affine case (2.2.8).

Affine-locally on X, π^{-1} takes a map $T \to \pi_0(X)$ of profinite sets to a pro-(Zariski localization) by (2.2.8), which is *a fortiori* weakly étale. Last, suppose we have a finite, jointly-surjective family $\{T_i \to T\}$ of profinite $\pi_0(X)$-sets. Then the maps $T_i \times_{\pi_0(X)} X \to T \times_{\pi_0(X)} X$ are affine, flat maps of affine X-schemes, and they coincide as maps of topological spaces with the maps between fiber products taken in the category of topological spaces. Since the family $\{T_i \to T\}$ is jointly surjective, the family $\{T_i \times_{\pi_0(X)} X \to T \times_{\pi_0(X)} X\}$ is too, hence is a fpqc covering. This shows that π^{-1} preserves coverings.

In summary, π^{-1} is a continuous [**SGAA**, III 1.6] left-exact functor, therefore gives rise to an adjoint pair $(\pi^*, \pi_*) := ((\pi^{-1})_!, (\pi^{-1})^*)$, and π^* is left exact by [**SGAA**, III 1.3 5)]. Therefore (π^*, π_*) is indeed a morphism of topos $\pi : X_{\text{proét}} \to \pi_0(X)_{\text{proét}} := \{*\}_{\text{proét}}/\pi_0(X)$.

One can add to the statement of (4.2.13):

5. $\pi_*F(S) = F(S \times_{\pi_0(X)} X)$ for any S in $\pi_0(X)_{\text{proét}}$ and F in $\tilde{X}_{\text{proét}}$.

One can restrict to looking over coherent U in $X_{\text{proét}}$ for the purposes of proving (4.2.13), as $\tilde{X}_{\text{proét}} \simeq \tilde{X}^{\text{aff}}_{\text{proét}}$ (note to 4.2.5). In fact, one can restrict to arrows $U \to X$ in $X_{\text{proét}}$ which are affine morphisms, and that is what I will do in what follows. So for the rest of the discussion of the proof of (4.2.13), $X_{\text{proét}}$ will denote the full subcategory of $X_{\text{proét}}$ consisting of schemes affine over X, with the induced topology, so that $\pi_0 : X_{\text{proét}} \to \pi_0(X)_{\text{proét}}$ is defined on all of $X_{\text{proét}}$.

To prove (1), note that by [**SGAA**, I §5], π^*F is the sheafification of the presheaf $(\pi^{-1})_! iF$, and sections over an affine morphism $U \to X$ of the latter presheaf are given in terms of a colimit over (the opposite of) a category $I^U_{\pi^{-1}}$ with objects given by pairs $(S, U \to \pi^{-1}(S))$, where $S \in \pi_0(X)_{\text{proét}}$. The category $I^U_{\pi^{-1}}$ has an initial object $(\pi_0(U), U \to \pi^{-1}(\pi_0(U)))$ given by the unit of adjunction. So indeed $((\pi^{-1})_! iF)(U) = F(\pi_0(U))$.

The assertion that $F(\pi_0(U))$ is already a sheaf whenever F is a sheaf on $\pi_0(X)_{\text{proét}}$ is equivalent to the statement that $\pi_0 : X_{\text{proét}} \to \pi_0(X)_{\text{proét}}$ is a continuous functor. I don't see how the claim follows from the claim about coequalizers, since Yoneda generally doesn't preserve coequalizers, and the map $\pi_0(U \times_V U) \to \pi_0(U) \times_{\pi_0(V)} \pi_0(U)$ is in general neither an injection nor a surjection.

The functor $\pi_{0!} : \hat{X}_{\text{proét}} \to \widehat{\pi_0(X)}_{\text{proét}}$[35] is defined as a colimit over (the opposite of) a category $I^S_{\pi_0}$ with objects given by pairs $(Y, S \to \pi_0(Y))$, where Y is in $X_{\text{proét}}$. To show that π_0 is continuous, it suffices to show that for every object Y of $X_{\text{proét}}$ and covering sieve $R \hookrightarrow Y$, the morphism $\pi_{0!}(R) \to \pi_0(Y)$ is bicouvrant [**SGAA**, III 1.2]. A section of $\pi_{0!}R$ over S in $\pi_0(X)_{\text{proét}}$ is the data (modulo an equivalence relation) of a scheme Z in $X_{\text{proét}}$, a map $z : S \to \pi_0(Z)$, and a morphism $g : Z \to R$. The map $\pi_{0!}R \to \pi_0(U)$ sends the section (Z, z, g) to the section $S \xrightarrow{z} \pi_0(Z) \xrightarrow{\pi_0(g)} \pi_0(Y)$.

The map $\pi_{0!}R \to \pi_0(Y)$ is couvrant [**SGAA**, II 5.1] if for every $S \to \pi_0(Y)$ with S in $\pi_0(X)_{\text{proét}}$, the morphism $\pi_{0!}R \times_{\pi_0(Y)} S \to S$ has image a covering sieve of S. So let $\{f_i : Y_i \to Y\}$ be a finite, jointly surjective family of maps in $X_{\text{proét}}$;[36] every covering sieve of Y contains the image of such a covering. Sections over T of the image of $\pi_{0!}R \times_{\pi_0(Y)} S \to S$ consist of those maps $T \to S$ in $\pi_0(X)_{\text{proét}}$ that when composed with $S \to \pi_0(Y)$ factor through $\pi_{0!}R$; i.e. so that there is some $g : Z \to Y_i$ in $X_{\text{proét}}$ and $S \to \pi_0(Z)$ so that $T \to S$ coincides with the map $T \to \pi_0(Z) \xrightarrow{\pi_0(g)} \pi_0(Y_i) \to \pi_0(Y)$.[37] For each i, the map $(S \times_{\pi_0(Y)} \pi_0(Y_i)) \times_{\pi_0(Y_i)} Y_i \to S$ composed with $S \to \pi_0(Y)$ coincides with the projection onto Y_i composed with f. Therefore, the maps $S \times_{\pi_0(Y)} \pi_0(Y_i) \to S$ are in the image of $\pi_{0!}R \times_{\pi_0(Y)} S \to S$, so $\pi_{0!}R \to \pi_0(Y)$ is couvrant.

The map $\varphi : \pi_{0!}R \to \pi_0(Y)$ is bicouvrant if

> it is couvrant and for every S in $\pi_0(X)_{\text{proét}}$ and pair of maps
> $u_1, u_2 : S \rightrightarrows \pi_{0!}R$ so that $\varphi u_1 = \varphi u_2$, the kernel of (u_1, u_2) (∗)
> is a covering sieve of S [**SGAA**, II 5.3].

[35]To define these categories of presheaves, the sites must be $\mathcal{U}$-small, so we are implicitly restricting to $\mathcal{U}$-small subcategories of these sites.

[36]The f_i are automatically quasi-compact since by our assumption on $X_{\text{proét}}$ the f_i are affine.

[37]This is not the same as $R' \times_{\pi_0(Y)} S \hookrightarrow S$, where $R' \hookrightarrow \pi_0(Y)$ is the crible generated by the family $\{\pi_0(f_i) : \pi_0(Y_i) \to \pi_0(Y)\}$.

The sections u_b ($b = 1, 2$) correspond to the data of maps $g_b : \mathrm{Z}_b \to \mathrm{R}$ in $\hat{\mathrm{X}}_{\text{proét}}$ and maps $z_b : \mathrm{S} \to \pi_0(\mathrm{Z}_i)$ in $\pi_0(\mathrm{X})_{\text{proét}}$. The condition that $\varphi u_1 = \varphi u_2$ means that the maps $\mathrm{S} \xrightarrow{z_b} \pi_0(\mathrm{Z}_b) \xrightarrow{\pi_0(f \circ g_b)} \pi_0(\mathrm{Y})$ coincide for $b = 1, 2$. A map $\mathrm{T} \to \mathrm{S}$ is in the kernel of (u_1, u_2) if the sections $\mathrm{T} \to \mathrm{S} \xrightarrow{u_b} \pi_{0!}\mathrm{R}$ coincide. This means that there exists an $n \in \mathbf{N}$ and commutative diagrams

$$
\begin{tikzcd}
 & & & & \mathrm{S} \arrow[llll, bend right, "z_1"'] \arrow[rrrr, bend left, "z_2"] & & & & \\
 & & & & \mathrm{T} \arrow[u] \arrow[dlll, bend right] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[dddr, bend left, phantom] \arrow[dl] & & & & \\
 & \pi_0(\mathrm{W}_1) \arrow[dl] \arrow[dr] & & \pi_0(\mathrm{W}_3) \arrow[dl] \arrow[dr] & & \pi_0(\mathrm{W}_{2n-3}) \arrow[dl] \arrow[dr] & & \pi_0(\mathrm{W}_{2n-1}) \arrow[dl] \arrow[dr] & \\
\pi_0(\mathrm{Z}_1) & & \pi_0(\mathrm{W}_2) & & \cdots & & \pi_0(\mathrm{W}_{2n-2}) & & \pi_0(\mathrm{Z}_2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathrm{W}_1 \arrow[dl] \arrow[dr] \arrow[ddrrr, bend right] & & \mathrm{W}_3 \arrow[dl] \arrow[dr] \arrow[ddr] & & \mathrm{W}_{2n-3} \arrow[dl] \arrow[dr] \arrow[ddl] & & \mathrm{W}_{2n-1} \arrow[dl] \arrow[dr] \arrow[ddlll, bend left] & \\
\mathrm{Z}_1 \arrow[drrrr, bend right, "g_1"'] & & \mathrm{W}_2 \arrow[drr] & & \cdots & & \mathrm{W}_{2n-2} \arrow[dll] & & \mathrm{Z}_2 \arrow[dllll, bend left, "g_2"] \\
 & & & & \mathrm{R} & & & &
\end{tikzcd}
\tag{†}
$$

where the zig-zag in the first diagram is obtained by applying π_0 to the zig-zag in the second.

It is enough to check the condition ($*$) on extremally disconnected S. For such S, the condition that $\ker(u_1, u_2)$ be a covering sieve is equivalent to the existence of a disjoint union decomposition $\mathrm{S} = \coprod_a \mathrm{S}_a$ so that taking $\mathrm{T} = \mathrm{S}_a \to \mathrm{S}$ to be the immersion, the above W_j and commutative diagrams exist. The condition that $\pi_{0!}\mathrm{R} \to \pi_0(\mathrm{Y})$ be an injection, which is stronger than the condition ($*$), is equivalent to the existence of W_j and these commutative diagrams in the case that $\mathrm{T} = \mathrm{S} \xrightarrow{\mathrm{id}} \mathrm{S}$.

It's not a sufficient condition, but we see that a necessary condition for the kernel of (u_1, u_2) to be a covering sieve is that for every point of S, there is a map $\{*\} \to \ker(u_1, u_2)$ hitting that point of S. Replacing the schemes in the above diagrams by their connected components corresponding to this map with their canonical scheme structures (04PX), we see that a necessary condition for $\ker(u_1, u_2)$ to be a covering sieve is that for every connected Y in $X_{\text{proét}}$, every covering sieve $R \hookrightarrow Y$, and every choice of connected schemes Z_b $(b = 1, 2)$ and maps $Z_b \to R$ in $\hat{X}_{\text{proét}}$, there exists a zig-zag as in (†) with the W_j connected schemes in $X_{\text{proét}}$. I believe this condition is secured.

In conclusion, I don't see that $\pi_0 : X_{\text{proét}} \to \pi_0(X)_{\text{proét}}$ is continuous.

Intermezzo: 099R.

09XN. This is needed for 0GLZ. To complete the proof, we have to check that the sections $t_j|_{W_j}, t_k|_{W_k}$ agree on $W_j \times_T W_k$ for $j, k \in I$. It suffices to show that they go to the same element in the stalks at geometric points of $W_j \times_T W_k$. As $(T_i \times_T W_j \times_T W_k \to W_j \times_T W_k)_{i \in I}$ is a fpqc covering, it suffices to show that $t_j|_{W_j}$ and $t_k|_{W_k}$ pull back to the same section of the pullback of $\mathscr{G}$ to $T_i \times_T W_j \times_T W_k$. This is clear as both pullback to the restriction of s_i.

0GLZ. We analyze the counit $\varepsilon : a\epsilon_* \to \text{id}$. Given a sheaf $\mathscr{G}$ on $X_{\text{proét}}$ and $f : Y \to X$ in $X_{\text{proét}}$,

$$f_{\text{ét}}^* \epsilon_* \mathscr{G}(Y) = \operatorname{colim}_b \mathscr{G}(U/X), \qquad \begin{tikzcd} Y \arrow[r, "b"] \arrow[dr, "f"'] & U \arrow[d] \\ & X, \end{tikzcd} \qquad U \to X \text{ étale.}$$

Therefore $\Gamma(Y_{\text{proét}}, \varepsilon)$ is given by the map

$$\operatorname{colim}_b \mathscr{G}(U/X) \to \mathscr{G}(Y)$$

obtained by restriction along b. When f is in $X_{\text{ét}}$, of course f is the initial object in the diagram opposite to the one indexing this colimit, so we find $\varepsilon : \epsilon_* a \epsilon_* \mathscr{G} \to \epsilon_* \mathscr{G}$ is the inverse to the unit $\eta : \epsilon_* \mathscr{G} \to \epsilon_* a \epsilon_* \mathscr{G}$. Given a morphism $\mathscr{F} \to \epsilon_* \mathscr{G}$, it's therefore clear that we recover this morphism as

$$\mathscr{F} \to \epsilon_* a \mathscr{F} \to \epsilon_* a \epsilon_* \mathscr{G} \to \epsilon_* \mathscr{G}.$$

On the other hand, if we start from a morphism $a\mathscr{F} \to \mathscr{G}$, the fact that we recover this morphism as

$$a\mathscr{F} \to a\epsilon_* a\mathscr{F} \to a\epsilon_*\mathscr{G} \to \mathscr{G}$$

follows from the observation that any morphism $a\mathscr{F} \to \mathscr{G}$ factors on sections over Y as

$$f_{\text{ét}}^*\mathscr{F}(\mathrm{Y}) = \operatorname{colim}_b \mathscr{F}(\mathrm{U}/\mathrm{X}) \to \operatorname{colim}_b \mathscr{G}(\mathrm{U}/\mathrm{X}) \to \mathscr{G}(\mathrm{Y}).$$

This is true since given a morphism $b : \mathrm{Y} \to \mathrm{U}$ over X with $\mathrm{U} \to \mathrm{X}$ étale, the square

$$\begin{tikzcd} a\mathscr{F}(\mathrm{Y}) \arrow[r] & \mathscr{G}(\mathrm{Y}) \\ \mathscr{F}(\mathrm{U}) \arrow[r] \arrow[u] & \mathscr{G}(\mathrm{U}) \arrow[u] \end{tikzcd}$$

commutes, where the vertical arrows are restriction along b.

In conclusion, $a \dashv \epsilon_*$ form an adjoint pair.

099U. Note for $i_1, \dots, i_p \in \{1, \dots, n\}$, $\mathrm{U}_b \times_{\mathrm{U}_{b_0}} \mathrm{U}_{i_0,b_0} \times_{\mathrm{U}_{b_0}} \mathrm{U}_{i_1,b_0} \times_{\mathrm{U}_{b_0}} \dots \times_{\mathrm{U}_{b_0}} \mathrm{U}_{i_n,b_0} = \mathrm{U}_{i_0,b} \times_{\mathrm{U}_b} \mathrm{U}_{i_1,b} \times_{\mathrm{U}_b} \dots \times_{\mathrm{U}_b} \mathrm{U}_{i_p,b}$ and likewise with $\mathrm{U}_b, \mathrm{U}_{*,b}$ replaced with U, U_*.

099Z. Follows directly from `0D7U`, `099Y`, & `099V`.

5.2.3. 1. If U is in $\mathrm{X}_{\text{ét}}$, then $\nu^{-1}\mathrm{U}$ just means U (or the sheaf represented by U) considered as an object (or sheaf) of/on $\mathrm{X}_{\text{proét}}$. If K is in $\mathrm{D}(\mathrm{X}_{\text{proét}})$, $\mathrm{R}\Gamma(\mathrm{U}, \nu_*\mathrm{K}) = \mathrm{R}\Gamma(\nu^{-1}\mathrm{U}, \mathrm{K})$ [**SGAA**, V 5.3], so we have one inclusion. The other amounts to the statement that L in $\mathrm{D}(\mathrm{X}_{\text{ét}})$ is null if $\mathrm{R}\Gamma(\mathrm{U}, \mathrm{L}) = 0$ for all U in $\mathrm{X}_{\text{ét}}$. This is immediate from the fact that $\mathrm{X}_{\text{ét}}$ is generated by sheaves of the form $j_{\mathrm{U}!}\mathbf{Z}$ for U in $\mathrm{X}_{\text{ét}}$, since $\mathrm{Hom}(j_{\mathrm{U}!}\mathbf{Z}[-n], \mathrm{L}) = \mathrm{H}^n(\mathrm{U}, \mathrm{L})$.

3. The definition the $\mathrm{M}^n(\mathrm{K})$ make it clear that we're working in the stable ∞-category $\mathrm{D}(\mathrm{X}_{\text{proét}})$ which indeed has all (small) colimits (Lurie, *Higher Algebra* (1.3.5.21)). Suppose, instead, we define $\mathrm{M}^\infty(\mathrm{K})$ as $\operatorname{hocolim}_n \mathrm{M}^n(\mathrm{K})$. To show that $\nu_*\mathrm{L}(\mathrm{K}) = 0$, it suffices to show that $\mathrm{R}\Gamma(\nu^{-1}\mathrm{U}, \mathrm{K}) = 0$ for U coherent in $\mathrm{X}_{\text{ét}}$. As $\mathrm{R}\Gamma(\nu^{-1}\mathrm{U}, \mathrm{L}(\mathrm{K})) = \operatorname{hocolim}_n \mathrm{R}\Gamma(\nu^{-1}\mathrm{U}, \mathrm{M}^n(\mathrm{K}))$ for such U, it indeed suffices to show that $\mathrm{R}\Gamma(\nu^{-1}\mathrm{U}, \mathrm{K}) \to \mathrm{R}\Gamma(\nu^{-1}\mathrm{U}, \mathrm{M}(\mathrm{K}))$ is the null map for any K in $\mathrm{D}(\mathrm{X}_{\text{proét}})$.

As $R\Gamma(U, F) := R\,\mathrm{Hom}(\mathbf{Z}_U, F)$ and $\nu^*\mathbf{Z}_U = \mathbf{Z}_{\nu^{-1}U}$ [**SGAA**, IV 13.4 b)], the isomorphism $R\Gamma(U, \nu_*K) \simeq R\Gamma(\nu^{-1}U, K)$ can be written as $R\,\mathrm{Hom}(\mathbf{Z}_U, \nu_*K) \simeq R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K)$. We have a commutative square (`079P`)

$$\begin{tikzcd} K \arrow[r] & I_0 \arrow[r, "c"] & I_1 \\ & \nu^*\nu_*I_0 \arrow[u] \arrow[r] & J \arrow[u] \end{tikzcd}$$

where the complexes I_0, I_1, and J are K-injective and the horizontal arrows are quasi-isomorphisms. The maps

$$\mathrm{Hom}(\mathbf{Z}_U, \nu_*I_0) \to \mathrm{Hom}(\nu^*\mathbf{Z}_U, \nu^*\nu_*I_0) \to \mathrm{Hom}(\nu^*\mathbf{Z}_U, I_0) \to \mathrm{Hom}(\nu^*\mathbf{Z}_U, I_1)$$

give the isomorphism $R\,\mathrm{Hom}(\mathbf{Z}_U, \nu_*K) \xrightarrow{\sim} R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K)$ followed by an automorphism $c \in \mathrm{Aut}(R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K))$. The commutativity of the above square shows that this composite factors though $R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, \nu^*\nu_*K) \to R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K)$. Therefore the identity map of $R\Gamma(\nu^{-1}U, K)$ (represented by $\mathrm{Hom}(\nu^*\mathbf{Z}_U, I_1)$) factors as

$$\begin{aligned} R\Gamma(\nu^{-1}U, K) \simeq R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K) &\xrightarrow{c^{-1}} R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K) \\ &\simeq R\,\mathrm{Hom}(\mathbf{Z}_U, \nu_*K) \to R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, \nu^*\nu_*K) \\ &\to R\,\mathrm{Hom}(\nu^*\mathbf{Z}_U, K) \simeq R\Gamma(\nu^{-1}U, K). \end{aligned}$$

Later in the proof, assuming P parasitic, we have to show that $\mathrm{Hom}(K, P) = \lim \mathrm{Hom}(M^n(K), P)$ is isomorphic to $\mathrm{Hom}(L(K), P)$. We have an exact sequence (`0A5K`)

$$0 \to R^1 \lim \mathrm{Hom}(M^n(K), P[-1]) \to \mathrm{Hom}(L(K), P) \to \lim \mathrm{Hom}(M^n(K), P) \to 0,$$

and since $\mathrm{Hom}(M^n(K), P[-1]) \simeq \mathrm{Hom}(M^{n+1}K, P[-1])$, the $R^1 \lim$ vanishes.

4. If P is parasitic in $D(X_{\text{proét}})$, then $\mathrm{hocolim}_n M^n(P) = \mathrm{colim}_n M^n(P) = P$, since for any K in $D(X_{\text{proét}})$, the sequence

$$0 \to R^1 \lim \mathrm{Hom}(M^n(P), K[-1]) \to \mathrm{Hom}(L(P), K) \to \lim \mathrm{Hom}(M^n(P), K) \to 0$$

is exact, and as $M^n(P) \simeq M^{n+1}(P)$, the $R^1 \lim$ vanishes.

Intermezzo: A simple identity. I couldn't find the following simple identity written down in the usual places, so I'm writing it down here even though it has surely already appeared in these notes. Let f be a morphism of ringed topos.

$$f_*\mathrm{R}\mathscr{H}\!om(f^*\mathrm{K}, \mathrm{L}) = \mathrm{R}\mathscr{H}\!om(\mathrm{K}, f_*\mathrm{L}),$$

since for any M, `07A4` gives

$$\begin{aligned}\mathrm{Hom}(\mathrm{M}, f_*\mathrm{R}\mathscr{H}\!om(f^*\mathrm{K}, \mathrm{L}) &= \mathrm{Hom}(f^*\mathrm{M}, \mathrm{R}\mathscr{H}\!om(f^*\mathrm{K}, \mathrm{L}) \\ &= \mathrm{Hom}(f^*\mathrm{M} \otimes^{\mathbf{L}} f^*\mathrm{K}, \mathrm{L}) \\ &= \mathrm{Hom}(f^*(\mathrm{M} \otimes^{\mathbf{L}} \mathrm{K}), \mathrm{L}) \\ &= \mathrm{Hom}(\mathrm{M} \otimes^{\mathbf{L}} \mathrm{K}, f_*\mathrm{L}) \\ &= \mathrm{Hom}(\mathrm{M}, \mathrm{R}\mathscr{H}\!om(\mathrm{K}, f_*\mathrm{L})).\end{aligned}$$

5.2.6. This proposition gives more than a semiorthogonal decomposition; recall $\mathrm{D}^+(\mathrm{X}_{\text{proét}})$ has a semiorthogonal decomposition into (1) $\mathrm{D}^+_p(\mathrm{X}_{\text{proét}})$ and (2) $\mathrm{D}^+(\mathrm{X}_{\text{ét}})$ if every morphism from the object of the latter to one of the former is null (this is satisfied since $\mathrm{Hom}(\nu^*\mathrm{K}, \mathrm{P}) = \mathrm{Hom}(\mathrm{K}, \nu_*\mathrm{P}) = 0$) and the smallest strictly full triangulated subcategory of $\mathrm{D}^+(\mathrm{X}_{\text{proét}})$ is containing both is $\mathrm{D}^+(\mathrm{X}_{\text{proét}})$ itself (this is true since for any K in $\mathrm{D}^+(\mathrm{X}_{\text{proét}})$ we have an exact triangle $\nu^*\nu_*\mathrm{K} \to \mathrm{K} \to \mathrm{L}(\mathrm{K}) \to$).

5.2.9. The ∞-category $\mathrm{D}(\mathrm{X}_{\text{proét}})$ has all (small) colimits by Lurie, *Higher Algebra* (1.3.5.21).

5.3. The pullback η^* factors through τ since $\eta^*\mathrm{K} \xrightarrow{\sim} \mathrm{R}\lim \tau\eta^*\mathrm{K} \simeq \mathrm{R}\lim \eta^*\tau\mathrm{K}$, so the functor $\hat{\mathrm{D}}(\mathrm{X}_{\text{ét}}) \to \mathrm{D}_{cc}(\mathrm{X}_{\text{proét}})$ is given by $\mathrm{R}\lim \eta^*$.

Recall that by (`09B1`), $\nu^* : \mathrm{D}^+(\mathrm{X}_{\text{ét}}) \to \mathrm{D}^+_{cc}(\mathrm{X}_{\text{proét}})$ is an equivalence with quasi-inverse ν_*.

5.3.2. Recall that if $f \dashv g$ are adjoint functors between categories $\mathrm{C} \underset{g}{\overset{f}{\leftrightarrows}} \mathrm{D}$ and f is an equivalence, then g is also an equivalence: g is essentially surjective since η is a natural isomorphism, and fully faithful since any a in C is isomorphic to $f(b)$ for some b in D, so $(a, c) = (f(b), c) = (b, g(c)) = (gf(b), g(c)) = (g(a), g(c))$, again since η is a natural isomorphism. In conclusion, for (K_n) in $\hat{D}(X_{\text{proét}})$, we have $\tau_{\geq -n} R\lim K_n \xrightarrow{\sim} K_n$ for any n, so $\mu((K_n)) = R\lim \nu^*(K_n)$ indeed belongs to $D_{cc}(X_{\text{proét}})$.

Next, since $H^{-n-1}K$ is in the essential image of ν^* and $\nu_*\nu^* \simeq \mathrm{id}$ in $D^+(X_{\text{ét}})$, applying ν_* to the distinguished triangle

$$H^{-n-1}K \to \tau_{\geq -n-1}K \to \tau_{\geq -n}K \to$$

gives a triangle $(\nu_* H^{-n-1}K, \nu_*\tau_{\geq -n-1}K, \nu_*\tau_{\geq -n}K)$ with the first term in $D^{\leq -n-1}$ and the last in $D^{> -n-1}$. Therefore $\nu_*\tau_{\geq -n}K \simeq \tau_{\geq -n}\nu_*\tau_{\geq -n-1}K$, and $\gamma(K)$ indeed lands in $\hat{D}(X_{\text{ét}})$. (N.B. To define γ so that it lands in $D(X^{\mathbf{N}}_{\text{proét}})$ rather than $D(X_{\text{proét}})^{\mathbf{N}}$, one can use (070F).)

Combining the fact that $\tau \dashv R\lim$ give an adjoint equivalence $\hat{D}(X_{\text{proét}}) \leftrightarrows D(X_{\text{proét}})$ and $\nu^* \dashv \nu_*$ give an adjoint equivalence $D^+_{cc}(X_{\text{proét}}) \leftrightarrows D^+(X_{\text{ét}})$, we can write

$$\mu\gamma(K) = R\lim \nu^*\nu_*\tau_{\geq -n}K = R\lim \tau_{\geq -n}K = K \quad \text{and}$$
$$\gamma\mu((K_n)) = (\nu_*\tau_{\geq -n} R\lim \nu^*(K_n)) = (\nu_*\nu^*(K_n)) = (K_n).$$

In conclusion, $D^+_{cc}(X_{\text{proét}}) \simeq D^+(X_{\text{ét}})$ and $D_{cc}(X_{\text{proét}}) \simeq \hat{D}(X_{\text{ét}})$.

5.4.1. The map $f_{\text{proét}}$ is the morphism of sites defined by the continuous functor $Y_{\text{proét}} \to X_{\text{proét}}$ that sends $U \to Y$ weakly étale to $Y \times_Y X \to X$. This functor commutes with fiber products so is continuous by the criterion of [**SGAA**, III 1.6] (022D). Since every object of $\tilde{Y}_{\text{ét}}$ is a colimit of representable objects and $f^*_{\text{ét}}, f^*_{\text{proét}}, \nu^*_X, \nu^*_Y$ commute with colimits, it amounts to showing that the two different compositions of functors $Y_{\text{ét}} \to X_{\text{proét}}$ coincide. Both of course send $(U \to Y)$ étale to $U \times_Y X \to X$, considered as a weakly étale morphism. This proves the identity on pullbacks and also the commutativity of the square by [**SGAA**, IV 3.2].

5.4.2. Given a site $\mathscr{C}$, the assignment $X \mapsto \widetilde{(\mathscr{C}/X)}$ defines a stack over $\mathscr{C}$ (example 4.11 in Vistoli's notes). As f_* commutes with restriction,[38] we reduce to the affine case,

[38]Let $U \to Y$ be weakly étale. The functor f^{-1} extends in an obvious way to a continuous functor $Y_{\text{proét}}/U \to X_{\text{proét}}/f^{-1}U$ determining a morphism of sites. The identity $j^*_U \circ f_* = f_* \circ j^*_{f^{-1}U}$ follows since the sections of both over $V \to U$ in $Y_{\text{proét}}/U$ are given by sections over $f^{-1}V$.

where $f^* \dashv f_*$ form an adjoint equivalence since f^{-1} sets up an equivalence of sites. (The underlying categories are equivalent since they are Ind of equivalent categories by topological invariance of the étale site, and the topologies obviously coincide since a family of maps (g_i) is covering if finitely many among them are jointly surjective, which can be checked before or after applying the homeomorphism f^{-1}.)

5.4.3. *Caveat lector:* S_2 is acting randomly on the appearances of elements of the set $\{X, Y\}$ in this proof. Let's insist $f : Y \to X$, in which case the identity we're trying to show is

$$\nu_X^* \circ f_{\text{ét},*} \xrightarrow{\sim} f_{\text{proét},*} \circ \nu_Y^*.$$ [39]

Let Y be a scheme. We say Y has property $(*)$ if

$$\text{Whenever } U, V \subset Y \text{ are affine open, } U \cap V \text{ is affine.} \tag{$*$}$$

Step 1: X, Y affine $\rightsquigarrow$ As written when F is a sheaf of sets. When K is in $D^+(Y_{\text{ét}})$, one can also reason following (`099V`), so that after replacing K by a bounded below complex of injectives I, $f_{\text{proét},*}\nu_Y^* K$ is computed by ${}^\circ f_{\text{proét},*}\nu_Y^* I$, and now the desired identity follows from the case of a sheaf of sets.

Step 2: X affine, Y is quasi-compact and has $(*)$ $\rightsquigarrow$ Induction on the least integer n so that Y can be covered by so many affine opens, the case $n = 1$ covered by Step 1. Suppose $Y = U \cup V$ with U affine open and V the union of strictly fewer than n affine opens. There is a distinguished triangle (`0EVY` & `01EC`)

$$0 \to f_* K \to (f|_U)_*(K|_U) \oplus (f|_V)_*(K|_V) \to (f|_{U\cap V})_*(K|_{U\cap V}) \to \quad ;$$

taking the long exact sequence and using the inductive hypothesis, we conclude.

When F is a sheaf of sets, we have instead that the diagram

$$f_* F \to (f|_U)_*(F|_U) \amalg (f|_V)_*(F|_V) \rightrightarrows (f|_{U\cap V})_*(F|_{U\cap V})$$

is exact, since for any weakly étale $W \to Y$, the diagram

$$F(X \times_Y W) \to F(U \times_Y W) \amalg F(V \times_Y W) \rightrightarrows F(U \cap V \times_Y W)$$

is. The same inductive argument works, since in this case $\nu_X^* f_{\text{ét},*}\mathrm{F}$ and $f_{\text{proét},*}\nu_Y^*\mathrm{F}$ are both equalizers of isomorphic pairs of arrows.

Step 3: X affine, Y coherent $\rightsquigarrow$ We repeat the argument of Step 2, but now if U is affine and V is the union of $< n$ affine opens, $\mathrm{U}\cap\mathrm{V}$ is quasi-compact and has property $(*)$, so we get $f|_{\mathrm{U}\cap\mathrm{V}}$ from Step 2, while the inductive hypothesis gives $f|_{\mathrm{V}}$.

5.4.6. The spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{R}^q f_*\mathscr{H}^p(\tau_{\geq -n}\mathrm{K}) \Rightarrow \mathrm{R}^{p+q} f_*\tau_{\geq -n}\mathrm{K}$$

combined with Lemma 5.4.3 and the assumption $\mathrm{cd}\, f_* \leq d$ implies $f_*\tau_{\geq -n}\mathrm{K}$ is in $\mathrm{D}_{cc}^{\leq k+d}(\mathrm{Y}_{\text{proét}})$. For $n \geq d - i$, $\mathscr{H}^i(f_*\tau_{<-n}\mathrm{K}) = 0$, so the distinguished triangle

$$f_*\tau_{<-n}\mathrm{K} \to f_*\mathrm{K} \to f_*\tau_{\geq -n}\mathrm{K} \to$$

shows that $\mathscr{H}^i(f_*\tau_{\geq -n}\mathrm{K}) = \mathscr{H}^i(f_*\mathrm{K})$ for such n. By repleteness (`07KY`), we have an exact sequence

$$0 \to \mathrm{R}^1 \lim_n \mathscr{H}^{i-1}(f_*\tau_{\geq -n}\mathrm{K}) \to \mathscr{H}^i\mathrm{K} \to \lim_n \mathscr{H}^i(f_*\tau_{\geq -n}\mathrm{K}) \to 0,$$

and since for each i, the system $\mathscr{H}^i(f_*\tau_{\geq -n}\mathrm{K})$ is essentially constant with value $\mathscr{H}^i(f_*\mathrm{K})$, we find $\mathscr{H}^i\mathrm{K} \simeq \lim_n \mathscr{H}^i(f_*\tau_{\geq -n}\mathrm{K}) = \mathscr{H}^i(f_*\mathrm{K})$.

6.1.2. In the note to (2.2.14), we showed that given a surjective ring map $\mathrm{B} \to \mathrm{B}/\mathrm{J}$ and an ind-étale B/J-algebra B_0, $\mathrm{Hens}_\mathrm{B}(\mathrm{B}_0)$ is indeed henselian along $\mathrm{J}\,\mathrm{Hens}_\mathrm{B}(\mathrm{B}_0) = \ker(\mathrm{Hens}_\mathrm{B}(\mathrm{B}_0) \to \mathrm{B}_0))$. In our case, $\mathrm{X} = \mathrm{Spec}\,\mathrm{B}$ and Z is cut out by J.

Suppose $\mathrm{V}' = \mathrm{Spec}(\mathrm{A}_0') \to \mathrm{V}$ is surjective and let $\tilde{\mathrm{V}}' = \mathrm{Spec}\,\mathrm{A}'$. Then $\mathrm{A} \to \mathrm{A}'$ is ind-étale (`08HS`) and $\mathrm{V} = \mathrm{Spec}\,\mathrm{A} \otimes_\mathrm{B} \mathrm{B}/\mathrm{J} \leftarrow \mathrm{Spec}\,\mathrm{A}' \otimes_\mathrm{B} \mathrm{B}/\mathrm{J}$ is surjective, so the image of $\tilde{\mathrm{V}}' \to \tilde{\mathrm{V}}$ contains V. If $\tilde{\mathrm{V}}' = \lim_i \mathrm{W}_i$ with W_i étale over $\tilde{\mathrm{V}}$, then $\tilde{\mathrm{V}}' \to \tilde{\mathrm{V}}$ factors through each W_i, so necessarily the image of $\mathrm{W}_i \to \tilde{\mathrm{V}}$ contains V for each i. If $\mathrm{W}_i \to \tilde{\mathrm{V}}$ is surjective for all i, then $\tilde{\mathrm{V}}' \to \tilde{\mathrm{V}}$ is surjective,[40] so we are indeed reduced to the claim that any étale map $\mathrm{W} \to \tilde{\mathrm{V}}$ whose image contains V is surjective.

[40]The base change to any point of $\tilde{\mathrm{V}}$ is a cofiltered limit of nonempty schemes.

6.1.3. If we let v denote the left adjoint to $u := i^{-1}$, then by [**SGAA**, I 5.5], $v^* = u_!$ as functors on presheaves. The functor i^{-1} is continuous since it commutes with fiber products and takes surjections to surjections, and therefore i^{-1} induces a functor $u_s : \tilde{Z}^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}} \to \tilde{X}^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ whose left adjoint $u^s := i^*$ is given by $\underline{a}u_! = \underline{a}v^*$, where $v^*F(V) := F(\tilde{V})$.[41] Sheafification is trivial on any w-contractible object V since any pro-étale covering of V can be refined by a finite decomposition of V into clopens (`098F`); henselization commutes with finite disjoint unions as a left adjoint, and F sends disjoint unions to products since it is already a sheaf.

6.1.6. We have $i^*\mathcal{O}_X = S$ and $i^*I = IS$ since étale pullback of quasi-coherent sheaves coincides with ordinary (Zariski) pullback (`04I4` & `03DR`).

6.1.7. The functor i^* has a left adjoint since it commutes with small limits [**SGAA**, IV 1.8], and this does not require X to be affine. It is not true in general that $i_\#$ sends any V in $Z^{\mathrm{aff}}_{\mathrm{pro\acute{e}t}}$ to $\tilde{V}$; that is equivalent to the statement

$$F(\tilde{V}) = i^*F(V),$$

which by Lemma 6.1.3 we only know is true when X is affine and V is w-contractible ($\tilde{V}$ hasn't even been defined when X isn't affine). As $i_\#$ commutes with restriction to an affine open $U \subset X$ iff j_{U*} commutes with i^*, which seems unlikely, I don't think we yet have enough to characterize $i_\#$ when X is not affine.

6.1.8. As in the case of (3.1.9) and its note, exactness of i_* means commuting with finite colimits, while it's only proved here that i_* preserves surjections, so is exact as a functor on abelian sheaves.

If $f \in F(W)$ lifts g, it means in particular that f is in the kernel of $F(W) \rightrightarrows F(W \times_{Y_0} W)$. In order for f to be a section of $i_*F(\tilde{W} \coprod Y_U)$ ($Y_U := Y \times_X U$) lifting g, f must belong to the kernel of

$$F(W) = i_*F(\tilde{W} \amalg Y_U) \rightrightarrows i_*F((\tilde{W} \amalg Y_U) \times_Y (\tilde{W} \amalg Y_U)).$$

[41] Collision of notation: here tilde means both 'sheaves' and 'henselization.'

As $(\tilde{W} \amalg Y_U) \times_Y (\tilde{W} \amalg Y_U) \simeq (\tilde{W} \times_Y \tilde{W}) \coprod (Y_U \times_Y \tilde{W}) \coprod (\tilde{W} \times_Y Y_U) \coprod (Y_U \times_Y Y_U)$ and i^{-1} of this is $i^{-1}(\tilde{W} \times_Y \tilde{W}) = W \times_{Y_0} W$, f is indeed a section lifting g.

6.1.9. The isomorphisms $i_* i^* F(V) = i^* F(V_0) = F(\tilde{V}_0)$ hold when X is affine and V_0 in $Z^{\mathrm{aff}}_{\mathrm{proét}}$ is w-contractible. As the pro-étale topos is indifferent to the scheme structure on Z, the validity of these isomorphisms is independent of this choice. Therefore by (6.1.3), it suffices to find an ideal $I \subset \Gamma(X, \mathcal{O}_X)$ so that Z is cut out setwise by I and V_0 is w-contractible whenever V is. By (2.4.10), this is true if I can be chosen to be finitely generated. This is true since U is quasi-compact, so $U = \cup_{i=1}^n D(f_i)$ and Z coincides as a subset of X with $V(f_1, \ldots, f_n)$.

As F is pointed, given $f \in i_* i^* F(V) = F(\tilde{V}_0)$, we can find a section of $F(V_U)$, together defining a section of $F(\tilde{V}_0 \coprod V_U) = F(\tilde{V}_0) \times F(V_U)$. As $i^{-1}\tilde{V}_0 = V_0$,

$$i_* i^* F(\tilde{V}_0 \amalg V_U) = i_* i^* F(\tilde{V}_0) \times i_* i^* F(V_U) = i^* F(V_0) \times \{*\} = F(\tilde{V}_0) = i_* i^* F(V)$$

whenever V is w-constructible and I is chosen as above, so our chosen section goes to $f \in i_* i^* F(V)$, and we're done.

6.1.10. Every V in $X_{\mathrm{proét}}$ has at most one X-morphism to U. Therefore $i_* i^* F(V) = i_* \mathbf{Z}(V) = \mathbf{Z}(V_Z)$ (`00Y2` & `03CE`), while on the other hand (`03HT`),

$$F(V) = i_! \mathbf{Z}(V) = \begin{cases} \mathbf{Z}(V) = \mathbf{Z}(V_Z) & \text{if } V \to X \text{ factors through Z, and} \\ \emptyset & \text{otherwise.} \end{cases}$$

So F is a subsheaf of $i_* i^* F$, but they are not the same, since $F(X) = \emptyset$ while $i_* i^* F(X) = F(Z) = \mathbf{Z}^{\pi_0(Z)}$.

6.1.11. First, let's check that all the functors $j_!, j^*, i_*, i^*$ commute with localization. The commutativity of the square of topos (`0EYV`) shows j^* does, and the adjoint of the formula $j^*_{V'/V} j_{U/V,*} = j_{U'/V',*} j^*_{U'/U}$ shows $j_!$ does. Then (`03CF`) takes care of i_* and i^*.

Here is a slightly different proof. We replace X by any affine open and let V in $X^{\mathrm{aff}}_{\mathrm{proét}}$ be w-contractible. Since U is quasi-compact, as in the argument in the note to (6.1.9), Z is cut out by a finitely-generated ideal, so V_0 is w-contractible by (2.4.10). Then $\tilde{V}_0 \coprod V_U \to V$ is a cover in $X_{\mathrm{proét}}$, and every object of $X_{\mathrm{proét}}$ can be covered by schemes obtained via this construction, for varying V. By [**SGAA**, II 4.4, 4.9], the (sheaves represented by) schemes of this type form a generating family; i.e. the family of functors ($h_{\tilde{V}_0 \coprod V_U} := \mathrm{Hom}(\tilde{V}_0 \coprod V_U, -)$) as V varies among w-constructible objects of $X^{\mathrm{aff}}_{\text{proét}}$ form a conservative and faithful family. If we can show that

$$0 \to j_! j^* F(\tilde{V}_0 \amalg V_U) \to F(\tilde{V}_0 \amalg V_U) \to i_* i^* F(\tilde{V}_0 \amalg V_U) \to 0 \tag{‡}$$

is exact for all such V, then by the faithfulness of the family we can conclude that the composition $j_! j^* F \to i_* i^* F$ is 0,[42] so we get a map $j_! j^* F \to \ker(F \to i_* i^* F)$. As this map is an bijection on sections over all the $\tilde{V}_0 \coprod V_U$, it is an isomorphism since this family of functors is conservative. Therefore it suffices to show (‡).

Since every object W of $X_{\text{proét}}$ has at most one X-morphism to U, (03HT) says that

$$j_! j^* F(W) = \begin{cases} F(W) & \text{if } W \to X \text{ factors through U} \\ * & \text{otherwise.} \end{cases}$$

On the other hand, when $W_0 = \emptyset$, $i_* i^* F(W) = *$, and if V is in $X^{\mathrm{aff}}_{\text{proét}}$ and is w-constructible, then $i_* i^* F(V) = F(\tilde{V}_0)$ by (6.1.3). In conclusion, for w-constructible V in $X^{\mathrm{aff}}_{\text{proét}}$ we have

$$\begin{aligned} j_! j^* F(\tilde{V}_0) &= * & \qquad i_* i^* F(\tilde{V}_0) &= F(\tilde{V}_0) \\ j_! j^* F(V_U) &= F(V_U) & \qquad i_* i^* F(V_U) &= *. \end{aligned}$$

The exactness of (‡) is now clear.

6.1.12. The reference for the second part of (1) is (00Y2). For $i^* i_* \xrightarrow{\sim} \mathrm{id}$, we reduce to X affine and test on w-contractible V in $Z^{\mathrm{aff}}_{\text{proét}}$ as described.

6.1.15. This lemma, like many others, is stated for $f : (\tilde{Y}_{\text{proét}}, \mathbf{Z}) \to (\tilde{X}_{\text{proét}}, \mathbf{Z})$, but is needed later with, e.g., $\mathbf{Z}$ replaced by $\hat{R}$. It holds for $\hat{R}$ or $\mathcal{O}_{E,X}$ or E_X of §6.8, since all these sheaves have the property that $i^*\hat{R} = \hat{R}$, $i^*\mathcal{O}_{E,X} = \mathcal{O}_{E,Z}$, etc. by Corollary 6.1.5, so i^* (and j^*) coincide with i^{-1} and j^{-1} and we still have $i^* i_* = \mathrm{id} = j^* j_!$. The given commutative diagram gives rise to a commutative diagram of topos ringed by $\hat{R}$.

[42]Let 0 denote the distinguished element in the pointed set $\mathrm{Hom}(j_! j^* F, i_* i^* F)$.

6.1.16. The existence of $i^!$ is guaranteed by Brown representability (`0A8G`). L is defined by the distinguished triangle $L \to K \to j_*j^*K \to$. After that, from the perspective of the ordinary triangulated category, all the R Homs should rather be Homs; the conclusion holds that $\mathrm{Hom}_{D(X_{\text{proét}})}(i_*M, i_*i^!K) = \mathrm{Hom}_{D(X_{\text{proét}})}(i_*M, L)$, and as $L = i_*i^*L$, for any M in $D(Z_{\text{proét}})$ we have $\mathrm{Hom}_{D(Z_{\text{proét}})}(M, i^!K) = \mathrm{Hom}_{D(Z_{\text{proét}})}(M, i^*L)$ by full faithfulness of i_*, so it follows that $i^!K = i^*L$ and hence $\eta : i_*i^!K \to i_*i^*L = L$ is an isomorphism.

6.1.17. Let $0 \to F \to G \to H \to 0$ be an exact sequence in $\mathrm{Ab}(X_{\text{proét}})$, and let V in $Y_{\text{proét}}$ be the spectrum of a w-contractible ring. Then $f^{-1}V$ is the same by (2.4.10), so $0 \to F(f^{-1}V) \to G(f^{-1}V) \to H(f^{-1}V) \to 0$ is exact (`098H` & `090J`). As $F(f^{-1}V) = f_*F(V)$, etc., and there are enough weakly contractible objects in $Y_{\text{proét}}$, we conclude that $0 \to f_*F \to f_*G \to f_*H \to 0$ is exact. (To be more explicit, replacing Y by an affine open, the full subcategory of $Y^{\text{aff}}_{\text{proét}}$ of w-contractible objects with the induced topology[43] is endowed with a continuous functor u to $Y_{\text{proét}}$ inducing an equivalence of topos [**SGAA**, III 4.1].)

6.2. As X is coherent, an open $U \subset X$ is retrocompact iff it's quasi-compact, in which case it's automatically coherent. Therefore the functors i and j and subsets U and Z of (6.2.1) satisfy the hypotheses of §6.1. Moreover, if X is coherent and U is quasi-compact, then Z is also coherent: given two quasi-compact opens $P_0, Q_0 \subset Z$, write $P_0 = P \cap Z$ and $Q_0 = Q \cap Z$ for P and Q open in X. While P and Q needn't be quasi-compact, $P \cup U$ and $Q \cup U$ are, so $P_0 \cap Q_0 = (P \cup U) \cap (Q \cup U) \cap Z$ is quasi-compact as X is quasi-separated, so the same is true of Z.

6.2.1. Let $W = T \cap V^c$ with T, V quasi-compact opens of X. We replace $\overline{W}$ with V^c in what follows. A complex in $D(V^c_{\text{proét}})$ is supported on W if its restriction to $T^c \cap V^c \to V^c$ is acyclic, while a complex in $D(X_{\text{proét}})$ is supported on W if its restriction

[43]By [**SGAA**, III 3.3] this is the usual topology where a family of morphisms in $Y^{\text{aff}}_{\text{proét}}$ is covering if finitely many among them are jointly surjective.

to $a : \mathrm{V} \cup \mathrm{T}^c \to \mathrm{X}$ is acyclic. The cartesian square

$$\begin{tikzcd} \mathrm{V}^c \cap \mathrm{T}^c \arrow[r, "a"] \arrow[d, "g"] & \mathrm{V}^c \arrow[d, "g"] \\ \mathrm{V} \cup \mathrm{T}^c \arrow[r, "a"] & \mathrm{X} \end{tikzcd}$$

and (6.1.15) allow us to write $a^*g_* = g_*a^*$, so K in $\mathrm{D}(\mathrm{V}^c_{\text{proét}})$ is supported on W iff $g_*\mathrm{K}$ is, and $g^* \dashv g_*$ restrict to adjoint equivalences $\mathrm{D}_{\mathrm{W}}(\mathrm{V}^c_{\text{proét}}) \leftrightarrows \mathrm{D}_{\mathrm{W}}(\mathrm{X}_{\text{proét}})$. The rest is the same after replacing $\overline{\mathrm{W}}$ with V^c.

The functor $k_!$ only depends on the choice of T and V up to natural isomorphism since its quasi-inverse k^* is independent of this choice.

6.2.3. These results are stated for **Z**-coefficients but hold more generally.

Of course (3) should read $k_!\mathrm{K} \otimes \mathrm{L} \simeq k_!(\mathrm{K} \otimes k^*\mathrm{L})$. When k is weakly étale,

$$\begin{aligned} \mathrm{Hom}(k_!\mathrm{K} \otimes \mathrm{L}, \mathrm{M}) &= \mathrm{Hom}(k_!\mathrm{K}, \mathrm{R}\mathscr{H}\!om(\mathrm{L}, \mathrm{M})) \\ &= \mathrm{Hom}(\mathrm{K}, k^*\mathrm{R}\mathscr{H}\!om(\mathrm{L}, \mathrm{M})) \\ &= \mathrm{Hom}(\mathrm{K}, \mathrm{R}\mathscr{H}\!om(k^*\mathrm{L}, k^*\mathrm{M})) \\ &= \mathrm{Hom}(\mathrm{K} \otimes k^*\mathrm{L}, k^*\mathrm{M}) \\ &= \mathrm{Hom}(k_!(\mathrm{K} \otimes k^*\mathrm{L}), \mathrm{M}) \end{aligned}$$

(c.f. [**SGAA**, IV 12.3 b)] for the third equality).

4. When k is open, follows from (`03CF`). The commutativity of ν^* with i^* follows more generally from a commutative square of morphisms of topos, which can be verified by checking that the two compositions of the continuous functors inducing ν and i commute, which is trivial in this case.

5. Explicitly, $k_! = g_*f_! \dashv f^*g^!$.

6.3. Let's note once and for all that a locally closed constructible subset of a coherent scheme X can be written as $\mathrm{U} \cap \mathrm{V}^c$ for U, V quasi-compact opens of X (conjugate `0F2K`, `005H` & `09YH` and see also the note to 6.2).

6.3.1. It's common to call a complex locally constant if its cohomology sheaves are, but in this definition, $K|_{X_i}$ is locally constant if there is an étale cover $(U_{ij} \to X_i)$ so that $K|_{U_{ij}}$ is constant in the sense of 6.3; i.e. in the essential image of the pullback $D(F) \to D((U_{ij})_{\text{ét}}, F)$; i.e. can be represented by a complex of locally constant sheaves. (Typo: it's meant that $K|_{X_i}$ is locally constant with perfect values on $(X_i)_{\text{ét}}$.) Note that any (6.3.1)-constructible complex is bounded since a perfect complex is (`0657`).

6.3.2. This definition coincides with $D^b_{\text{ctf}}(X_{\text{ét}}, F)$ in the setting where the latter is usually defined; i.e. when X and F are noetherian (Deligne, *Cohomologie Etale, Rapport* 4.6). In that case, any $K \in \operatorname{ob} D^b_{\text{ctf}}(X_{\text{ét}}, F)$ is (6.3.1)-constructible by combining *Rapport* (4.6) with `066E`, `0658`, and `094G`, and vice versa.

6.3.4. 'Flat dimension' = 'tor-dimension.'

6.3.5. The (derived) tensor product of perfect complexes is perfect, as the tensor product and direct sum of finite projective modules is finite projective.

6.3.6. Let $b : (X_{\text{ét}}, F) \to (\text{Set}, F)$ be the morphism to the final topos. The unit of adjunction gives a map $K \to b_* b^* K = R\Gamma(X_{\text{ét}}, \underline{K})$ in $D(F)$. If U is in $X_{\text{ét}}$, the triangle

$$\begin{tikzcd}
(U_{\text{ét}}, F) \arrow[rr, "j_U"] \arrow[dr, "b"'] & & (X_{\text{ét}}, F) \arrow[dl, "b"] \\
& (\text{Set}, F) &
\end{tikzcd}$$

commutes. We have that if I is a K-injective complex representing $\underline{K}$, $K_x = I_x = \operatorname{colim} I(U)$, so $H^0 K = \operatorname{colim} H^0(U_{\text{ét}}, \underline{K})$. In other words, as $bx = \text{id} : (\text{Set}, F) \to (\text{Set}, F)$, $\text{id} \to b_* x_* x^* b^*$ is an isomorphism. Given $s \in H^0(X_{\text{ét}}, \underline{K})$, we get a section $s_i \in H^0 K$; i.e. a map $F \to H^0 K$ that we lift to a morphism of complexes $s_i : F \to K$ with the property that $H^0 x^* b^* s_i : F \to \operatorname{colim} H^0(U_{\text{ét}}, \underline{K})$ coincides with the image of s. For $V \to U$ in $X_{\text{ét}}$, the units $b_* b^* \to b_* j_{V*} j_V^* b^*$ provide maps $R\Gamma(U_{\text{ét}}, \underline{K}) \to R\Gamma(V_{\text{ét}}, \underline{K})$. So there is some U in $X_{\text{ét}}$ so that

$$F \xrightarrow{s} H^0(X_{\text{ét}}, \underline{K}) = H^0 b_* b^* K \to H^0 b_* j_{U*} j_U^* b^* K = H^0(U_{\text{ét}}, \underline{K})$$

coincides with

$$\mathrm{F} \to \mathrm{H}^0 b_* j_{\mathrm{U}*} j_{\mathrm{U}}^* b^* \mathrm{F} \xrightarrow{s_i} \mathrm{H}^0 b_* j_{\mathrm{U}*} j_{\mathrm{U}}^* b^* \mathrm{K} = \mathrm{H}^0(\mathrm{U}_{\text{ét}}, \underline{\mathrm{K}}).$$

6.3.7. N.B.: Implicitly, K in $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{F})$ is constant if K is in the essential image of the functor $b^* : \mathrm{D}(\mathrm{F}) \to \mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{F})$; i.e. can be represented by a complex of constant sheaves.

The proof of this lemma rests on a purported identity

$$\underline{\mathrm{R}\,\mathrm{Hom}(\mathrm{A}, \mathrm{B})} = \mathrm{R}\mathscr{H}om(\underline{\mathrm{A}}, \underline{\mathrm{B}})$$

for A, B in $\mathrm{D}^b(\mathrm{F})$. The underived version of this statement is proved in `093T` with the additional assumption that A be of finite presentation. I can't substantiate it without a finiteness hypothesis. Compare also with `094G`, which adds some finiteness hypotheses.

6.3.8. Write l.c.c. for locally closed constructible. Implicit in the statement is that the strata are l.c.c. Let's suppose $\mathrm{W} \subset \mathrm{X}$ and $\mathrm{Y} \subset \mathrm{W}$ are l.c.c., so $\mathrm{W} = \mathrm{U} \cap \mathrm{V}^c$ and $\mathrm{Y} = \mathrm{A} \cap \mathrm{B}^c$ for U, V quasi-compact open in X and A, B quasi-compact open in W (by the note to 6.2, V^c hence W are coherent). There is a quasi-compact open A' of X so that $\mathrm{A}' \cap \mathrm{Y} = \mathrm{A}$: take the union of V with any open of X that intersects V^c in A. Similarly we find B' for B. Then $\mathrm{U} \cap \mathrm{A}'$ and $\mathrm{V} \cup \mathrm{B}'$ are quasi-compact opens of X and

$$(\mathrm{U} \cap \mathrm{A}') \cap (\mathrm{V} \cup \mathrm{B}')^c = \mathrm{W} \cap (\mathrm{A}' \cap \mathrm{B}'^c) = \mathrm{A} \cap \mathrm{B}^c = \mathrm{Y},$$

so Y is a constructible subset of X contained in W. Conversely, as the immersion of a l.c.c. subset is quasi-compact, the intersection of any l.c.c. subset of X with W is a l.c.c. subset of X.

6.3.9. Recall that the full subcategory of perfect objects in D(F) is a saturated (= épaisse), strictly full and triangulated subcategory of D(F) (`0ATI`). Any saturated subcategory of an idempotent complete triangulated category is idempotent complete, and the derived category of any Grothendieck abelian category is idempotent complete (`05QW` & `07D9`).

To see that the cone of $\mathrm{K} \to \mathrm{L}$ is locally constant (if so it has perfect values by the previous paragraph), we argue as in the note to (6.3.12) with simplifications since

K' is perfect. As explained in detail there, it suffices to show that $\underline{\mathrm{R\,Hom}(\mathrm{A},\mathrm{B})} = \mathrm{R}\mathscr{H}\!\mathit{om}(\underline{\mathrm{A}},\underline{\mathrm{B}})$ when A is now a finite projective F-module. The lemma there shows that it suffices to show the statement after dropping the Rs, whence the conclusion by the same references.

To see that $\mathrm{D}_c(\mathrm{X}_{\text{ét}},\mathrm{F}) := \mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{ét}},\mathrm{F})$ is idempotent complete, it suffices to show that it is a saturated subcategory of $\mathrm{D}(\mathrm{X}_{\text{ét}},\mathrm{F})$, since this latter category is idempotent complete by the remarks of the first paragraph. So suppose $\mathrm{A}\oplus\mathrm{B}$ is in $\mathrm{D}_c(\mathrm{X}_{\text{ét}},\mathrm{F})$, or, restricting, suppose $\mathrm{A}\oplus\mathrm{B} = \underline{\mathrm{C}}$ is constant with perfect values on $\mathrm{X}_{\text{ét}}$, and let e_1, e_2 denote the two obvious idempotents on $\mathrm{A}\oplus\mathrm{B}$. The argument in the previous paragraph implies that there is an étale cover $(\mathrm{U}_i \to \mathrm{X})$ and idempotents $d_{1,i}, d_{2,i}$ on C so that $e_1|_{\mathrm{U}_i} = b^*d_{1,i}$, and likewise for e_2 (b as in the note to (6.3.6)). Replacing X by U_i, these idempotents split as $\mathrm{D}_{\text{perf}}(\mathrm{F})$ is idempotent complete, so $\mathrm{C} = \mathrm{C}_1 \oplus \mathrm{C}_2$, with $\mathrm{C}_1, \mathrm{C}_2$ perfect. Therefore both $\underline{\mathrm{C}}_1$ and A are limits (and colimits) of the diagram

$$\mathrm{A}\oplus\mathrm{B} = \underline{\mathrm{C}} \underset{\mathrm{id}}{\overset{e_1=b^*d_1}{\rightrightarrows}} \underline{\mathrm{C}},$$

so are isomorphic, and similarly for $\underline{\mathrm{C}}_2$ and B. In conclusion, A and B are constructible.

6.3.10. Note that if K is constructible then $f_i^*\mathrm{K}$ is constructible by `054I`. The converse is also true, but the proof given is not enough for two reasons: in general there doesn't exist such a stratification as described, and even when f is finite étale, I don't see how we can use Galois theory since that usually requires the base to be connected, and I don't think a connected component of X is in general a constructible subset. However, we can follow the proof of [**SGAA**, IX 2.8] to prove the stronger

Lemma. — Given K *in* $\mathrm{D}(\mathrm{X}_{\text{ét}},\mathrm{F})$ *and a surjective map* $f : \mathrm{Y} \to \mathrm{X}$ *locally of finite presentation,* K *is constructible if and only if* $f^*\mathrm{K}$ *is.*

(Note that this implies Lemma 6.3.10.) The key ingredient is

Lemma [**SGAA**, IX 2.8.1]. — *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a surjective morphism locally of finite presentation, with* X *coherent. Then there exists a finite partition of* X *into subschemes* X_i *of finite presentation over* X *(in particular, each* X_i *is constructible in* X*) and for each i, finite surjective morphisms* $\mathrm{X}_i'' \xrightarrow{g_i} \mathrm{X}_i' \xrightarrow{h_i} \mathrm{X}_i$*, with* h_i *étale and* g_i *locally* *free (i.e. flat and of finite presentation, in addition to being finite) and radicial, and finally a* X*-morphism* $X_i'' \to Y$.

We're now in a position to prove the first lemma. Suppose $f : Y \to X$ is surjective étale and f^*K is constructible. The above lemma gives us a finite partition of X into locally closed[44] constructible subsets, and replacing X by one of them[45] we may assume f finite radicial or finite étale. The former case being trivial by the topological invariance of the étale site, we treat the latter. As f^*K is constructible there's a finite partition $\coprod Y_i$ of Y into constructible subsets of finite presentation over Y so that the restriction of f^*K to Y_i is locally constant with perfect values. Replacing Y by $\coprod Y_i$ and applying the above lemma again, we find a finite partition of X into locally closed constructible subsets; replacing X with one of them we reduce[46] to the case f finite étale and f^*K locally constant with perfect values, which is trivial.

By way of addendum, I record the following simple

Lemma. — Given a finite partition $\coprod Y_i$ *of any scheme* Y *into locally closed constructible subsets* Y_i*, then one of the* Y_i *contains a nonempty open of* Y.

Proof. — Indeed, if $\overline{Y}_i = Y$ for some i, Y_i is open in Y, and if not, then there is a minimal finite collection of $Y_1, \ldots, Y_n$ so that $Y = \overline{Y}_1 \cup \ldots \cup \overline{Y}_{n-1}$, with $n > 1$. Then $V := Y \setminus (\overline{Y}_1 \cup \ldots \cup \overline{Y}_{n-1})$ is nonempty, open in Y and contained in $\overline{Y}_n$. As $Y_n = \overline{Y}_n \cap U$ for a nonempty open $U \subset Y$, $V \cap U$ is a nonempty open of Y contained in Y_n since Y_n is dense in $\overline{Y}_n$. □

6.3.11. The partition of X over which j becomes finite étale ⇝ `03S0`, the étale-local splitting of revêtements étales ⇝ `04HN`. Now given a constructible K on $\coprod_{i=1}^n X$, we can find a common refinement of the n partitions of X, and are reduced to showing the claim for K locally constant with perfect values, which is obvious, since $j_! = j_*$ here and a finite direct sum of perfect complexes is perfect (`0665`).

[44] Recall that a subscheme is locally closed by definition (EGA I 4.1.3).
[45] By the note to (6.2), a constructible subset of a coherent scheme is coherent.
[46] If $A \subset B$ is locally closed constructible and $B \subset C$ is locally closed constructible, then $A \subset C$ is locally closed (obvious) and constructible (`09YJ`).

6.3.12. I can prove this statement when F/I is pseudo-coherent as an F-module. This is true, for example, when F is noetherian.

We avoid the use of (6.3.7) and argue as follows to find an open cover of X over which K restricts to constant complexes of the form $\underline{L}$. Once we know this, L is seen to be perfect via (07LU), since $L \otimes_F F/I$ is perfect.

Assume $I^2 = 0$ and $K_1 = \underline{L}_1$ for L_1 in $D_{perf}(F/I)$. We have a distinguished triangle

$$K \otimes_F I \longrightarrow K \longrightarrow K_1 \longrightarrow$$

and $K \otimes_F I = K_1 \otimes_{F/I} I = \underline{L_1 \otimes_{F/I} I}$. We want to conclude that K is locally constant in the sense of 6.3; i.e. that there is an étale cover of X over which K is in the essential image of b^* (b as in the note to (6.3.6)). It will suffice to show that there exists such a cover over which the map $s : K_1[-1] \to K \otimes_F I$ is b^* of a map $L_1 \to L_1 \otimes_F I$. We have a global section $s \in \mathrm{Hom}_{D(X_{ét},F)}(K_1[-1], K \otimes_F I) = H^0 R\mathscr{H}om_F(K_1[-1], K \otimes_F I)$, and $K_1[-1] = \underline{L}_1[-1]$ with L_1 perfect in $D(F/I)$.[47] If we can show that

$$\underline{R\,\mathrm{Hom}_F(L_1[-1], L_1 \otimes_{F/I} I)} = R\mathscr{H}om_F(\underline{L}_1[-1], \underline{L_1 \otimes_{F/I} I}),$$

the result would then follow from (6.3.6), since we would be guaranteed a cover $(U_i \to X)$ with the property that our

$$s \in \mathrm{Hom}_{D(X_{ét},F)}(\underline{L}_1[-1], \underline{L_1 \otimes_{F/I} I}) = H^0(X_{ét}, R\mathscr{H}om_F(\underline{L}_1[-1], \underline{L_1 \otimes_{F/I} I}))$$

would restrict over U_i to

$$s_i \in \mathrm{Hom}_{D(U_{ét},F)}(\underline{L}_1[-1], \underline{L_1 \otimes_{F/I} I}) = H^0(U_{ét}, \underline{R\,\mathrm{Hom}_F(L_1[-1], L_1 \otimes_{F/I} I)})$$

arising via pullback from $\overline{s}_i \in \mathrm{Hom}_{D(F)}(L_1[-1], L_1 \otimes_{F/I} I) = H^0 R\,\mathrm{Hom}_F(L_1[-1], L_1 \otimes_{F/I} I)$, so the cone of $b^*\overline{s}_i$ would coincide with the cone of s_i in $D(U_i, F)$, and this cone would therefore be constant.

[47]It's tempting to try to argue along these lines using that L_1 is perfect in $D(F/I)$ to remove the hypothesis that F/I is pseudo-coherent as an F-module, but this fails since the map $K_1[-1] \to K \otimes_F I$ in $D(F)$ isn't in the image of a map in $D(F/I)$: if it were, K would necessarily have I-torsion cohomology sheaves.

As both L_1 and $L_1 \otimes_{F/I} I$ are bounded, dévissage along stupid truncations of both reduces us to showing

$$\underline{\mathrm{R}\,\mathrm{Hom}_{\mathrm{F}}(\mathrm{A}, \mathrm{B})} = \mathrm{R}\mathscr{H}\!om_{\mathrm{F}}(\underline{\mathrm{A}}, \underline{\mathrm{B}})$$

where A is a finite projective F/I-module and B is any F/I-module. We can represent A in D(F) by a bounded above complex $\mathrm{P}^{\bullet}$ of finite projective F-modules (`064Z` & `064U`). Then $\mathrm{P}^{\bullet}$ computes both sides (we verify this for the right-hand side below); i.e. we must check

$$\underline{\mathrm{Hom}_{\mathrm{F}}(\mathrm{P}^{\bullet}, \mathrm{B})} = \mathscr{H}\!om_{\mathrm{F}}(\underline{\mathrm{P}}^{\bullet}, \underline{\mathrm{B}}).$$

We conclude by (`058R` & `093T`).

The following lemma holds with no conditions on F or the scheme X.

Lemma. — Let K *be in* $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{F})$ *and* L *be a bounded above complex of finite projective* F*-modules. Suppose either (a)* L *is bounded, or (b)* K *is in* $\mathrm{D}^+(\mathrm{X}_{\text{ét}}, \mathrm{F})$*. Then* $\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K})$ *computes* $\mathrm{R}\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K})$*.*

(The conditions ensure that the double complex computing $\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K})$ has only finitely many nonzero terms in each (total) degree. N.B.: 'L bounded above' means $\mathrm{L}^n = 0$ for $n \gg 0$. 'L bounded' means additionally $\mathrm{L}^n = 0$ for $n \ll 0$.)

Proof. — If A is a finitely presented F-module and $\mathscr{F}$ is any sheaf, we have

$$\mathscr{H}\!om(\underline{\mathrm{A}}, \mathscr{F})_x = \operatorname{colim} \mathrm{Hom}(\mathrm{A}, \mathscr{F}(\mathrm{U})) = \mathrm{Hom}(\mathrm{A}, \mathscr{F}_x).$$

Let $\mathrm{K} \to \mathrm{I}$ be a quasi-isomorphism into a K-injective complex. We have a map $\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K}) \to \mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{I}) = \mathrm{R}\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K})$. Since the terms of both complexes are finite direct sums of sheaves of the form $\mathscr{H}\!om(\underline{\mathrm{L}}^n, ?^m)$ for ? = K or I, the map on stalks is given by

$$\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K})_x = \mathrm{Hom}(\mathrm{L}, \mathrm{K}_x) \to \mathrm{Hom}(\mathrm{L}, \mathrm{I}_x) = \mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{I})_x$$

since finite projective modules are finitely presented (`058R`). As L is a bounded above complex of projective F-modules, $\mathrm{Hom}(\mathrm{L}, -)$ computes $\mathrm{R}\,\mathrm{Hom}(\mathrm{L}, -)$. As $\mathrm{K}_x \to \mathrm{I}_x$ is a quasi-isomorphism, $\mathrm{R\,Hom}(\mathrm{L}, \mathrm{K}_x) = \mathrm{Hom}(\mathrm{L}, \mathrm{K}_x) \to \mathrm{Hom}(\mathrm{L}, \mathrm{I}_x) = \mathrm{R\,Hom}(\mathrm{L}, \mathrm{I}_x)$ is a quasi-isomorphism. □

6.3.13. I can't make this proof work because $\mathrm{Y} \to \mathrm{Y}_i$ is in general not surjective. So, as explained below, I can't make a reduction suggested by the proof work and can't justify the conclusion of the lemma. Below, I provide justification for some other steps in the proof (descending a stratification, defining the morphism $f^*\underline{\mathrm{L}} \to f^*\mathrm{K}$) over some Y_i), as they may be of independent interest.

We may assume X is affine by `09Y4`.[48] Constructibility is stable under pullback by an arbitrary morphism (`054I`). Given our cover $f = \coprod_i f_i$ in $\mathrm{X}_{\text{proét}}$, there are finitely many i and finitely many open affines $\mathrm{X}_{ij} \subset \mathrm{X}_i$ so that the map $\coprod_{ij} \mathrm{X}_{ij} \to \mathrm{X}$ defined by f is a cover in $\mathrm{X}_{\text{proét}}$. This cover can be refined by a pro-étale morphism in $\mathrm{X}^{\text{aff}}_{\text{proét}}$ by (2.3.4), which is how we find our pro-étale affine $f : \mathrm{Y} = \lim_i \mathrm{Y}_i \to \mathrm{X}$ covering X with $f^*\mathrm{K}$ constructible. Let $g_i : \mathrm{Y} \to \mathrm{Y}_i$ denote the projection.

To descend a stratification: taking inspiration from `09Y4` & `09Y2`, let $\coprod \mathrm{Y}_j \to \mathrm{Y}$ be a finite partition of Y into locally closed constructible subsets $\mathrm{Y}_j = \mathrm{U}_j \cap \mathrm{V}_j^c$ with $\mathrm{U}_j, \mathrm{V}_j \subset \mathrm{Y}$ quasi-compact opens. The complement of a quasi-compact open in an affine scheme can be given a closed subscheme structure so that the corresponding closed immersion is of finite presentation, and conversely the complement of a closed subscheme of finite presentation in an affine scheme is a quasi-compact open.[49] Therefore the $\mathrm{U}_j^c, \mathrm{V}_j^c$ can be given closed subscheme structures of finite presentation over Y. There is an i so that for each j there are closed immersions of finite presentation $\mathrm{U}_j'^c, \mathrm{V}_j'^c \hookrightarrow \mathrm{Y}_i$ whose complements in Y_i are quasi-compact opens $\mathrm{U}_j', \mathrm{V}_j'$ and so that $\mathrm{U}_j^c = g^{-1}(\mathrm{U}_j'^c)$ and $\mathrm{V}_j^c = g^{-1}(\mathrm{V}_j'^c)$ (`01ZM` & `01ZP`).

Let S be the finite set of closed subsets of Y_i consisting of $\emptyset, \mathrm{Y}, \mathrm{U}_j'^c, \mathrm{V}_j'^c$, and finite intersections of such, so $\mathrm{S} = \{\mathrm{Z}_\lambda\}_{\lambda \in \Lambda}$ for some finite set Λ. For each $\lambda \in \Lambda$, Z_λ is a constructible closed subset of Y_i (`005H`), and for all $\lambda, \lambda' \in \Lambda$, $\mathrm{Z}_\lambda \cap \mathrm{Z}_{\lambda'} \in \mathrm{S}$. Define a partial order on S by inclusion, and let Λ inherit this partial order. Set $\mathrm{Y}_\lambda := \mathrm{Z}_\lambda \setminus \bigcup_{\lambda' < \lambda} \mathrm{Z}_{\lambda'}$, so Y_λ is locally closed constructible. Suppose $g^{-1}(\mathrm{Y}_\lambda) \cap \mathrm{U}_j \neq 0$

[48]Note that the X_i that appear in the statement of `09Y4` are moreover locally closed constructible.
[49]Follows from `01TV`, since an open is quasi-compact iff it can be covered by finitely many $\mathrm{D}(f)$.

for some λ and some j. Then $Z_\lambda \cap U_j'^c$ is a proper closed subset of Z_λ, so it equals $Z_{\lambda'}$ for some $\lambda' < \lambda$, and so we see $Y_\lambda \cap U_j'^c = \emptyset$ and $Y_\lambda \subset U_j'$. Suppose $g^{-1}(Y_\lambda) \cap V_j^c \neq 0$ for some j. Then $Y_\lambda \cap V_j'^c \neq \emptyset$, so $Z_\lambda \cap V_j'^c$ is not a proper subset of Z_λ; i.e. $Y_\lambda \subset Z_\lambda \subset V_j'^c$. Putting this together, we find that for each j, Y_j is a finite union of the locally closed constructible $g^{-1}(Y_\lambda)$. Therefore $\coprod_{\lambda\in\Lambda} Y_\lambda \to Y_i$ is a finite stratification of Y_i into locally closed constructible subsets so that $\coprod_{\lambda\in\Lambda} g^{-1}Y_\lambda \to Y$ refines the given partition $\coprod_j Y_j \to Y$.

By (6.3.10), we may replace X by Y_i since $f_i : Y_i \to X$ is surjective étale (f factors through f_i). However, the pro-étale map $g_i : Y \to Y_i$ needn't be surjective, so replacing X by Y_i isn't a valid reduction. Here's an example of what can happen: let $X = \mathbf{A}^1$ and $Y = \operatorname{Spec} \mathcal{O}_{\mathbf{A}^1,0} \coprod \mathbf{A}^1 \smallsetminus \{0\}$, then $Y = \lim Y_i$ where $Y_i = U_i \coprod \mathbf{A}^1 \smallsetminus \{0\}$ for U_i a fundamental basis of affine neighborhoods of $0 \in \mathbf{A}^1$.

Once reduced to the case that $f^*K = \underline{L}$ for L in $D_{\mathrm{perf}}(F)$; replacing L by a bounded complex of finite projective F-modules, note that for any scheme S and any ? in $D(S_{\text{ét}}, F)$, $\mathcal{H}om(\underline{L}, ?)$ computes $R\mathcal{H}om(\underline{L}, ?)$ by the lemma in the note to (6.3.12). The natural map[50] $g^*\mathcal{H}om(\underline{L}, K) \to \mathcal{H}om(\underline{L}, g^*K)$ is an isomorphism for any morphism of schemes $g : S \to X$, as can be seen on stalks: if s is a geometric point of S,

$$\begin{aligned}(g^*\mathcal{H}om(\underline{L}, K))_s &= \mathcal{H}om(\underline{L}, K)_{g(s)} = \operatorname{Hom}(L, K_{g(s)})\\ &= \mathcal{H}om(L, (g^*K)_s) = \mathcal{H}om(\underline{L}, g^*K)_s;\end{aligned}$$

here we use that each term of the total complex describing $\mathcal{H}om(\underline{L}, K)$ is a finite direct sum of sheaves of the type $\mathcal{H}om(\underline{P}, K^n)$, where P is a finite projective F-module (see

[50] $\operatorname{Hom}(g^*\mathcal{H}om(A,B), \mathcal{H}om(g^*A, g^*B)) = \operatorname{Hom}(g^*(\mathcal{H}om(A,B)\otimes A), g^*B)$ and $\operatorname{Hom}(\mathcal{H}om(A,B)\otimes A, B) = \operatorname{Hom}(\mathcal{H}om(A,B), \mathcal{H}om(A,B))$.

the proof of the lemma in the note to (6.3.12)). Passage to the limit (0EZM) finds

$$\begin{aligned}\mathrm{Hom}_{D(Y_{\text{ét}},F)}(\underline{L}, f^*K) &= H^0(Y_{\text{ét}}, R\mathscr{H}om(\underline{L}, f^*K))\\ &= H^0(Y_{\text{ét}}, f^*\mathscr{H}om(\underline{L}, K))\\ &= \operatorname{colim}_i H^0((Y_i)_{\text{ét}}, f_i^*\mathscr{H}om(\underline{L}, K))\\ &= \operatorname{colim}_i H^0((Y_i)_{\text{ét}}, \mathscr{H}om(\underline{L}, f_i^*K))\\ &= \operatorname{colim}_i H^0((Y_i)_{\text{ét}}, R\mathscr{H}om(\underline{L}, f_i^*K))\\ &= \operatorname{colim}_i \mathrm{Hom}_{D((Y_i)_{\text{ét}},F)}(\underline{L}, f_i^*K).\end{aligned}$$

Therefore the morphism $\underline{L} \to f^*K$ is the pullback of a morphism $\underline{L} \to f_i^*K$ for some i. One would like to check on stalks that this morphism is an isomorphism, which could be done after pulling back via a surjective map. Although this morphism becomes an isomorphism upon pulling back via $Y \to Y_i$, this morphism is not surjective, so it's not enough to conclude.

6.3.14.

1. By the lemma in the note to (6.3.12), representing M by a bounded complex of finite projective F-modules, $\mathscr{H}om(\underline{M}, -)$ computes $R\mathscr{H}om(\underline{M}, -)$ and $\mathscr{H}om(\underline{M}, -)_x = \mathrm{Hom}(M, -_x)$ for any geometric point x of X. Then the result follows as direct sums in $D(X_{\text{ét}}, F)$ are computed termwise (07D9), $\mathrm{Hom}(M, -)$ commutes with filtered colimits (058R & 0G8P), and taking stalks commutes with colimits, so for K_α in $D(X_{\text{ét}}, F)$,

$$\begin{aligned}R\mathscr{H}om(\underline{M}, \oplus_\alpha K_\alpha)_x &= \mathrm{Hom}(M, \oplus_\alpha (K_\alpha)_x)\\ &= \bigoplus_\alpha \mathrm{Hom}(M, (K_\alpha)_x)\\ &= \Big(\bigoplus_\alpha R\mathscr{H}om(\underline{M}, K_\alpha)\Big)_x.\end{aligned}$$

2. Suppose M is in $\mathscr{C}_{\mathrm{U}}$ and the K_α are in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{ét}}, \mathrm{F})$; then

$$\begin{aligned} \mathrm{R}\mathscr{H}om(j_!\mathrm{M}, \oplus_\alpha \mathrm{K}_\alpha) &= j_*\mathrm{R}\mathscr{H}om(\mathrm{M}, j^*(\oplus_\alpha \mathrm{K}_\alpha)) \\ &= j_*\mathrm{R}\mathscr{H}om(\mathrm{M}, \oplus_\alpha j^*\mathrm{K}_\alpha) \\ &= \oplus_\alpha j_*\mathrm{R}\mathscr{H}om(\mathrm{M}, j^*\mathrm{K}_\alpha) \\ &= \oplus_\alpha \mathrm{R}\mathscr{H}om(j_!\mathrm{M}, \mathrm{K}_\alpha); \end{aligned}$$

here the first and last equalities are by [**SGAA**, IV 12.3 d)] since j^* of a K-injective complex is K-injective (`08BJ`), and the third follows as j_* commutes with direct sums since the $\mathrm{R}^p j_*$ commute with filtered colimits of uniformly bounded below complexes (`0GIV`),[51] and we have the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{R}^p j_* \mathscr{H}^q \mathrm{K}_\alpha \Rightarrow \mathrm{R}^{p+q} j_* \mathrm{K}_\alpha. \tag{$*$}$$

3. Follows since if $j : \mathrm{U} \to \mathrm{X}$ is étale, j^* commutes with direct sums and $j^*\mathrm{R}\mathscr{H}om(\mathrm{A}, \mathrm{B}) = \mathrm{R}\mathscr{H}om(j^*\mathrm{A}, j^*\mathrm{B})$ [**SGAA**, IV 12.3 b)].
4. When U in $\mathrm{X}_{\text{ét}}$ is coherent, $\mathrm{R}\Gamma(\mathrm{U}_{\text{ét}}, -)$ commutes with direct sums in $\mathrm{D}^{\geq 0}(\mathrm{U}_{\text{ét}}, \mathrm{F})$ since $\mathrm{H}^q(\mathrm{U}_{\text{ét}}, -)$ does on sheaves of F-modules (`03Q5`) and we have the spectral sequence ($*$).

As

$$\mathrm{R}\,\mathrm{Hom}(\mathrm{M}|_{\mathrm{U}}, -) = \mathrm{R}\Gamma(\mathrm{X}_{\text{ét}}, j_*\mathrm{R}\mathscr{H}om(\mathrm{M}|_{\mathrm{U}}, -)) = \mathrm{R}\Gamma(\mathrm{X}_{\text{ét}}, \mathrm{R}\mathscr{H}om(\mathrm{M}, j_*-)),$$

it suffices to show that j_* commutes with direct sums in $\mathrm{D}^{\geq 0}(\mathrm{U}_{\text{ét}}, \mathrm{F})$; we already saw this in (2) when j is coherent. But U in $\mathrm{X}_{\text{ét}}$ is coherent iff the structural arrow $\mathrm{U} \to \mathrm{X}$ is coherent,[52] so we conclude one direction. The other follows since

$$\mathrm{R}\,\mathrm{Hom}(\mathrm{M}|_{\mathrm{U}}, -|_{\mathrm{U}}) = \mathrm{R}\Gamma(\mathrm{U}_{\text{ét}}, \mathrm{R}\mathscr{H}om(\mathrm{M}, -))$$

and the functors $\mathrm{R}\Gamma(\mathrm{U}_{\text{ét}}, -)$ with U coherent in $\mathrm{X}_{\text{ét}}$ commute with direct sums and form a conservative family of functors.

'passing to a cover of X refining g over Y' $\rightsquigarrow$ `04FW`.

---

[51]Also recall `07D9`. This is where the assumption that j is coherent is used.

[52]This is immediate from `01KV`, `03GI`, and the fact that a scheme X is quasi-compact iff $\mathrm{X} \to \operatorname{Spec} \mathbf{Z}$ is.

6.3.16. First note that the description given in `0GLZ` of $\nu^*\mathscr{F}$ for $\mathscr{F}$ in $X_{\text{ét}}$ makes it clear that for $j : U \to X$ in $X_{\text{proét}}$, $j^*\nu^* = \nu^* j^*$, where on the left hand side, j^* denotes restriction, and on the right hand side, it denotes pullback along $U_{\text{ét}} \to X_{\text{ét}}$: given $g : Z \to U$ in $X_{\text{proét}}$,

$$j^*\nu^*\mathscr{F}(Z/U) = \nu^*\mathscr{F}(Z/X) = \Gamma(Z_{\text{ét}}, g^*j^*\mathscr{F}) = \nu^*j^*\mathscr{F}(Z/U).$$

Therefore (using the simple identity)

$$\begin{aligned} R\Gamma(U_{\text{proét}}, R\mathscr{H}om(\nu^*K, \nu^*L)) &= R\operatorname{Hom}(j^*\nu^*K, j^*\nu^*L) = R\operatorname{Hom}(\nu^*j^*K, \nu^*j^*L) \\ &= R\operatorname{Hom}(j^*K, \nu_*\nu^*j^*L) = R\operatorname{Hom}(j^*K, j^*L) \\ &= R\Gamma(U_{\text{ét}}, R\mathscr{H}om(j^*K, j^*L)). \end{aligned} \tag{†}$$

To do this proof without invoking ∞-categories, it amounts to writing down a map and then checking it induces a quasi-isomorphism on pro-étale affine sections, using that everything in sight is bounded below and filtered colimits are exact. As

$$\begin{aligned} &\operatorname{Hom}_{D(X_{\text{proét}})}(\nu^*R\mathscr{H}om(K,L), R\mathscr{H}om(\nu^*K, \nu^*L)) \\ &= \operatorname{Hom}_{D(X_{\text{proét}})}(\nu^*(R\mathscr{H}om(K,L) \otimes_F K), \nu^*L) \\ &= \operatorname{Hom}_{D(X_{\text{ét}})}(R\mathscr{H}om(K,L), \nu_*\nu^*L) \\ &= \operatorname{Hom}_{D(X_{\text{ét}})}(R\mathscr{H}om(K,L), L) \\ &= \operatorname{Hom}_{D(X_{\text{ét}})}(R\mathscr{H}om(K,L), R\mathscr{H}om(K,L)), \end{aligned}$$

we get a map

$$\nu^*R\mathscr{H}om(K,L) \longrightarrow R\mathscr{H}om(\nu^*K, \nu^*L)$$

It will suffice to show that the map is an isomorphism after applying $R\Gamma(U_{\text{proét}}, -)$ for pro-étale affine U, as the $j_!\underline{F}_U$ generate $D(X_{\text{ét}}, F)$. Using `099W` (plus $j^*\nu^* = \nu^*j^*$) and (†), we find the map is

$$R\Gamma(U_{\text{ét}}, j^*R\mathscr{H}om(K,L)) \longrightarrow R\Gamma(U_{\text{ét}}, R\mathscr{H}om(j^*K, j^*L)).$$

By `0EZM`, the map on cohomology is

$$\operatorname{colim}_i \operatorname{Hom}_{D(X_{\text{ét}})}(K, j_{i*}j_i^*L) \longrightarrow \operatorname{Hom}_{D(X_{\text{ét}})}(K, j_*j^*L),$$

and this is the map we must show is an isomorphism. Replacing L by a bounded below complex $I^\bullet$ of injectives, $\mathrm{colim}_i\, j_{i*}j_i^*I^\bullet \to j_*j^*I^\bullet$ is a quasi-isomorphism.[53] Therefore it suffices to show

Lemma. — Let $(F_i^\bullet)$ *denote a directed system of abelian sheaves on* $X_{\text{ét}}$ *with* $F_i^n = 0$ *for* $n < a$ *independent of* i, *let* $F^\bullet$ *denote the pointwise colimit, and let* K *be in* $D_{\mathrm{cons}}(X_{\text{ét}}, F)$. *Then the natural map*

$$\mathrm{colim}_i\, \mathrm{Hom}_{D(X_{\text{ét}})}(K, F_i^\bullet) \to \mathrm{Hom}_{D(X_{\text{ét}})}(K, F^\bullet)$$

is a bijection.

Proof. — Via the spectral sequence (`015J`)

$$E_2^{pq} = H^p(X_{\text{ét}}, H^q R\mathscr{H}om(K, -)) \Rightarrow H^{p+q} R\,\mathrm{Hom}(K, -),$$

it suffices by `03Q5` to show that the natural map

$$\mathrm{colim}_i\, H^0 R\mathscr{H}om(K, F_i^\bullet) \to H^0 R\mathscr{H}om(K, F^\bullet)$$

is an isomorphism. Now we proceed along the lines of Lemma 6.3.14; let $\mathscr{E}_X$ denote the full (triangulated) subcategory of $D^b(X_{\text{ét}}, F)$ spanned by those M for which $H^0 R\mathscr{H}om(M, -)$ commutes with filtered colimits as in the lemma ($a \in \mathbf{Z}$ is arbitrary). Then (1–3) of (6.3.14) continue to hold; for (2) the argument uses the spectral sequence

$$E_2^{pq} = R^p j_* H^q R\mathscr{H}om(M, j^*-) \Rightarrow H^{p+q} R j_* R\mathscr{H}om(M, j^*-).$$

In conclusion, $D_{\mathrm{cons}}(X_{\text{ét}}, F) \subset \mathscr{E}_X$. □

This exercise shows that it's considerably more convenient to be able to have all (homotopy) colimits in the relevant derived ∞-categories. This allows one to completely avoid the use of spectral sequences in the above. A reference for the fact used in Lemma 6.3.16 that $R\,\mathrm{Hom}(K, -)$ commutes with filtered colimits if it commutes with direct sums is Lurie, *Higher Algebra* Proposition 1.4.4.1.

6.4.2. The citation for the Mayer-Vietoris is `0EVY`. In this context, the first condition is satisfied since $F(V_1) \times F(V_2) \rightrightarrows F(W)$ is exact. When U is separated and quasi-compact,

[53]If I is an injective sheaf, j^*I is acyclic for j_* by `09Z1` so $j_*j^*I^\bullet$ computes Rj_*j^*L (`015E`); now use `0GIV`.

the lemma follows by induction on the number of affines needed to cover U: let $(U_i)_{i=1}^n$ be affine opens covering U. Then $U_1 \cup \ldots \cup U_{n-1} \cap U_n$ can be covered by $n-1$ affines. When U is coherent, then $U_1 \cup \ldots \cup U_{n-1} \cap U_n$ is separated (`01L7`), so again induction on n works.

6.4.5. By `08U1`, with $\mathscr{A}$ the Grothendieck abelian category of modules over $(X_{\text{ét}}, F)$, $\mathscr{B}$ the category of F-modules, and $F = \Gamma(X_{\text{ét}}, -)$, we find $R\Gamma(X_{\text{ét}}, -)$ commutes with $R\lim$. As $\tau_{<-n}K = R\lim_m \tau_{\geq -m}\tau_{<-n}K$, $R\Gamma(X_{\text{ét}}, \tau_{\geq -m}\tau_{<-n}K)$ is in $D^{<N_X-n}(F)$, so $R\Gamma(X_{\text{ét}}, \tau_{<-n}K) = R\lim_m R\Gamma(X_{\text{ét}}, \tau_{\geq -m}\tau_{<-n}K)$ is in $D^{\leq N_X-n}(F)$ as countable products are exact in F-mod.

6.4.6. It suffices to show $R\mathscr{H}om(i_*K, -)$ commutes with direct sums since by Lemma 6.4.5 (1), $R\Gamma(X_{\text{ét}}, -)$ commutes with direct sums.

'passing to a suitable cover' $\rightsquigarrow$ `04FW`.

6.4.7. '$i_*\underline{F} \simeq \operatorname{colim} j_*\underline{F}$' $\rightsquigarrow$ `005F`. Of course $?_*\underline{F} = {}^\circ ?_*\underline{F}$ for $? = i, j$ since both are closed immersions.

'all maps $i_*\underline{F} \to j_*\underline{F}$ are constant' $\rightsquigarrow$ both complexes are concentrated in degree zero and maps from $i_*\underline{F}$ are determined by where $1 \in H^0(S, i_*\underline{F}) = F$ goes, but by support considerations it must go to $0 \in H^0(S, j_*\underline{F})$. In conclusion $\operatorname{Hom}(i_*\underline{F}, j_*\underline{F}) = 0$ but $\operatorname{Hom}(i_*\underline{F}, \operatorname{colim} j_*\underline{F})$ contains an isomorphism.

6.4.8. A more direct way to see that $D(X_{\text{ét}}, F)$ is compactly generated is just note that if $K \neq 0$ then $H^pK \neq 0$ for some p, so $(H^pK)(U) \neq 0$ for some U in $X_{\text{ét}}$, defining a nonzero morphism $j_{U!}\underline{F}[-p] \to K$.[54]

'$D_c(X_{\text{ét}}, F)$ is the smallest idempotent complete triangulated subcategory of $D(X_{\text{ét}}, F)$ that contains the coherent objects' $\rightsquigarrow$ let $\mathscr{C}$ denote this latter category; as `09QH`, `07D9`, `05QW` & `09SH` show that $D_c(X_{\text{ét}}, F)$ is idempotent complete and triangulated, we have $\mathscr{C} \subset D_c(X_{\text{ét}}, F)$. Let X be an object of $D(X_{\text{ét}}, F)$. Then $X = \operatorname{hocolim} X_n$ as in `09SN`, and if X is moreover compact, the morphism $X \to \operatorname{hocolim} X_n$ factors through

[54]Let $I^\cdot$ be a K-flat complex representing K. As $(H^pI^\cdot)(U) \neq 0$ implies $H^p(U, K) = H^p(I^\cdot(U)) \neq 0$, we get a nonzero morphism in $\operatorname{Hom}(j_{U!}\underline{F}[-p], K) = \operatorname{Hom}(F, R\Gamma(U, K[p]))$.

$X_n \to \operatorname{hocolim} X_n$ for some n (`094A`), and this morphism in turn factors through $X \to E \to X_n$ for some E in $\langle C\rangle$ where C is a *finite* direct sum of coherent objects; in other words, E is in $\mathscr{E}$. Since the morphisms $X \to E \to X_n \to \operatorname{hocolim} X_n \to X$ composes to give the identity, it gives rise to an idempotent of E, so X is a direct summand of E (`09SH`) and therefore in $\mathscr{E}$ since $\mathscr{E}$ is saturated.

6.5. 'The sheaf $\hat{R}_X := \lim R/\mathfrak{m}^n$ coincides with $\mathscr{F}_R$ for the topological ring R' $\rightsquigarrow$ given U in $X_{\text{proét}}$, sections over U of the constant sheaf $R/\mathfrak{m}^n$ on $X_{\text{proét}}$ clearly include the locally constant functions $U \to R/\mathfrak{m}^n$, where $R/\mathfrak{m}^n$ is topologized discretely. By Lemma 4.2.12, however, this assignment already describes a sheaf, so $R/\mathfrak{m}^n = \mathscr{F}_{R/\mathfrak{m}^n}$. As $\lim_n \operatorname{Maps}_{\text{cont}}(U, R/\mathfrak{m}^n) = \operatorname{Maps}_{\text{cont}}(U, R)$, $\hat{R}$ coincides with $\mathscr{F}_R$.

6.5.1. Recall the definition of '$\mathfrak{m}$-adically complete' in Definition 3.5.2.

'$D_{\text{cons}}(X_{\text{proét}}, \hat{R})$ is a triangulated subcategory of $D(X_{\text{proét}}, \hat{R})$' $\rightsquigarrow$ this is clear for 'derived $\mathfrak{m}$-complete,' so follows for 'constructible' by `099V` and the fact that $D_{\text{cons}}(X_{\text{ét}}, R/\mathfrak{m})$ is triangulated (6.3.9) as $D_{\text{cons}}(X_{\text{proét}}, \hat{R}) \subset D^b(X_{\text{proét}}, \hat{R})$ by Lemma 6.5.3.

To see that ν induces an equivalence $D_{\text{cons}}(X_{\text{proét}}, R/\mathfrak{m}^n) \simeq D_{\text{cons}}(X_{\text{ét}}, R/\mathfrak{m}^n)$: ν^* is fully faithful (`099V`) and essentially surjective (`09B2`).

It will be useful to be able to use the results of `09C0`, although the definition of $D_{\text{cons}}(X_{\text{proét}}, \hat{R})$ differs slightly from the one there.

Via the equivalence $D_{\text{comp}}(X_{\text{proét}}, \hat{R}) \simeq D_{\text{comp}}(X_{\text{proét}}, R)$ of Lemma 3.5.6, the following lemma assures us that K in $D_{\text{cons}}(X_{\text{proét}}, \hat{R})$ can be regarded as a constructible object of $D_{\text{comp}}(X_{\text{proét}}, R)$ in the sense of `09C0`

Lemma. — If K *is in* $D_{\text{cons}}(X_{\text{proét}}, \hat{R})$*,* K_1 *has finite tor-dimension.*

Proof. — Conjugating Lemmas 6.3.3 and 6.2.3 (4), we're reduced to the case that $K_1 = k_!L$, $k : W \hookrightarrow X$, $L = \nu^*L'$ for some L' in $D(W_{\text{ét}}, R/\mathfrak{m})$ locally constant with perfect values. It will suffice by Lemma ✠ below to show L has finite tor-dimension on $W_{\text{proét}}$, so to that end let $\mathscr{F}$ be in $\operatorname{Mod}(X_{\text{proét}}, R/\mathfrak{m})$. Localizing on $W_{\text{ét}}$, we may assume

$L' = \underline{M}$ for M in $D_{\mathrm{perf}}(R/\mathfrak{m})$. If P is a finite projective $R/\mathfrak{m}$-module, then $\underline{P}$ on $W_{\mathrm{pro\acute{e}t}}$ is flat,[55] and the conclusion follows. □

Lemma (✠). — *Given a coherent scheme* X *and* K *in* $D(X_{\mathrm{pro\acute{e}t}})$, *suppose* $\coprod X_i \to X$ *is a partition of* X *into constructible locally closed subsets, and suppose* $K|_{X_i} = 0$ *for all* i. *Then* $K = 0$.

Proof. — We proceed inductively as in the proof of [**SGAA**, IX 2.5], using Lemma 6.1.11. For W as in the proof, $K|_W = 0$, so we find $K \to K_Y$ is an isomorphism, but the inductive hypothesis says $K|_Y = 0$. □

6.5.2. This is a special case of Corollary 6.5.7.

6.5.3. Suppose $K_1 = \nu^* L$ and L has tor amplitude in $[a, b]$. From the distinguished triangle

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n \longrightarrow$$

and the isomorphisms $K \otimes^{\mathbf{L}}_{\hat{R}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = K_1 \otimes^{\mathbf{L}}_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = \nu^*(L \otimes^{\mathbf{L}}_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^n)$, where $\nu : (X_{\mathrm{pro\acute{e}t}}, R/\mathfrak{m}) \to (X_{\mathrm{\acute{e}t}}, R/\mathfrak{m})$ is the morphism of ringed topos, we see that $R/\mathfrak{m}^{n+1}$ is acyclic off $[a, b]$.

6.5.5. The easy part: let K_1, L_1 in $D_{\mathrm{cons}}(X_{\mathrm{\acute{e}t}}, R/\mathfrak{m})$ be defined by $K \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m} = \nu^* K_1$, and likewise for L_1. Then $K \otimes^{\mathbf{L}}_{\hat{R}} L \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m} = (\nu^* K_1) \otimes^{\mathbf{L}}_{R/\mathfrak{m}} (\nu^* L_1) = \nu^*(K_1 \otimes^{\mathbf{L}}_{R/\mathfrak{m}} L_1)$, and $D_{\mathrm{cons}}(X_{\mathrm{\acute{e}t}}, R/\mathfrak{m})$ is closed under tensor products (Lemma 6.3.5).

Now we justify the reduction.

Lemma. — *Let* X *be a coherent scheme and* K *be in* $D_{\mathrm{cons}}(X_{\mathrm{pro\acute{e}t}}, \hat{R})$. *Suppose given a finite partition* $\coprod X_j \to X$ *of* X *into locally closed constructible subsets so that the restriction of* $L \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}$ *to* X_i *is étale locally constant (as* K_1 *comes from a constructible étale* $R/\mathfrak{m}$*-complex, such a partition always exists). Then there is a finite filtration of* K *with graded pieces of the form* $i_! L$, *with* $i : X_j \hookrightarrow X$ *ranging through the given partition, and* $L \in D_{\mathrm{cons}}(Y, \hat{R})$ *with* $L \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}$ *étale locally constant with perfect values.*

[55] Write $(R/\mathfrak{m})^{\oplus m} = P \oplus Q$; then if f is an injection in $\mathrm{Mod}(X_{\mathrm{\acute{e}t}}, R/\mathfrak{m})$, $0 = \ker(f \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})^{\oplus m}) = \ker(f \otimes_{R/\mathfrak{m}} P) \oplus \ker(f \otimes_{R/\mathfrak{m}} Q)$.

(‘Filtration’ in this sense means that there are distinguished triangles (G_n, C_n, C_{n+1}), with $C_0 = K$, G_n of the form $i_!L$, and $C_m = 0$ for $m \gg 0$.)

Proof. — We follow [**SGAA**, IX 2.5] according to the given partition of X, using Lemma 6.1.11. If $Y \hookrightarrow X$ is locally closed, then $K|_Y = R\lim K|_Y \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n$ by Corollary 6.1.5, and $K|_Y \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m} = K_1|_Y$ (this uses 1) of Lemma 6.5.8), so $K|_Y$ is constructible. (Keep in mind `09YJ` and the note to 6.2.) □

Now let K and L be as in the statement of Lemma 6.5.5. There is a finite partition $\coprod X_j \to X$ of X into locally closed constructible subsets $X_j \hookrightarrow X$ so that $K_1|_{X_j}$ and $L_1|_{X_j}$ are étale locally constant on X_j with perfect values. Applying the above lemma for the partition $\coprod_i X_j$, we find finite filtrations of K and L with graded pieces of the form $i_!M$, with M_1 étale locally constant with perfect values and i the immersion of an X_j. It suffices to verify the conditions of the lemma for $i_!M \otimes^{\mathbf{L}}_{\hat{R}} i'_!M'$. When $i \neq i'$, $i_!M \otimes^{\mathbf{L}}_{\hat{R}} i'_!M'$ is zero. To see this, by Definition 6.2.2, $i_!M$ is supported on some X_j and $i'_!M'$ on some $X_{j'}$ and by assumption $j \neq j'$. Therefore the restrictions of $i_!M \otimes^{\mathbf{L}}_{\hat{R}} i'_!M'$ to X_j and to X_j^c are null, so $i_!M \otimes^{\mathbf{L}}_{\hat{R}} i'_!M' = 0$ by Lemma ✠ in the note to Definition 6.5.1.

It remains to treat the case $i = i'$; i.e. show that the conclusions of Lemma 6.5.5 hold for $i_!M \otimes^{\mathbf{L}}_{\hat{R}} i_!N$ with i the immersion of some X_j, M, N in $D_{\mathrm{cons}}((X_j)_{\text{proét}}, \hat{R})$, and M_1, N_1 étale locally constant with perfect values on X_j.

Let $X_j = U_j \cap V_j^c$ for U_j, V_j quasi-compact opens in X.[56] As the question is local, covering U_j by finitely many affine opens, we may assume X is affine and $i : Z \hookrightarrow X$ is closed of finite presentation. Next, we replace X by a w-contractible, ind-étale, faithfully flat cover by Lemma 2.4.9. Then $i : Z \hookrightarrow X$ being closed of finite presentation implies that Z is also w-contractible by Lemma 2.4.10. By `09C3`, there is a finite clopen decomposition of $Z = \coprod Z_k$ so that $M|_{Z_k}$ can be represented by a complex of the form

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^b})^\wedge$$

where M^a through M^b are finite projective R-modules (however, the maps $(\underline{M^a})^\wedge \to (\underline{M^{a+1}})^\wedge$ etc. needn’t be constant), and likewise for N. Finding a common refinement of the two clopen partitions of Z, replacing X by a quasi-compact open of X that meets Z in

[56]Recall the note to 6.3.

a single stratum, and then filtering the restrictions of M and N along stupid truncations, we reduce by Lemma 6.2.3 2) to the case that $i_!\mathrm{M} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_!\mathrm{N}$ is of the form $i_*\underline{\hat{\mathrm{M}}} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*\underline{\hat{\mathrm{N}}}$, for M, N in $\mathrm{D}_{\mathrm{perf}}(\mathrm{R})$ (actually for M, N finite projective R-modules), where $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is the closed immersion of a constructible closed subset of a coherent scheme (see the note to Lemma 6.1.15 for the matter of coefficients).

Both M and N can be realized as bounded complexes of finite projective R-modules. Given such a complex $\mathrm{P}^\bullet$ with P^m finite projective and zero for almost all m, we can find Q^m so that $\mathrm{P}^m \oplus \mathrm{Q}^m$ is finite free (00NX) and zero for almost all m. This creates a complex $\mathrm{Q}^\bullet$ with zero differentials so that $\mathrm{P}^\bullet \oplus \mathrm{Q}^\bullet$ is isomorphic to a finite complex of finite free R-modules. As direct sums in D(R) are taken termwise (07D9), we've found M' so that $\mathrm{M} \oplus \mathrm{M}'$ can be realized by a finite complex of finite free R-modules, and likewise for N. As

$$i_*(\underline{\widehat{\mathrm{M} \oplus \mathrm{M}'}}) \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*(\underline{\widehat{\mathrm{N} \oplus \mathrm{N}'}}) = i_*\underline{\hat{\mathrm{M}}} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*\underline{\hat{\mathrm{N}}} \oplus i_*\underline{\hat{\mathrm{M}}'} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*\underline{\hat{\mathrm{N}}} \oplus i_*\underline{\hat{\mathrm{M}}} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*\underline{\hat{\mathrm{N}}'} \oplus i_*\underline{\hat{\mathrm{M}}'} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*\underline{\hat{\mathrm{N}}'},$$

it suffices to show that $i_*(\underline{\widehat{\mathrm{M} \oplus \mathrm{M}'}}) \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*(\underline{\widehat{\mathrm{N} \oplus \mathrm{N}'}})$ is derived $\mathfrak{m}$-complete. Dévissage along stupid truncations and then finite filtrations reduces to the case $i_*\hat{\mathrm{R}} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} i_*\hat{\mathrm{R}}$.

Intermezzo: 09BS & 09C0.

09BY. The limit (which computes $\mathrm{R}\lim$ by Proposition 3.1.10) is nonzero since it surjects onto $\mathscr{G}$ by repleteness.

09BZ. The decomposition $\mathrm{X} = \coprod \mathrm{U}_i$ is by 098F and the fact that $\mathscr{F}/\mathrm{I}\mathscr{F}$ is locally constant.

To find the sections s_{ij}: we can find sections $\bar{s}_{ij} \in \mathscr{F}/\mathrm{I}\mathscr{F}(\mathrm{X})$ restricting to m_{ij} since $\mathscr{F}/\mathrm{I}\mathscr{F}(\mathrm{X}) = \prod \mathscr{F}/\mathrm{I}\mathscr{F}(\mathrm{U}_i)$. These lift to global sections of $\mathscr{F}$ since X is weakly contractible (090K).

In the notation of §6.5, $(\underline{\Lambda}^\wedge)^{\oplus t}$ is just $\hat{\mathrm{R}}^{\oplus t}$.

09C3. The sheaf $\mathscr{F}/\mathrm{I}\mathscr{F}$ coincides with $\mathscr{H}^b(\mathrm{K} \otimes^{\mathbf{L}}_{\Lambda} \Lambda/\mathrm{I})$, as can be seen from the hypertor spectral sequence (061Z).

Quick proof of `093S` (3): localizing, we may assume $\varphi : \underline{M} \to \underline{N}$ is a map of constant sheaves with $\underline{M}$ of finite type; i.e. M a finite Λ-module (`093N`). This map is determined by adjunction by a map $\underline{M}^p \to \underline{N}$, where $\underline{M}^p$ is the constant presheaf associated to M. This map, in turn, is determined by where the finitely many global sections corresponding to generators of M are sent. Localizing again, we can assume they are all sent to global sections of $\underline{N}^p$, and we've found our map of Λ-modules $M \to N$.

After applying `093S`, we have a covering $(U_i \to X)$ in $X_{\text{proét}}$ and Λ/I-modules $\overline{M}_i$ so that over U_i, $\mathcal{F}/I\mathcal{F}$ restricts to $\underline{\overline{M}_i}$ and $\rho|_{U_i}$ mod I is of the form $\underline{\alpha}_i$ for some $\alpha_i : (\Lambda/I)^{\oplus t} \to M_i$. We can then refine this covering by some $\coprod_{i=1}^n U_i \to X$ with U_i Zariski clopen in X (`098F`), and the same things hold over each U_i as before.

Let $(\Lambda/I)^{\oplus t} \underset{s}{\overset{\alpha_i}{\rightleftarrows}} \overline{M}_i$ be such that $\alpha_i s = \mathrm{id}$. Then $\overline{p}_i := s\alpha_i$ makes the commutative diagram

$$\begin{tikzcd} & & \overline{M}_i \\ (\Lambda/I)^{\oplus t} \arrow[r, "\alpha_i"] & \overline{M}_i \arrow[ur, "\mathrm{id}"] \arrow[r, "s"] & (\Lambda/I)^{\oplus t}, \arrow[u, "\alpha_i"'] \end{tikzcd}$$

from which one checks $\overline{p}_i^2 = \overline{p}_i$ and that the image of $\overline{p}_i$ maps isomorphically onto $\overline{M}_i$.

Now let's build a tower of projectors. Suppose we have the projector $p_{i,n} : (\Lambda/I^n)^{\oplus t} \to (\Lambda/I^n)^{\oplus t}$, then we let e be any map making a commutative square

$$\begin{tikzcd} (\Lambda/I^{n+1})^{\oplus t} \arrow[r, "e"] \arrow[d] & (\Lambda/I^{n+1})^{\oplus t} \arrow[d] \\ (\Lambda/I^n)^{\oplus t} \arrow[r, "p_{i,n}"] & (\Lambda/I^n)^{\oplus t}. \end{tikzcd}$$

Then e^2 followed by the projection coincides with the projection followed by $p_{i,n}$ since $p_{i,n}$ is idempotent, so $x := e^2 - e$ belongs to $(I^n/I^{n+1})^{\oplus t}$, and $x \in \mathrm{End}((\Lambda/I^{n+1})^{\oplus t}) =: A$ is nilpotent. By `05BU`, there exists an idempotent e' of the form $e' = e + x(\sum a_{ij} e^i x^j)$ with $a_{ij} \in A$. In other words, the square above still commutes after replacing e with e'. In this way we obtain projectors $p_{i,n}$ defining an idempotent $p_i := (\Lambda^\wedge)^{\oplus t} \to (\Lambda^\wedge)^{\oplus t}$.

Letting $\mathrm{M}_i := \operatorname{im} p_i$, we have that M_i is a direct summand of $(\Lambda^\wedge)^{\oplus t}$, so $\mathrm{M}_i \cap \mathrm{I}(\Lambda^\wedge)^{\oplus t} = \mathrm{IM}_i$, and the composition $(\Lambda^\wedge)^{\oplus t} \xrightarrow{p_i} \mathrm{M}_i \hookrightarrow (\Lambda^\wedge)^{\oplus t} \to (\Lambda/\mathrm{I})^{\oplus t}$ coincides with $(\Lambda^\wedge)^{\oplus t} \xrightarrow{p_i} \mathrm{M}_i \to \mathrm{M}_i/\mathrm{IM}_i \hookrightarrow (\Lambda/\mathrm{I})^{\oplus t}$. As p_i followed by the projection $(\Lambda^\wedge)^{\oplus t} \to (\Lambda/\mathrm{I})^{\oplus t}$ coincides with the projection followed by $\overline{p}_i =: p_{i,1}$, which has image $\overline{\mathrm{M}}_i$, we find $\mathrm{M}_i/\mathrm{IM}_i = \overline{\mathrm{M}}_i$. As M_i is a finite $\Lambda^\wedge$-module, it is automatically I-adically complete (`00MA`). As a direct summand of a finite free $\Lambda^\wedge$-module, M_i is a finite projective $\Lambda^\wedge$ module. It follows that $\underline{\mathrm{M}_i}^\wedge = \lim_n \underline{\mathrm{M}}_i \otimes_\Lambda \Lambda/\mathrm{I}^n$ computes $\mathrm{R}\lim \underline{\mathrm{M}}_i \otimes^{\mathbf{L}}_\Lambda \Lambda/\mathrm{I}^n$ by Proposition 3.1.10 and Leray's acyclicity lemma (`015E`); in particular, $\underline{\mathrm{M}_i}^\wedge$ is derived complete. Lemma 3.5.5 then gives that we recover $\underline{\mathrm{M}_i/\mathrm{I}^n\mathrm{M}_i} = \underline{\mathrm{M}}_i \otimes^{\mathbf{L}}_\Lambda \Lambda/\mathrm{I}^n$ from $\underline{\mathrm{M}_i}^\wedge \otimes^{\mathbf{L}}_{\Lambda^\wedge} \Lambda^\wedge/\mathrm{I}^n\Lambda^\wedge = \underline{\mathrm{M}_i}^\wedge \otimes_\Lambda \Lambda/\mathrm{I}^n = \underline{\mathrm{M}_i}^\wedge/\mathrm{I}^n\underline{\mathrm{M}_i}^\wedge$. In particular, $\underline{\overline{\mathrm{M}}_i} = \underline{\mathrm{M}_i}^\wedge/\mathrm{I}^n\underline{\mathrm{M}_i}^\wedge$.

In the proof of the case $b > a$, $\Gamma(\mathrm{X}_{\text{proét}}, -)$ is an exact functor as X is weakly contractible. Of course $(\Lambda/\mathrm{I})^{\oplus t}[-b] = (\underline{\Lambda}^\wedge)^{\oplus t}[-b] \otimes^{\mathbf{L}}_\Lambda \Lambda/\mathrm{I}$ has tor-amplitude $[b, b]$ and surjects onto $\mathscr{H}^b(\mathrm{K} \otimes^{\mathbf{L}}_\Lambda \Lambda/\mathrm{I}) = \mathscr{F}/\mathrm{I}\mathscr{F}$, so L is in $\mathrm{D}^{<b}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \Lambda)$. Since the finite type, locally constant sheaves form a weak Serre subcategory (`093U`), L satisfies the conditions of the inductive hypothesis.

The morphism $\mathrm{L} \to (\underline{\Lambda}^\wedge)^{\oplus t}[1-b]$ gives a map on $\mathscr{H}^{b-1}$ which determines a map $\upsilon : \mathscr{H}^{b-1}\mathrm{L} \to (\underline{\Lambda}^\wedge)^{\oplus t}$ and hence a map $(\underline{\mathrm{M}^{b-1}})^\wedge \to \mathscr{H}^{b-1}\mathrm{L} \xrightarrow{\upsilon} (\underline{\Lambda}^\wedge)^{\oplus t}$. Letting L' denote the resulting complex

$$(\underline{\mathrm{M}^a})^\wedge \to \dots \to (\underline{\mathrm{M}^{b-1}})^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t},$$

we have a morphism $\varrho : \mathrm{K} \to \mathrm{L} \to \mathrm{L}'$ in $\mathrm{D}(\mathrm{X}_{\text{proét}}, \Lambda)$. We have that $\mathrm{K} \to \mathrm{L}$ induces an isomorphism on cohomology away from degrees $b-1$ and b, and we have the exact sequence

$$0 \to \mathscr{H}^{b-1}\mathrm{K} \to \mathscr{H}^{b-1}\mathrm{L} \xrightarrow{\upsilon} (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathscr{H}^b\mathrm{K} \to 0.$$

As $\mathscr{H}^{b-1}\mathrm{L}' = \ker(\upsilon)$, $\mathscr{H}^{b-1}\varrho$ is an isomorphism. As $\mathscr{H}^b\mathrm{K} = \operatorname{coker} \upsilon = \mathscr{H}^b\mathrm{L}'$, $\mathscr{H}^b\varrho$ is also an isomorphism, so ϱ is an isomorphism in $\mathrm{D}(\mathrm{X}_{\text{proét}}, \Lambda)$.

6.5.6. Let K_n be defined by $\nu^*\mathrm{K}_n := \mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n$, and recall that K_n is in $\mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$ by Lemma 6.3.12. Suppose we have the case when X is strictly henselian local, and suppose $(\mathrm{K}_n)_x = \mathrm{L}$ for L in $\mathrm{D}(\mathrm{R}/\mathfrak{m}^n)$. We have

$$\begin{aligned}\mathrm{R}\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K}_n)_x &= \mathrm{R}\,\mathrm{Hom}(\mathrm{L}, (\mathrm{K}_n)_x), \quad \text{and}\\ \mathrm{H}^0(\mathrm{R}\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K}_n)_x) &= \mathrm{colim}_{\mathrm{U}}\, \mathrm{H}^0(\mathrm{U}_{\text{ét}}, \mathrm{R}\mathscr{H}\!om(\underline{\mathrm{L}}, \mathrm{K}_n))\\ &= \mathrm{colim}_{\mathrm{U}}\, \mathrm{Hom}_{\mathrm{D}(\mathrm{U}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)}(\underline{\mathrm{L}}|_{\mathrm{U}}, \mathrm{K}_n|_{\mathrm{U}}),\end{aligned}$$

where the colimit is over étale neighborhoods U of a geometric point x, and the equality $\mathrm{H}^0\mathrm{J}_x = \mathrm{colim}_{\mathrm{U}}\, \mathrm{H}^0(\mathrm{U}_{\text{ét}}, \mathrm{J})$ for any J in $\mathrm{D}(\mathrm{X}_{\text{ét}})$ comes from replacing J by a K-injective complex $\mathrm{I}^{\bullet}$, writing $\mathrm{I}^{\bullet}_x$ as the pointwise colimit $\mathrm{colim}_{\mathrm{U}}\, \mathrm{I}^{\bullet}(\mathrm{U})$, and using that filtered colimits are exact (`00DB`). So we get a morphism $\underline{\mathrm{L}}|_{\mathrm{U}} \to \mathrm{K}_n|_{\mathrm{U}}$ over some étale neighborhood U of x in $\mathrm{X}_{\text{ét}}$ which we can take to be affine. The cone C is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{U}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$, so the points $u \in \mathrm{U}$ where $\mathrm{C}_{\overline{u}} = 0$[57] form a constructible subset stable under generalization; i.e. an open subset of U (`0903`).

In the strictly henselian local case, the proof is clear; using the observations in the note to (6.5.3), and the full faithfulness of the pullback $\mathrm{D}(\mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$, we see that the morphism $\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n[-1] \to \mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathrm{K}_1 \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ has source and target in the essential image of the pullback, hence is the pullback of a morphism in $\mathrm{D}(\mathrm{R}/\mathfrak{m}^{n+1})$, so the pullback of the cone on this morphism is isomorphic to $\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^{n+1}$.

6.5.7.

To check that pullback sends $\mathrm{D}_{\mathrm{perf}}(\mathrm{R})$ *to* $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$*:* let $f : (\tilde{\mathrm{X}}_{\text{proét}}, \hat{\mathrm{R}}) \to (*, \mathrm{R})$ denote the morphism of ringed topos to the final topos ringed by R, and let $f_n : (\tilde{\mathrm{X}}_{\text{proét}}, \mathrm{R}/\mathfrak{m}^n) \to (*, \mathrm{R}/\mathfrak{m}^n)$ denote its reduction modulo $\mathfrak{m}^n$.[58] First note that $f^*(\mathrm{R}/\mathfrak{m}^n) = \mathrm{R}/\mathfrak{m}^n \otimes_{\mathrm{R}} \hat{\mathrm{R}} = \mathrm{R}^\wedge/\mathfrak{m}^n\mathrm{R}^\wedge = \mathrm{R}/\mathfrak{m}^n$, where the last isomorphism is by `093M`. For any K in $\mathrm{D}_{\mathrm{perf}}(\mathrm{R})$, therefore,

$$f_n^*(\mathrm{K} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n) = f^{-1}\mathrm{K} \otimes^{\mathbf{L}}_{\mathrm{R}} f^{-1}(\mathrm{R}/\mathfrak{m}^n) = f^*\mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n \tag{$*$}$$

by `07A4`. Therefore, $f^*\mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$ is ν^* of the pullback of $\mathrm{K} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}$ to $\mathrm{X}_{\text{ét}}$. As $\mathrm{K} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}$ is a perfect complex of $\mathrm{R}/\mathfrak{m}$-modules, its pullback to $\mathrm{X}_{\text{ét}}$ is constructible.

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[57] For some geometric point $\overline{u}$ centered on u.
[58] Continuing to denote $\underline{\mathrm{R}}, \underline{\mathfrak{m}^n}$ by $\mathrm{R}, \mathfrak{m}^n$, etc.

It remains to check that $f^*\mathrm{K}$ is $\mathfrak{m}$-adically complete. Given an R-module P, let $\underline{\hat{\mathrm{P}}} := \lim_n \underline{\mathrm{P}} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n = \lim_n f_n^*(\mathrm{P} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n)$. (When $\mathrm{P} = \mathrm{R}$, we write $\hat{\mathrm{R}}$ for $\underline{\hat{\mathrm{R}}}$.)

Lemma. — Let $g : \mathrm{A} \twoheadrightarrow \mathrm{B}$ *be a surjection of* R*-modules. Then* $\hat{g} : \underline{\hat{\mathrm{A}}} \to \underline{\hat{\mathrm{B}}}$ *is epimorphic.*

Proof. — As $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n \to \mathrm{B} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n$ is surjective for each n, it will suffice by Lemma 3.1.8 and the exactness of f_n^* to show that

$$\mathrm{A} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^{n+1} \longrightarrow \mathrm{A} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n \times_{\mathrm{B}\otimes_{\mathrm{R}}\mathrm{R}/\mathfrak{m}^n} \mathrm{B} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^{n+1}$$

is also surjective. Let $a_n \in \mathrm{A}/\mathfrak{m}^n\mathrm{A}$ and $b_{n+1} \in \mathrm{B}/\mathfrak{m}^{n+1}\mathrm{B}$ be chosen so that they go to the same element $b_n \in \mathrm{B}/\mathfrak{m}^n\mathrm{B}$. The choice of a_n gives rise to an equivalence class $a + \mathfrak{m}^n\mathrm{A} \subset \mathrm{A}$. As the image of $\mathfrak{m}^n\mathrm{A}$ in B is $\mathfrak{m}^n\mathrm{B}$, $g(a + \mathfrak{m}^n\mathrm{A}) = g(a) + \mathfrak{m}^n\mathrm{B}$. Any representative b of b_{n+1} in B lies in $g(a) + \mathfrak{m}^n\mathrm{B}$, since both $g(a)$ and b go to b_n. □

Lemma. — Let P *be a finite* R*-module. Then* $\underline{\mathrm{P}} \otimes_{\mathrm{R}} \hat{\mathrm{R}} = \underline{\hat{\mathrm{P}}}$.

Proof. — We have an exact sequence $0 \to \mathrm{K} \to \mathrm{R}^{\oplus a} \to \mathrm{P} \to 0$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{K}} \otimes_{\mathrm{R}} \hat{\mathrm{R}} \arrow[r] \arrow[d] & \mathrm{R}^{\oplus a} \otimes_{\mathrm{R}} \hat{\mathrm{R}} \arrow[r] \arrow[d] & \underline{\mathrm{P}} \otimes_{\mathrm{R}} \hat{\mathrm{R}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\hat{\mathrm{K}}} \arrow[r] & (\mathrm{R}^{\oplus a})^\wedge \arrow[r] & \underline{\hat{\mathrm{P}}} \arrow[r] & 0.
\end{tikzcd}$$

Note that $\mathrm{R}^{\oplus a} \otimes_{\mathrm{R}} \hat{\mathrm{R}} = \hat{\mathrm{R}}^{\oplus a} = (\mathrm{R}^{\oplus a})^\wedge$, so the middle arrow is an isomorphism. The upper row is exact (`093M`), and $(\mathrm{R}^{\oplus a})^\wedge \to \underline{\hat{\mathrm{P}}}$ is an epimorphism by the previous lemma, so we deduce that $\underline{\mathrm{P}} \otimes_{\mathrm{R}} \hat{\mathrm{R}} \to \underline{\hat{\mathrm{P}}}$ is an epimorphism, and hence the same for K, since K is also finite. We'll be done by the five lemma if we can show that the bottom row is exact. We have $\hat{\mathrm{R}}^{\oplus a} \twoheadrightarrow \underline{\hat{\mathrm{P}}}$, so, letting N stand in for $\mathrm{R}^{\oplus a}$, we have an exact sequence

$$0 \longrightarrow \lim_n \underline{\mathrm{K}/\mathfrak{m}^n\mathrm{N} \cap \mathrm{K}} \longrightarrow \hat{\mathrm{R}}^{\oplus a} \longrightarrow \underline{\hat{\mathrm{P}}} \longrightarrow 0.$$

Artin-Rees (`00IN`) allows us to find some integer c so that $\mathfrak{m}^n\mathrm{K} \subset \mathfrak{m}^n\mathrm{N} \cap \mathrm{K} \subset \mathfrak{m}^{n-c}\mathrm{K}$ for $n \geq c$, so $\lim_n \underline{\mathrm{K}/\mathfrak{m}^n\mathrm{N} \cap \mathrm{K}} = \lim_n \underline{\mathrm{K}}/\underline{\mathfrak{m}^n\mathrm{K}} = \lim_n f_n^*(\mathrm{K} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n) = \underline{\hat{\mathrm{K}}}$. □

Lemma (♦). — *Let* X *and* R *be as in §6.5,* $f : (X_{\text{proét}}, \hat{R}) \to (*, R)$ *be the morphism of ringed topos, and* K *be in* $D_{\text{perf}}(R)$. *Then* f^*K *is* $\mathfrak{m}$*-adically complete:* $f^*K \xrightarrow{\sim} \underline{\hat{K}}$.

Proof. — Fix K in $D_{\text{perf}}(R)$, and represent K by a bounded complex of finite projective R-modules $P^{\bullet}$. Proposition 3.1.10 says that an inverse system $(F_n)_{n\in\mathbf{N}^\circ}$ in $\mathrm{Mod}(X_{\text{proét}}, R)$ or $\mathrm{Mod}(X_{\text{proét}}, \hat{R})$ is right acyclic for $\lim_n$ if the transition morphisms $F_{n+1} \to F_n$ are surjective. As the functor $\lim_n$ of $\mathbf{N}^\circ$-indexed limits in $\mathrm{Mod}(X_{\text{proét}}, R)$ has finite cohomological dimension, any complex consisting of right acyclic objects for $\lim_n$ computes $\mathrm{R}\lim_n$ (07K7), so

$$\begin{aligned}\mathrm{R}\lim(f^*K \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n) &= \mathrm{R}\lim(\underline{K} \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n) = \mathrm{R}\lim f_n^*(K \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n)\\ &= \lim_n \underline{P}^{\bullet} \otimes_R R/\mathfrak{m}^n = \lim_n f_n^*(P^{\bullet} \otimes_R R/\mathfrak{m}^n),\end{aligned}$$

where the $\lim_n$ are computed pointwise. So in order to show $f^*K \to \mathrm{R}\lim(f^*K \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n)$ is an isomorphism, since $f^*K = f^{-1}K \otimes^{\mathbf{L}}_{R} \hat{R}$ is represented by $\underline{P}^{\bullet} \otimes_R \hat{R}$, it suffices to show $\underline{P} \otimes_R \hat{R} = \lim_n \underline{P} \otimes_R R/\mathfrak{m}^n = \underline{\hat{P}}$. This is the content of the previous lemma. □

To check that pullback is fully faithful: Let K be in $D_{\text{perf}}(R)$, and continue to let $f : (\tilde{X}_{\text{proét}}, \hat{R}) \to (*, R)$ denote the morphism of ringed topos and $f_n : (\tilde{X}_{\text{proét}}, R/\mathfrak{m}^n) \to (*, R/\mathfrak{m}^n)$ its reductions modulo $\mathfrak{m}^n$, each of which factors as $(\tilde{X}_{\text{proét}}, R/\mathfrak{m}^n) \xrightarrow{\nu} (\tilde{X}_{\text{ét}}, R/\mathfrak{m}^n) \to (*, R/\mathfrak{m}^n)$. As f^*K is $\mathfrak{m}$-adically complete, $f^*K = \mathrm{R}\lim f_n^*(K \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n)$ by (∗), so if $P^{\bullet}$ is a finite complex of finite projective R-modules representing K,

$$\begin{aligned}\mathrm{R}\Gamma(X_{\text{proét}}, f^*K) &= \mathrm{R}\lim \mathrm{R}\Gamma(X_{\text{proét}}, f_n^*(K \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n))\\ &= \mathrm{R}\lim \mathrm{R}\Gamma(X_{\text{ét}}, \underline{K} \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n)\\ &= \mathrm{R}\lim(K \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n)\\ &= \lim_n P^{\bullet} \otimes_R R/\mathfrak{m}^n = P^{\bullet} = K.\end{aligned}$$

The second equality by 099W and the third since X is strictly henselian local. The lim on the last line is the pointwise limit; it computes $\mathrm{R}\lim(K \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n)$ by 091D, since $P^{\bullet} \otimes_R R/\mathfrak{m}^n$ computes $K \otimes^{\mathbf{L}}_{R} R/\mathfrak{m}^n$, while $(P^m \otimes_R R/\mathfrak{m}^n)$ is ML. Finally, $\lim_n P^m \otimes_R R/\mathfrak{m}^n = P^m$ by 00MA, as $R = R^\wedge$ is noetherian and $\mathfrak{m}$-adically complete, and P^m is a finite R-module. In conclusion, pullback is fully faithful.

To check that pullback is essentially surjective: Fix K in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$ and let $\nu^*\mathrm{K}_1 := \mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$ for K_1 in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{\acute{e}t}}, \mathrm{R}/\mathfrak{m})$. By hypothesis, $\nu^*\mathrm{K}_1$ is locally constant on $\mathrm{X}_{\mathrm{pro\acute{e}t}}$. This implies that $\nu^*\mathrm{K}_1$ is constant as follows. By hypothesis there is a pro-étale covering $(\mathrm{U}_i \to \mathrm{X})$ so that $\nu^*\mathrm{K}_1|_{\mathrm{U}_i} \simeq \underline{\mathrm{M}_i}$ for M_i in $\mathrm{D}(\mathrm{R}/\mathfrak{m})$. As X is connected, the argument in the first paragraph of `099Y` shows that $\mathrm{M}_i \simeq \mathrm{M}$ for some M in $\mathrm{D}(\mathrm{R}/\mathfrak{m})$ and for all i. Base changing over the closed point of X, as one of the U_i covers this point, and there is a section $x \to \mathrm{U}_i \times_{\mathrm{X}} x$, we see that the constant value M coincides with the stalk of K_1 (as an étale sheaf) at x. We get a morphism $\underline{\mathrm{M}} \to \mathrm{K}_1$ by adjunction from $\mathrm{M} \to \mathrm{R}\Gamma(\mathrm{X}_{\mathrm{\acute{e}t}}, \mathrm{K}_1) = (\mathrm{K}_1)_x$, and hence a morphism $\underline{\mathrm{M}} \to \nu^*\mathrm{K}_1$ which is an isomorphism over each U_i, and hence an isomorphism in $\mathrm{D}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \mathrm{R}/\mathfrak{m})$.

The previous paragraph shows that if K is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$ and $\mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$ is locally constant for the pro-étale topology, in fact $\mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$ is constant. Lemma 6.5.6 implies the same is true for $\mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n$ for all n. Denote the constant values by K_n. We have that $f_n^*(\mathrm{K}_{n+1} \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}^{n+1}} \mathrm{R}/\mathfrak{m}^n) = f_{n+1}^*\mathrm{K}_{n+1} \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}^{n+1}} \mathrm{R}/\mathfrak{m}^n = f_n^*\mathrm{K}_n$, so we find $\mathrm{K}_{n+1} \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}^{n+1}} \mathrm{R}/\mathfrak{m}^n = \mathrm{K}_n$ upon applying $\mathrm{R}\Gamma(\mathrm{X}_{\mathrm{pro\acute{e}t}}, -)$ as above. Note also that K_1 is perfect as K is constructible.

Lemma ($\Diamond$). — *Let* R *be a noetherian ring,* $\mathfrak{m} \subset \mathrm{R}$ *an ideal, and suppose* R *is* $\mathfrak{m}$*-adically complete. Suppose given an* $\mathbf{N}^\circ$*-indexed inverse system* (K_n) *in* $\mathrm{D}(\mathrm{R})$ *with* K_n *coming via restriction from* $\mathrm{D}(\mathrm{R}/\mathfrak{m}^n)$ *and the transition morphisms* $\mathrm{K}_{n+1} \to \mathrm{K}_n$ *coming via restriction from* $\mathrm{D}(\mathrm{R}/\mathfrak{m}^{n+1})$ *and inducing isomorphisms* $\mathrm{K}_{n+1} \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}^{n+1}} \mathrm{R}/\mathfrak{m}^n \xrightarrow{\sim} \mathrm{K}_n$*, and suppose* K_1 *is in* $\mathrm{D}_{\mathrm{perf}}(\mathrm{R}/\mathfrak{m})$*. Then each* K_n *is in* $\mathrm{D}_{\mathrm{perf}}(\mathrm{R}/\mathfrak{m}^n)$*,* $\mathrm{R}\lim \mathrm{K}_n$ *is in* $\mathrm{D}_{\mathrm{perf}}(\mathrm{R})$*, and* $(\mathrm{R}\lim \mathrm{K}_n) \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n = \mathrm{K}_n$*.*

Proof. — It follows from the proof of `07LU` that since K_1 can be represented by a bounded complex $\mathrm{E}_1^\bullet$ of finite projective $\mathrm{R}/\mathfrak{m}$-modules zero outside degrees $[a, b]$ (as K is constructible), then K_n can be represented by a bounded complex $\mathrm{E}_n^\bullet$ of finite projective $\mathrm{R}/\mathfrak{m}^n$ modules zero outside degrees $[a, b]$ with the property that $\mathrm{E}_{n+1}^\bullet/\mathfrak{m}^n\mathrm{E}_{n+1}^\bullet = \mathrm{E}_n^\bullet$. In particular the systems $(\mathrm{E}_n^i)_{n\in\mathbf{N}^\circ}$ for fixed i are surjective. Therefore $\mathrm{E} := \mathrm{R}\lim \mathrm{K}_n$ can be computed as the pointwise limit $\lim_n \mathrm{E}_n^\bullet$ (`091D`), and $\lim_n \mathrm{E}_n^i$ is a finite projective R-module for each i by the lemma below, so $\mathrm{E} \in \mathrm{D}_{\mathrm{perf}}(\mathrm{R})$. The isomorphism $\mathrm{E} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n = \mathrm{K}_n$ follows from this and `031C`. $\square$

Lemma. — *Let* R *be a noetherian ring,* $\mathfrak{m} \subset \mathrm{R}$ *an ideal, and suppose* R *is* $\mathfrak{m}$*-adically complete. Let* $\mathscr{C}$ *be the category of* $\mathbf{N}^{\circ}$*-indexed inverse systems* (M_n) *of finite* R*-modules, with the property that* M_n *is a projective* $\mathrm{R}/\mathfrak{m}^n$*-module, and the transition maps induce isomorphisms* $\mathrm{M}_{n+1} \otimes \mathrm{R}/\mathfrak{m}^n \xrightarrow{\sim} \mathrm{M}_n$. *Then* $\mathscr{C}$ *is equivalent to the category of finite projective* R*-modules via the functors* $(\mathrm{M}_n) \mapsto \lim \mathrm{M}_n$ *and* $\mathrm{M} \mapsto (\mathrm{M}/\mathfrak{m}^n\mathrm{M})$.

Proof. — This lemma with the word ‘projective’ dropped is `087W`. So, we have a fully faithful functor from the category $\mathrm{Proj}_{\mathrm{R}}$ to $\mathscr{C}$, and it suffices to check that if P is a finite R-module so that $\mathrm{P} \otimes_{\mathrm{R}} \mathrm{R}/\mathfrak{m}^n$ is a projective $\mathrm{R}/\mathfrak{m}^n$-module for all n, then P is a finite projective R-module. As $\mathfrak{m}$ is contained in the Jacobson radical of R (`05GI`), `0523` tells us that for every maximal ideal $\mathfrak{p} \subset \mathrm{R}$, $\mathrm{P}_{\mathfrak{p}}$ is flat over R, hence finite flat over $\mathrm{R}_{\mathfrak{p}}$ (`00HT`), hence finite free over $\mathrm{R}_{\mathfrak{p}}$ (`00NZ`), hence P is a finite projective R-module (`00NX`). □

It remains to show that $\mathrm{K} = \mathrm{R}\lim \underline{\mathrm{K}}_n$ coincides with $f^*\mathrm{E}$ in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$, where $\mathrm{E} := \mathrm{R}\lim \mathrm{K}_n$. We get a map $f^*\mathrm{E} \to \mathrm{K}$ via

$$\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \mathrm{K}) = \mathrm{R}\lim \mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, \underline{\mathrm{K}}_n) = \mathrm{R}\lim \mathrm{R}\Gamma(\mathrm{X}_{\text{ét}}, \underline{\mathrm{K}}_n) = \mathrm{R}\lim \mathrm{K}_n = \mathrm{E}.$$

This map induces the isomorphism

$$f^*\mathrm{E} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} = \underline{\mathrm{E}} \otimes^{\mathbf{L}}_{\mathrm{R}} \hat{\mathrm{R}} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} = f_1^*(\mathrm{E} \otimes^{\mathbf{L}}_{\mathrm{R}} \mathrm{R}/\mathfrak{m}) = f_1^*\mathrm{K}_1 = \underline{\mathrm{K}}_1 = \mathrm{K} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$$

modulo $\mathfrak{m}$. We finish with the following

Lemma. — *In the setting of §6.5, let* f *be a morphism in* $\mathrm{D}_{\mathrm{comp}}(\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$ *so that* $f \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$ *is an isomorphism. Then* f *is an isomorphism.*

Proof. — Let $\mathrm{C} := \mathrm{Cone}(f)$. Then C is in $\mathrm{D}_{\mathrm{comp}}(\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$ and $\mathrm{C} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} = 0$. The distinguished triangle

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathrm{R}/\mathfrak{m}^{n+1} \longrightarrow \mathrm{R}/\mathfrak{m}^n \longrightarrow$$

combined with the isomorphism $\mathrm{C} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathrm{C} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and induction on n show that $\mathrm{C} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = 0$ for all n, so $\mathrm{C} = \mathrm{R}\lim(\mathrm{C} \otimes^{\mathbf{L}}_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) = 0$. □

6.5.8. 1) This notation could be confusing. The assertion is that $k^{-1}(\hat{\mathrm{R}}_{\mathrm{X}}) = \hat{\mathrm{R}}_{\mathrm{W}}$ with the notation of [**SGAA**, IV 14.2.3].

For 2), you need that i^* commutes with homotopy-limits, which is true because it has a left adjoint by Remark 6.1.7 and `09T5`. Then by `07A4`, $k^*(K \otimes_{\hat{R}} R/\mathfrak{m}^n) = k^*K \otimes_{\hat{R}} k^*(R/\mathfrak{m}^n)$, and $k^*(R/\mathfrak{m}^n) = k^{-1}(R/\mathfrak{m}^n) \otimes^{\mathbf{L}}_{k^{-1}\hat{R}} \hat{R}$ (`08I6`), but since $k^{-1}\hat{R}_X = \hat{R}_W$, $k^*(R/\mathfrak{m}^n) = k^{-1}(R/\mathfrak{m}^n) = R/\mathfrak{m}^n$.

For 3), we need that $k_!$ commutes with homotopy-limits by Lemma 6.2.3.

6.5.9. 1. Via Lemma 3.5.6, $f_*K \otimes_{\hat{R}} R/\mathfrak{m}^n = f_*K \otimes_R R/\mathfrak{m}^n$ and $f_*(K \otimes_{\hat{R}} R/\mathfrak{m}^n) = f_*(K \otimes_R R/\mathfrak{m}^n)$. There is always a natural map $f_*K \otimes_R R/\mathfrak{m}^n \to f_*(K \otimes_R R/\mathfrak{m}^n)$ coming from adjunction. If $R/\mathfrak{m}^n$ were perfect in $D(R)$, by dévissage we could replace $R/\mathfrak{m}^n$ by a projective R-module, and then as a direct summand of a free module. As

$$\begin{aligned}&\mathrm{Cone}(f_*K \otimes_R (P \oplus Q) \to f_*(K \otimes_R (P \oplus Q)))\\ &\quad = \mathrm{Cone}(f_*K \otimes_R P \to f_*(K \otimes_R P)) \oplus \mathrm{Cone}(f_*K \otimes_R Q \to f_*(K \otimes_R Q)),\end{aligned}$$

and this cone is zero when $P \oplus Q$ is a finite free R-module, we'd be finished.

Suppose $\mathfrak{m} = (f_1, \ldots, f_m)$ and suppose we are in the setting of `0BKC`: if $K_n^\bullet := K_\bullet(R, f_1^n, \ldots, f_m^n)$ denotes the Koszul complex on $f_1^n, \ldots, f_m^n$, regarded as a cochain complex in degrees $[-m, 0]$, then $K_n^\bullet$ is a finite free complex and there are maps $K_{n+1}^\bullet \to K_n^\bullet$ compatible with the maps on $H^0(K_n^\bullet) = R/(f_1^n, \ldots, f_m^n)$ and the maps $R \to K_n^\bullet$. Then in fact the maps $K_n^\bullet \to H^0(K_n^\bullet) = R/(f_1^n, \ldots, f_m^n)$ determine a isomorphism $(K_n^\bullet) \xrightarrow{\sim} (R/(f_1^n, \ldots, f_m^n))$ in $\mathrm{Pro}(D(R))$ (`0921`). As the systems $(R/(f_1^n, \ldots, f_m^n))$ and $(R/\mathfrak{m}^n)$ are always pro-isomorphic, we have a pro-isomorphism $(K_n^\bullet) \xrightarrow{\sim} (R/\mathfrak{m}^n)$ and hence an isomorphism $(? \otimes_R R/\mathfrak{m}^n) \simeq (? \otimes_R K_n^\bullet)$ of pro-objects for $? = K$ or f_*K. We're done by the previous paragraph as $K_n^\bullet$ is a perfect complex in $D(R)$.

2. This is generalized by `0A0G`. Using `099I` and Lemma 3.5.6, we see that it follows from the constant noetherian case as

$$\begin{aligned}f_*\hat{K} &= f_* R\lim(K \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n) = f_* R\lim(K \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n)\\ &= R\lim(f_*K \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n) = R\lim(f_*K \otimes^{\mathbf{L}}_{\hat{R}} R/\mathfrak{m}^n) =: \widehat{f_*K}.\end{aligned}$$

3. There is a natural map $f_*K \otimes_{\hat{R}} \underline{\hat{L}} \to f_*(K \otimes_{\hat{R}} \underline{\hat{L}})$ coming from adjunction. By finding a complex L' so that $L \oplus L'$ can be represented as a finite complex of finite free

R-modules as in the proof of Lemma 6.5.5, the cone of the above map for L replaced by $L \oplus L'$ is the direct sum of the cones for L and L′ individually, so it suffices to show the result for L a finite complex of finite free R-modules; then dévissage along stupid truncations and along finite filtrations reduces to L = R.

5. We need Lemma 3.5.5 for the first isomorphism in

$$\begin{aligned} f^*_{\mathrm{comp}}(\mathrm{K}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} &= f^*(\mathrm{K}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} = f^{-1}\mathrm{K} \otimes_{f^{-1}\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} \\ &= f^{-1}(\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}) = f_1^*(\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}) \end{aligned}$$

(the second isomorphism is `08I6`). To conclude, recall that $\nu_{\mathrm{X}}^* \circ f_{\mathrm{et},*} = f_{\mathrm{pro\acute{e}t},*} \circ \nu_{\mathrm{Y}}^*$ by Lemma 5.4.1.

6.5.11. 1. The first equality is by `08YU` and Lemma 3.5.5.

Our hypothesis tells us that $f_* : \mathrm{Mod}(\mathrm{X}_{\acute{e}t}, \mathrm{R}/\mathfrak{m}) \to \mathrm{Mod}(\mathrm{Y}_{\acute{e}t}, \mathrm{R}/\mathfrak{m})$ has cohomological dimension $\leq d$. However, in the proof, the functor being discussed is the right derived functor of $f_* : \mathrm{Mod}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}}) \to \mathrm{Mod}(\mathrm{Y}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$. We know that cohomology on a ringed topos can be computed on underlying abelian sheaves (`079Y`), so the distinction shouldn't matter.[59] We have a commutative square of ringed topos

$$\begin{tikzcd} (\mathrm{X}_{\mathrm{pro\acute{e}t}}, \mathrm{R}/\mathfrak{m}^n) \arrow[r, "b"] \arrow[d, "d"] & (\mathrm{Y}_{\mathrm{pro\acute{e}t}}, \mathrm{R}/\mathfrak{m}^n) \arrow[d, "a"] \\ (\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}}) \arrow[r, "c"] & (\mathrm{Y}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}}). \end{tikzcd}$$

Direct image along both vertical morphisms (which are not flat) is given by restriction.

Lemma. — We have $a_*b_* = c_*d_*$ as functors $\mathrm{D}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}(\mathrm{Y}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$.

In other words, there's no ambiguity in using f_* for b_* or c_*.

Proof. — Since the square commutes, to show $a_*b_* = c_*d_*$ as functors $\mathrm{D}^+(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}^+(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$, it would suffice to know that both coincide with $(a \circ b)_* = (c \circ d)_*$ as

[59]If E is a topos, Y an object of E, and R a ring in E, then the fact that $\mathrm{H}^i(\mathrm{Y}, \mathrm{F}) = \mathrm{H}^i(\mathrm{Y}, \mathrm{F}_{\mathrm{ab}})$ for a R-module F implies that $\mathrm{R}\Gamma(\mathrm{Y}, \mathrm{K}) = \mathrm{R}\Gamma(\mathrm{Y}, \mathrm{K}_{\mathrm{ab}})$ when K is in $\mathrm{D}^+(\mathrm{E}, \mathrm{R})$. If $\mathrm{D}(\mathrm{E}, \mathrm{R})$ is left-complete or $\Gamma(\mathrm{Y}, -) : \mathrm{Ab}(\mathrm{X}) \to \mathrm{Ab}$ has finite cohomological dimension, then the above isomorphism of $\mathrm{R}\Gamma$ holds for K in $\mathrm{D}(\mathrm{E}, \mathrm{R})$.

functors $D^+(X_{\text{proét}}, R/\mathfrak{m}^n) \to D^+(X_{\text{proét}}, \hat{R})$, which is implied by `072Z`, `0731` & `015M`. As $D(X_{\text{proét}}, R/\mathfrak{m}^n)$ is left complete, for any M in $D(X_{\text{proét}}, R/\mathfrak{m}^n)$,

$$\begin{aligned} a_*b_*M = a_*b_*R\lim \tau_{\geq -n}M = R\lim a_*b_*\tau_{\geq -n}M \\ = R\lim c_*d_*\tau_{\geq -n}M = c_*d_*R\lim \tau_{\geq -n}M = c_*d_*M. \end{aligned}$$

□

It follows by Lemma 5.4.6 as stated in the text that b_* carries $D^{\leq k}_{cc}(X_{\text{proét}}, R/\mathfrak{m})$ $D^{\leq k+d+1}_{cc}(Y_{\text{proét}}, R/\mathfrak{m})$, so c_*d_* carries $D^{\leq k}_{cc}(X_{\text{proét}}, R/\mathfrak{m})$ to $D^{\leq k+d+1}(Y_{\text{proét}}, \hat{R})$ by the above. It only remains to show that if M is in $D^{\leq k}(X_{\text{proét}}, \hat{R})$ with $M_1 := M \otimes_{\hat{R}} R/\mathfrak{m}$ in the image of $\nu^* : D^{\leq k}(X_{\text{ét}}, R/\mathfrak{m}) \to D^{\leq k}(X_{\text{proét}}, R/\mathfrak{m})$, then $b_*(M_n)$ is in $D^{\leq k+d+1}(Y_{\text{proét}}, R/\mathfrak{m}^n)$. The distinguished triangle

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^{n+1} \longrightarrow R/\mathfrak{m}^n \longrightarrow$$

and the isomorphisms $M \otimes_{\hat{R}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = M_1 \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$, which is in the image of ν^*, hence *a fortiori* in $D^{\leq k}_{cc}(X_{\text{proét}}, R/\mathfrak{m})$, give the conclusion by induction on n.

2. In addition to Verdier's exercise (`05R0`), we need the following

Lemma. — Let $f : X \to Y$ *be a map of coherent schemes. Then* $f_* : D^{\geq 0}(X_{\text{proét}}) \to D^{\geq 0}(\mathrm{Ab})$ *commutes with direct sums.*

Proof. — For any coherent V in $Y_{\text{proét}}$, let $W := V \times_Y X$. As any morphism between coherent schemes is quasi-compact and quasi-separated,[60] and these properties are stable under base change, W is coherent. By `09A3` and `0D6H` we have

$$\begin{aligned} R\Gamma(V_{\text{proét}}, \oplus_s f_*K_s) = \oplus_s R\Gamma(V_{\text{proét}}, f_*K_s) = \oplus_s R\Gamma(W_{\text{proét}}, K_s|_W) \\ = R\Gamma(W_{\text{proét}}, \oplus_s K_s|_W) = R\Gamma(V_{\text{proét}}, f_*(\oplus_s K_s)). \end{aligned}$$

As this is true for all such V, we're done [**SGAA**, II 4.10]. □

3. Any K in $D(X_{\text{proét}}, R/\mathfrak{m}^n)$ is already $\mathfrak{m}$-adically complete, since this is the same as being derived $\mathfrak{m}$-complete by Lemma 3.5.6, so it suffices by `0998` to check that $R\lim(\cdots \xrightarrow{f} K \xrightarrow{f} K) = 0$ for all $f \in \mathfrak{m}$. But this system is pro-zero, so by the lemma in the note to `099M`, its homotopy limit is zero.

[60] `01KV` & `03GI`, as coherent = quasi-compact and quasi-separated over Spec **Z** by `01K4`.

6.6.1. To be clear, the site $\pi_0(Y)$ is the topological space $\pi_0(Y)$ (not the site $\pi_0(Y)_{\text{proét}}$ that appeared in §4). The continuous map $Y \to \pi_0(Y)$ determines via pullback a functor $\pi^{-1} : \pi_0(Y) \to Y_{\text{ét}}$ on the underlying categories, which evidently preserves finite limits and coverings, so is continuous [**SGAA**, III 1.6] and determines the stated morphism of sites. Given an open neighborhood U of $\overline{y}$, $U \cap Y^c$ is open in Y^c, hence determines an open subset of $\pi_0(Y)$ by Lemma 2.1.4, the preimage of which is contained in U.

To do the colimits with complexes, take a K-injective complex $I^\bullet$ representing π^*K; then $\pi_*I^\bullet$ is K-injective, as well as the restriction of the latter to an open in $\pi_0(Y)$ or the former to an arrow in $Y_{\text{ét}}$, so we have the displayed isomorphisms of colimits of complexes after dropping the R and replacing K by $I^\bullet$.

6.6.2. To be explicit, $C = R\lim C_n$ and the claim is that $L \simeq \underline{\hat{C}} := R\lim \underline{C}_n$.

In the fully faithful part of the note to Lemma 6.5.7, we showed that $L = \underline{E} \otimes_R \hat{R}$ under similar assumptions on a strictly henselian local scheme. Moreover, in Lemma ♦ in the note to Lemma 6.5.7, we showed that for X and R as in §6.5, $f : (X_{\text{proét}}, \hat{R}) \to (*, R)$ the morphism of ringed topos, and K in $D_{\text{perf}}(R)$, $f^*K \xrightarrow{\sim} \underline{\hat{K}}$ is an isomorphism. Therefore $\underline{\hat{K}}$ deserves to be called 'constant.'

To find C in $D_{\text{perf}}(R)$ with $C \otimes_R R/\mathfrak{m}^n = C_n$: we have $L \otimes_R R/\mathfrak{m}^n = \underline{C}_n$. We have maps $C_{n+1} \to C_n$ inducing isomorphisms $C_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \xrightarrow{\sim} C_n$: as $L_n = \nu^*\underline{C}_n$, the morphisms $L_{n+1} \to L_n$ on $S_{\text{proét}}$ give rise to maps $\underline{C}_{n+1} \to \underline{C}_n$ on $S_{\text{ét}}$ (`099V`) with the property that $\underline{C}_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \to \underline{C}_n$ is an isomorphism since the same is true with $\underline{C}$ replaced by L (the left hand side is bounded since if P is a projective $R/\mathfrak{m}^n$-module, $\underline{P}$ in $\mathrm{Mod}(S_{\text{ét}}, R/\mathfrak{m}^n)$ is flat). Pulling back to s and applying $R\Gamma(s_{\text{ét}}, -)$ gives the maps and isomorphisms without underlines; then use Lemma ◊ from the note to Lemma 6.5.7.

'$\mathrm{Ext}^i_{R/\mathfrak{m}^n}(\underline{C}_n, \underline{C}_n) = \mathrm{Maps}_{\text{cont}}(S, \mathrm{Ext}^i_{R/\mathfrak{m}^n}(C_n, C_n))$' $\rightsquigarrow$ in order to show

$$R\mathcal{H}om_{R/\mathfrak{m}^n}(\underline{C}_n, \underline{C}_n) = \underline{R\,\mathrm{Hom}_{R/\mathfrak{m}^n}(C_n, C_n)} =: \underline{\Xi}$$

on $S_{\text{proét}}$, it suffices by Lemma 6.3.16 to show the same on $S_{\text{ét}}$, where it follows from the lemma in the note to Lemma 6.3.12 and `093T`. We have $\mathcal{H}^i\underline{\Xi} = \underline{\mathrm{Ext}^i_{R/\mathfrak{m}^n}(C_n, C_n)}$ and the spectral sequence (`015J`)

$$E_2^{pq} = H^p(S_{\text{proét}}, \mathscr{H}^q\underline{\Xi}) \Rightarrow \operatorname{Ext}^{p+q}_{R/\mathfrak{m}^n}(\underline{C}_n, \underline{C}_n),$$

so to see $\operatorname{Ext}^i_{R/\mathfrak{m}^n}(\underline{C}_n, \underline{C}_n) = \Gamma(S_{\text{proét}}, \underline{\operatorname{Ext}^i_{R/\mathfrak{m}^n}(C_n, C_n)}) = \operatorname{Maps}_{\text{cont}}(S, \operatorname{Ext}^i_{R/\mathfrak{m}^n}(C_n, C_n))$ (the last equality by Lemma 4.2.12) we have to show that constant sheaves of $R/\mathfrak{m}^n$-modules are acyclic for $\Gamma(S_{\text{proét}}, -)$. Let M be some $R/\mathfrak{m}^n$-module, and write $S = \lim S_i$ as a limit of affine schemes finite over a separably closed field. Then

$$H^i(S_{\text{proét}}, \underline{M}) = H^i(S_{\text{ét}}, \underline{M}) = \operatorname{colim}_n H^i((S_i)_{\text{ét}}, \underline{M}),$$

(`099W` & `09YQ`) but $H^i((S_i)_{\text{ét}}, \underline{M}) = 0$ when $i > 0$ by Artin's theorem, as $\underline{M}$ is of torsion.

'the system $\{\operatorname{Ext}^i_{R/\mathfrak{m}^n}(\underline{C}_n, \underline{C}_n)\}$ satisfies ML' $\rightsquigarrow$ by the indicated lemmas it suffices to show that $R\operatorname{Hom}_R(K, L) \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n = R\operatorname{Hom}_{R/\mathfrak{m}^n}(K_n, L_n)$, when $K, L \in D_{\text{perf}}(R)$. By `07VI` & `08YU`,

$$R\operatorname{Hom}_R(K, L) \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n = K^\vee \otimes^{\mathbf{L}}_R L \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n = (K^\vee \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n) \otimes^{\mathbf{L}}_{R/\mathfrak{m}^n} (L \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n),$$

so it will suffice to show that $K^\vee \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n = (K \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n)^\vee$. As both are in $D_{\text{perf}}(R/\mathfrak{m}^n)$ it suffices to show their duals are the same. Indeed,

$$R\operatorname{Hom}_{R/\mathfrak{m}^n}(K^\vee \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n, R/\mathfrak{m}^n) = R\operatorname{Hom}_R(K^\vee, R/\mathfrak{m}^n) = (K^\vee)^\vee \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n = K \otimes^{\mathbf{L}}_R R/\mathfrak{m}^n.$$

'a map $f : \underline{C}_n \to \underline{C}_n$ is an automorphism iff it is so modulo $\mathfrak{m}$' $\rightsquigarrow$ the cone is in $D(S_{\text{proét}}, R/\mathfrak{m}^n)$ vanishes modulo $\mathfrak{m}$, but any $R/\mathfrak{m}^n$-complex is already derived $\mathfrak{m}$-complete (see the note to Lemma 6.5.11 3)).

'$\{\operatorname{Aut}_{R/\mathfrak{m}^n}(\underline{C}_n)\}$ has ML' $\rightsquigarrow$ $\{\operatorname{Hom}_{R/\mathfrak{m}^n}(\underline{C}_n)\}$ has ML by what has just been said, of course $\operatorname{Aut}_{R/\mathfrak{m}^n}(\underline{C}_n) \subset \operatorname{Hom}_{R/\mathfrak{m}^n}(\underline{C}_n)$, and since an endomorphism of $\underline{C}_n$ is an automorphism iff its reduction modulo $\mathfrak{m}$ is, letting φ_{mn} denote the transition maps when $m > n$,

$$\varphi_{mn}(\operatorname{Hom}_{R/\mathfrak{m}^m}(\underline{C}_m)) \cap \operatorname{Aut}_{R/\mathfrak{m}^n}(\underline{C}_n) = \varphi_{mn}(\operatorname{Aut}_{R/\mathfrak{m}^m}(\underline{C}_m)).$$

'As the evident map $\operatorname{Isom}_{R/\mathfrak{m}^n}(L_n, \underline{C}_n) \times \operatorname{Ext}^i_{R/\mathfrak{m}^n}(L_n, \underline{C}_n) \to \operatorname{Ext}^i_{R/\mathfrak{m}^n}(L_n, \underline{C}_n)$ is surjective' $\rightsquigarrow$ of course, for any $\alpha \in \operatorname{Isom}_{R/\mathfrak{m}^n}(L_n, \underline{C}_n)$, the map $\{\alpha\} \times \operatorname{Ext}^i_{R/\mathfrak{m}^n}(L_n, \underline{C}_n) \to \operatorname{Ext}^i_{R/\mathfrak{m}^n}(L_n, \underline{C}_n)$ is surjective. Suppose (A_n) and (B_n) are in $\mathrm{Set}^{\mathbf{N}^\circ}$ and both have ML.

Then for each i there is a $j > i$ so that if $a \in \mathrm{A}_i$ is in the image of A_j and $b \in \mathrm{B}_i$ is in the image of B_j, then a is in the image of A_n and b is in the image of B_n for $n \gg 0$, so $(\mathrm{A}_n \times \mathrm{B}_n)$ has ML: $(a, b) \in \mathrm{A}_i \times \mathrm{B}_i$ is in the image of $\mathrm{A}_n \times \mathrm{B}_n$ iff a is in the image of A_n and b is in the image of B_n iff a is in the image of A_j and b is in the image of B_j iff (a, b) is in the image of $\mathrm{A}_j \times \mathrm{B}_j$.

Perhaps it's worth writing

$$\mathrm{R}\,\mathrm{Hom}_{\hat{\mathrm{R}}}(\mathrm{L}, \underline{\hat{\mathrm{C}}}) = \mathrm{R}\lim \mathrm{R}\,\mathrm{Hom}_{\hat{\mathrm{R}}}(\mathrm{L}, \underline{\mathrm{C}}_n) = \mathrm{R}\lim \mathrm{R}\,\mathrm{Hom}_{\mathrm{R}/\mathfrak{m}^n}(\mathrm{L}_n, \underline{\mathrm{C}}_n).$$

Applying H^0, `0CQE` gives the exact sequence

$$0 \to \mathrm{R}^1 \lim_n \mathrm{Ext}^{-1}_{\mathrm{R}/\mathfrak{m}^n}(\mathrm{L}_n, \underline{\mathrm{C}}_n) \to \mathrm{Hom}_{\hat{\mathrm{R}}}(\mathrm{L}, \underline{\hat{\mathrm{C}}}) \to \lim_n \mathrm{Hom}_{\mathrm{R}/\mathfrak{m}^n}(\mathrm{L}_n, \underline{\mathrm{C}}_n) \to 0,$$

but $(\mathrm{Ext}^{-1}_{\mathrm{R}/\mathfrak{m}^n}(\mathrm{L}_n, \underline{\mathrm{C}}_n))$ has ML so is acyclic for $\lim_n$ (`091D`), and so we have the claim.

By completeness, a map $f : \mathrm{L} \to \underline{\hat{\mathrm{C}}}$ is an isomorphism iff $f \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}$ is one iff $f \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n$ is one for every n.

6.6.6. The category $\mathscr{C}$ is abelian since Ind of an abelian category $\mathscr{A}$ is abelian (*Categories and Sheaves,* Theorem 8.6.5), the opposite of an abelian category is abelian, and $\mathrm{Pro}(\mathscr{A}) = (\mathrm{Ind}(\mathscr{A}^{\mathrm{op}}))^{\mathrm{op}}$. Given an exact sequence

$$0 \longrightarrow \mathrm{K} \longrightarrow \mathrm{N} \longrightarrow \mathrm{M} \longrightarrow 0$$

of finite R-modules,

$$0 \longrightarrow \mathrm{K}/(\mathfrak{m}^n\mathrm{N} \cap \mathrm{K}) \longrightarrow \mathrm{N}/\mathfrak{m}^n\mathrm{N} \longrightarrow \mathrm{M}/\mathfrak{m}^n\mathrm{M} \longrightarrow 0$$

is exact, so it suffices to demonstrate that $(\mathrm{K}/(\mathfrak{m}^n\mathrm{N} \cap \mathrm{K}))$ and $(\mathrm{K}/\mathfrak{m}^n\mathrm{K})$ are pro-isomorphic. The Artin-Rees lemma (`00IN`) says that there exists a c so that $\mathfrak{m}^n\mathrm{K} \subset \mathfrak{m}^n\mathrm{N} \cap \mathrm{K} \subset \mathfrak{m}^{n-c}\mathrm{K}$ for $n \geq c$, which is enough to conclude.

6.6.8. Typo: $f^*\mathrm{K} \simeq \underline{\hat{\mathrm{C}}}$, not $f^*\mathrm{K} \simeq \underline{\mathrm{C}}$. The proof implies the following statement:

Lemma. — Let X *be a w-strictly local affine scheme. Fix* K *in* $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{proét}}, \hat{\mathrm{R}})$ *with* $\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = \underline{\mathrm{C}}_n$ *for* C_n *in* $\mathrm{D}_{\mathrm{perf}}(\mathrm{R}/\mathfrak{m}^n)$*. Then* $\mathrm{K} \simeq \underline{\hat{\mathrm{C}}}$ *for* $\mathrm{C} = \mathrm{R}\lim \mathrm{C}_n$.

That K_n is locally constant on $X_{\text{ét}}$ was proved in Lemma 6.5.6. Each $K_n = \nu^* K'_n$ for some étale sheaf K'_n in $D^+(X_{\text{ét}}, R/\mathfrak{m}^n)$. As ν^* commutes with étale localization and $\nu_* \underline{C}_n = \nu_* \nu^* \underline{C}_n = \underline{C}_n$ (099V), K'_n is étale locally constant with value C_n (i.e. over an étale cover we have $K'_n \simeq \underline{C}_n$). To apply Lemma 6.6.7, recall $f^*_{\text{proét}} \nu^* = \nu^* f^*_{\text{ét}}$ when f is a map in $X_{\text{proét}}$ (c.f. the note to Lemma 6.3.16). In conclusion, we get $f^* K_n \simeq \pi^* \underline{C}_n$. Therefore $L_n := \pi_* f^* K_n \simeq \pi_* \pi^* \underline{C}_n = \underline{C}_n$ by Lemma 6.6.1.

$L := \pi_* f^* K = R \lim L_n$ (f^* commutes with $R\lim$ since it is a localization map in $X_{\text{proét}}$, so commutes with all limits and colimits). (Note that by the lemma in the note to Lemma 6.5.11 1), whether we restrict $f^* K_n$ to $D(Y_{\text{proét}}, \hat{R})$ and then apply direct image along $D(Y_{\text{proét}}, \hat{R}) \to D(\pi_0(Y)_{\text{proét}}, \hat{R})$, or apply direct image along $D(Y_{\text{proét}}, R/\mathfrak{m}^n) \to D(\pi_0(Y)_{\text{proét}}, R/\mathfrak{m}^n)$ and then restrict to $D(\pi_0(Y)_{\text{proét}}, \hat{R})$, we get the same thing.)

We have morphisms of topos $(Y_{\text{proét}}, \hat{R}) \xrightarrow{\pi} (\pi_0(Y)_{\text{proét}}, \hat{R}) \xrightarrow{e} (*, R)$. Since $\underline{\hat{C}} = e^* \underline{C}$ on $\pi_0(Y)_{\text{proét}}$ by Lemma ♦ in the note to Corollary 6.5.7, $\pi^* \underline{\hat{C}} = \underline{\hat{C}}$ on $Y_{\text{proét}}$. As $\pi^* L \simeq \pi^* \underline{\hat{C}}$ is therefore in $D_{\text{cons}}(Y_{\text{proét}}, \hat{R})$, to demonstrate $f^* K \simeq \underline{\hat{C}}$ it will suffice by completeness to find a morphism $\pi^* L \to f^* K$ inducing an isomorphism modulo $\mathfrak{m}$. This morphism comes by adjunction from the definition $L := \pi_* f^* K$.

6.6.10. 3. Y must be coherent (qcqs) to get the desired stratification (03S0).

'As $Z \to X$ is finite, the function f_Z is upper semicontinuous' $\rightsquigarrow$ Let $\pi : Z \to X$ denote the finite morphism. Then $\pi_* \mathcal{O}_Z$ is a coherent $\mathcal{O}_X$-module of finite type, and as $U \to X$ is étale, $\dim_{k(x)}(\pi_* \mathcal{O}_Z \otimes k(x)) = f_Z(x)$.[61] We conclude from the following

Lemma. — Given a scheme X *and coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$*, the function*

$$x \mapsto \dim_{k(x)}(\mathcal{F} \otimes k(x))$$

is upper semicontinuous.

Proof. — Suppose $X = \operatorname{Spec} A$, $x \in X$, and $\mathcal{F} = \tilde{M}$ for M a finite A-module. Nakayama's lemma says that $n := \dim_{k(x)}(M \otimes k(x))$ is the minimal number of generators of M_x

[61] The scheme-theoretic fiber $\pi^{-1}(x)$ is a disjoint union of spectra of finite separable field extensions of $k(x)$, and the dimension of any one of these field extensions over $k(x)$ equals the number of geometric points in the geometric fiber sitting over that point.

as an A_x-module. Any such extend to sections over some neighborhood of x, so we may shrink X and assume there is a map $m : \oplus_{i=1}^n A \to M$ that induces a surjection after localizing at x. If $K := \mathrm{coker}(m)$, $K_x = 0$. As K is finitely generated, there is a neighborhood U of x over which K vanishes; i.e. over which m surjects. For every $y \in U$, therefore, $\dim_{k(y)}(M \otimes k(y)) \leq n$. □

The constancy of the function f_Z certainly implies the constancy of the function $x \mapsto |\pi^{-1}(x)|$, which in turn implies the constancy of Z.

6.6.11. For the reverse: Lemma 6.5.8 and the fact that $D_{\mathrm{cons}}(X_{\text{proét}}, \hat{R})$ is triangulated imply that it suffices to show K is constructible when it is, locally on $X_{\text{proét}}$, the constant $\hat{R}$-complex associated to a perfect R-complex (no need to assume X affine). As remarked, completeness is local for the pro-étale topology. It remains to show that $K_1 = \nu^*$ of something in $D_{\mathrm{cons}}(X_{\text{ét}}, R/\mathfrak{m})$. By the argument in the first paragraph of `099Y`, replacing X by a connected component, we may assume that there exists a pro-étale cover $Y \to X$ so that $K|_Y \simeq \underline{\hat{L}}$ for L in $D_{\mathrm{perf}}(R)$, and therefore also $K_1|_Y \simeq \underline{L}_1$. So $K_1 = \nu^* K_1'$ for some K_1' in $D^b_{flc}(X_{\text{ét}}, R/\mathfrak{m})$ by `099Z`. In order to conclude that $K_1' \simeq \underline{M}$ locally on $X_{\text{ét}}$ with M in $D_{\mathrm{perf}}(R/\mathfrak{m})$, it suffices to show that K_1' has finite tor-dimension (`09BI`). Let $\mathcal{F}$ be in $\mathrm{Mod}(X_{\text{ét}}, R/\mathfrak{m})$; then $\nu^*(K_1' \otimes^{\mathbf{L}}_{R/\mathfrak{m}} \mathcal{F}) = K_1 \otimes^{\mathbf{L}}_{R/\mathfrak{m}} \nu^*\mathcal{F}$ (`07A4`). Let's say L can be represented by a finite complex of projective R-modules zero outside degrees $[a, b]$. Then $K_1 \otimes^{\mathbf{L}}_{R/\mathfrak{m}} \nu^*\mathcal{F}$ is acyclic off $[a, b]$, since if P is a finite projective $R/\mathfrak{m}$-module, $\underline{P}$ is flat in $\mathrm{Mod}(X_{\text{proét}}, R/\mathfrak{m})$.[62] We conclude that $K_1' \otimes^{\mathbf{L}}_{R/\mathfrak{m}} \mathcal{F}$ is acyclic off $[a, b]$.[63] In conclusion, K_1' is in $D_{\mathrm{cons}}(X_{\text{ét}}, R/\mathfrak{m})$, so K is in $D_{\mathrm{cons}}(X_{\text{proét}}, \hat{R})$.

6.6.12. 'X represents $\hat{R}$' $\rightsquigarrow$ Let Y be a profinite set; then

$$\mathrm{Maps}_{\mathrm{cont}}(Y, X) = \lim_n \mathrm{Maps}_{\mathrm{cont}}(Y, X_n), \quad \text{while}$$
$$\hat{R}(Y) = \lim_n \mathbf{Z}/\ell^n(Y) = \lim_n \mathrm{Maps}_{\mathrm{cont}}(Y, \underline{\mathbf{Z}/\ell^n}),$$

where the last isomorphism is by Lemma 4.2.12.

[62]Write $(R/\mathfrak{m})^{\oplus m} = P \oplus Q$; then if f is an injection in $\mathrm{Mod}(X_{\text{ét}}, R/\mathfrak{m})$, $0 = \ker(f \otimes_{R/\mathfrak{m}} (R/\mathfrak{m})^{\oplus m}) = \ker(f \otimes_{R/\mathfrak{m}} P) \oplus \ker(f \otimes_{R/\mathfrak{m}} Q)$.

[63]If N is in $D(X_{\text{ét}}, \Lambda)$ and $\nu : (X_{\text{proét}}, \Lambda) \to (X_{\text{ét}}, \Lambda)$, then ν^* is exact so $\nu^* H^i N = H^i \nu^* N$; if this vanishes, then $H^i N = 0$ (`099T`).

The maps $\mathrm{K}_{n+1} \to \mathrm{K}_n$ are determined by the pullback of maps on X_{n+1}: the restriction of K'_{n+1} to $\alpha \in \mathrm{X}_{n+1}$ is given by the K-flat complex $(\mathbf{Z}/\ell^{n+1} \xrightarrow{\alpha} \mathbf{Z}/\ell^{n+1})$, and so $\mathrm{K}'_{n+1}|_\alpha \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}} \mathbf{Z}/\ell$ is the complex $(\mathbf{Z}/\ell^n \to \mathbf{Z}/\ell^n)$ where the map is given by multiplication by the image of α in $\mathbf{Z}/\ell^n$: this is the restriction to α of the pullback of K'_n to X_{n+1}. For the same reasons, the transition morphisms $\mathrm{K}_{n+1} \to \mathrm{K}_n$ are given by the reduction modulo ℓ^n of complexes; in other words, K_n is given by a two-term complex, say, in $[0,1]$, and $\mathrm{K}^p_{n+1} \twoheadrightarrow \mathrm{K}^p_n$ surjects for $p \in [0,1]$. Therefore the termwise limit computes $\mathrm{R}\lim \mathrm{K}_n$ in $\mathrm{D}(\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$ by Proposition 3.1.10 and Leray's acyclicity lemma, and K is represented by a two-term complex of the form $\hat{\mathrm{R}} \to \hat{\mathrm{R}}$ on $\mathrm{X}_{\text{proét}}$. This, or Lemma 3.5.5, guarantees $\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathbf{Z}/\ell^n = \mathrm{K}_n$. Given $\alpha \in \mathbf{Z}_\ell$ determining f_α, f_α^* commutes with limits (`09AA`), so $f_\alpha^*\mathrm{K} = \lim_n(\mathbf{Z}/\ell^n \xrightarrow{\overline{\alpha}} \mathbf{Z}/\ell^n) = (\mathbf{Z}_\ell \xrightarrow{\alpha} \mathbf{Z}_\ell)$, since the image of α in X_n is $\overline{\alpha} \in \mathbf{Z}/\ell^n$.

6.6.13. '$\mathrm{R}\Gamma(\mathrm{Y}_{\text{ét}}, \mathrm{M}) \simeq \mathrm{M}$' $\rightsquigarrow$ Let $(f_i : \mathrm{X}_i \to \mathrm{X})$ be the cofiltered system of finite étale coverings of X, so $\mathrm{Y} = \lim_i \mathrm{X}_i$. Translating into Galois cohomology, let η be a geometric point of X, say, centered on the generic point. Then X_i correspond to open normal subgroups $\mathrm{U}_i \subset \pi_1(\mathrm{X}, \eta) =: \mathrm{G}$, and $f_{i*}\mathrm{M} = \mathrm{CoInd}^{\mathrm{G}/\mathrm{U}_i}_{\{e\}} \mathrm{M} = \mathrm{CoInd}^{\mathrm{G}}_{\mathrm{U}_i} \mathrm{M}$. We have maps $f_{i*}\mathrm{M} \to f_{j*}\mathrm{M}$ whenever $j \to i$ coming from the units of adjunction, and $f_*\mathrm{M} = \mathrm{colim}_i\, f_{i*}\mathrm{M}$ (`0EYM`). So

$$\begin{aligned} \mathrm{R}\Gamma(\mathrm{X}_{f\text{ét}}, f_*\mathrm{M}) &= \mathrm{R}\Gamma(\mathrm{G}, \mathrm{colim}_i\, \mathrm{CoInd}^{\mathrm{G}/\mathrm{U}_i}_{\{e\}} \mathrm{M}) \\ &= \mathrm{colim}_j\, \mathrm{R}\Gamma(\mathrm{G}/\mathrm{U}_j, (\mathrm{colim}_i\, \mathrm{CoInd}^{\mathrm{G}}_{\mathrm{U}_i} \mathrm{M})^{\mathrm{U}_j}) \\ &= \mathrm{colim}_j\, \mathrm{R}\Gamma(\mathrm{G}/\mathrm{U}_j, \mathrm{CoInd}^{\mathrm{G}}_{\mathrm{U}_j} \mathrm{M}) = \mathrm{M}, \end{aligned}$$

where the first isomorphism is because X is a $\mathrm{K}(\pi, 1)$, the second isomorphism can be found in Weibel, Theorem 6.11.13, and the last isomorphism is Shapiro's lemma.

'$\mathrm{D}(\mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}(\mathrm{Y}_{\text{proét}}, \mathrm{R}/\mathfrak{m})$ is fully faithful' $\rightsquigarrow$ for M in $\mathrm{D}(\mathrm{R}/\mathfrak{m}^n)$,

$$\begin{aligned} \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \underline{\mathrm{M}}) &= \mathrm{R}\lim \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \tau_{\geq -n}\underline{\mathrm{M}}) = \mathrm{R}\lim \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \nu^*\tau_{\geq -n}\underline{\mathrm{M}}) \\ &= \mathrm{R}\lim \mathrm{R}\Gamma(\mathrm{Y}_{\text{ét}}, \underline{\tau_{\geq -n}\mathrm{M}}) = \mathrm{R}\lim \tau_{\geq -n}\mathrm{M} = \mathrm{M}, \end{aligned}$$

since $\mathrm{R}\Gamma(\mathrm{Y}_{\text{ét}}, \mathrm{M}) \simeq \mathrm{M}$ for M in $\mathrm{Mod}_{\mathrm{R}/\mathfrak{m}^n}$ implies the same for M in $\mathrm{D}^+(\mathrm{R}/\mathfrak{m}^n)$.

'$f^*\mathrm{K}_n \simeq \underline{\mathrm{C}}_n$' $\rightsquigarrow$ The claim is that if Λ is a noetherian ring and L is in $\mathrm{D}^b(\mathrm{Y}_{\text{proét}}, \Lambda)$ with $\mathcal{H}^i(\mathrm{L})$ equal to the constant sheaf associated to a finite Λ-module, then $\mathrm{L} = \underline{\mathrm{M}}$ for some M in $\mathrm{D}(\Lambda)$ (then automatically $\mathrm{M} = \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \mathrm{L})$). Immediate when L is concentrated in one degree, since in that case $\mathcal{H}^i(\mathrm{L})$ is constant (as $\pi_1(\mathrm{Y}, \xi) = 1$). In general, say L has amplitude in $[a, b]$. We have the distinguished triangle

$$\tau_{<b}\mathrm{L} \longrightarrow \mathrm{L} \longrightarrow \mathcal{H}^b\mathrm{L} \longrightarrow \tau_{<b}\mathrm{L}[1],$$

and the inductive hypothesis tells us that both $\mathcal{H}^b\mathrm{L}$ and $\tau_{<b}\mathrm{L}[1]$ are pulled back from $\mathrm{D}(\Lambda)$, so we're done if $\mathcal{H}^b\mathrm{L} \to \tau_{<b}\mathrm{L}[1]$ is the pullback of some morphism $\mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \mathcal{H}^b\mathrm{L}) \longrightarrow \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \tau_{<b}\mathrm{L}[1])$. This is immediate by the just-remarked fact that the pullback is fully faithful.

Now that we know that $f^*\mathrm{K}_n$ is in the essential image of $q^* : \mathrm{D}(\mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}(\mathrm{Y}_{\text{proét}}, \mathrm{R}/\mathfrak{m}^n)$, it's immediate that the counit $\underline{\mathrm{C}}_n := q^*q_*\mathrm{K}_n \to \mathrm{K}_n$ is an isomorphism,[64] and, as it is natural in n, induces an isomorphism of inverse systems. Since x^* is naturally isomorphic to $\mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, -)$ as functors on the essential image of the pullback $\mathrm{D}(\mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}(\mathrm{Y}_{\text{proét}}, \mathrm{R}/\mathfrak{m}^n)$ (x a geometric point of Y), we find that the maps $\mathrm{C}_{n+1} \to \mathrm{C}_n$ in $\mathrm{D}(\mathrm{R}/\mathfrak{m}^{n+1})$ found by adjunction $\mathrm{Hom}(\underline{\mathrm{C}}_{n+1}, \underline{\mathrm{C}}_n) = \mathrm{Hom}(\mathrm{C}_{n+1}, \mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, \underline{\mathrm{C}}_n)) = \mathrm{Hom}(\mathrm{C}_{n+1}, \mathrm{C}_n)$ induce isomorphisms $\mathrm{C}_{n+1} \otimes^{\mathbf{L}}_{\mathrm{R}/\mathfrak{m}^{n+1}} \mathrm{R}/\mathfrak{m}^n \xrightarrow{\sim} \mathrm{C}_n$. Then $\mathrm{R}\lim \mathrm{C}_n$ is perfect by Lemma $\Diamond$ in the note to Lemma 6.5.7.

6.7.1. $\boxed{\mathrm{D}_{\text{cons}} = \mathrm{D}^b_{ctf}}$ On a scheme X with underlying topological space which is noetherian, the notion of constructibility given in Definition 6.3.1 agrees with the classical notion of $\mathrm{D}^b_{ctf}(\mathrm{X}_{\text{ét}}, \Lambda)$ ('constructible + finite tor-dimension') when the coefficient ring Λ is constant and noetherian (`09BI`). Gabber shows in the non-proper case that under the given hypotheses, the $\mathrm{R}^i f_*$ are constructible and zero for $i \gg 0$. As f_* has finite cohomological dimension, it also preserves finite tor-dimension [**SGAA**, XVII 5.2.11].[65]

In the proper case, where the schemes are no longer presumed noetherian, it remains to show that f_* preserves constructibility in the sense of Definition 6.3.1.

[64]Indeed, $q^* \to q^*q_*q^* \to q^*$ is the identity.

[65]Note in the proof of [**SGAA**, XVII 5.2.11], both functors under consideration are way-out *left,* not right.

6.7.4. As $D_{\mathrm{comp}}(X_{\mathrm{proét}}, \hat{R}) = D_{\mathrm{comp}}(X_{\mathrm{proét}}, R)$, let's suppose the coefficients are constant with value R (so that $g^*(R/\mathfrak{m}^n) = R/\mathfrak{m}^n$), let $K_n := \nu_*(K \otimes_R R/\mathfrak{m}^n)$, and write

$$\begin{aligned} g^*_{\mathrm{comp}} f_* K = R\lim((g^* f_* K) \otimes_R R/\mathfrak{m}^n) &= R\lim g^*(f_* K \otimes_R R/\mathfrak{m}^n) \\ = R\lim g^* f_*(K \otimes_R R/\mathfrak{m}^n) &= R\lim g^* f_* \nu^* K_n \\ = R\lim \nu^* g^*_{\text{ét}} f_{\text{ét},*} K_n &= R\lim \nu^* f_{\text{ét},*} g^*_{\text{ét}} K_n \\ = R\lim f_* g^* \nu^* K_n &= R\lim f_* g^*(K \otimes_R R/\mathfrak{m}^n) \\ = R\lim f_*(g^* K \otimes_R R/\mathfrak{m}^n) &= f_* g^*_{\mathrm{comp}} K. \end{aligned}$$

6.7.7. We're in the following situation: we have compactifications $\overline{X}_1, \overline{X}_2$ of f and a morphism of compactifications (over Y)

$$\begin{tikzcd} & \overline{X}_2 \arrow[d, "p"] \\ X \arrow[ur, "j_2"] \arrow[r, "j_1"'] & \overline{X}_1. \end{tikzcd}$$

It will suffice to show that $p_* j_{2!} = j_{1!} : D_{\mathrm{cons}}(X_{\mathrm{proét}}, \hat{R}) \to D_{\mathrm{cons}}(\overline{X}_{1\,\mathrm{proét}}, \hat{R})$. If $i : Z_1 := \overline{X}_1 \smallsetminus X \hookrightarrow \overline{X}_1$ and $Z_2 := \overline{X}_2 \smallsetminus X$, both squares

$$\begin{tikzcd} X \arrow[r, "j_2"] \arrow[d, "\mathrm{id}"] & \overline{X}_2 \arrow[d, "p"] & Z_2 \arrow[l, hook', "i"'] \arrow[d, "p"] \\ X \arrow[r, "j_1"] & \overline{X}_1 & Z_1 \arrow[l, hook', "i"'] \end{tikzcd}$$

are cartesian, and we have that $j_1^* p_* j_{2!} = j_2^* j_{2!} = \mathrm{id}$ and $i^* p_* j_{2!} = p_* i^* j_{2!} = 0$ as functors $D_{\mathrm{cons}}(X_{\mathrm{proét}}, \hat{R}) \to D_{\mathrm{cons}}(X_{\mathrm{proét}}, \hat{R})$ by proper base change (Lemma 6.7.5; n.b. $j^* = j^*_{\mathrm{comp}}$ and $i^* = i^*_{\mathrm{comp}}$ by Lemma 6.5.8) and Lemma 6.1.12. We conclude by the standard distinguished triangle on $\overline{X}_1$ of Lemma 6.1.11.

6.7.8. As $Y = \operatorname{Spec} k$ is w-contractible when k is algebraically closed, $\Gamma(Y_{\mathrm{proét}}, -)$ is an exact functor on abelian sheaves. Therefore the derived functor of $F \mapsto \Gamma(\overline{X}_{\mathrm{proét}}, j_! F) = \Gamma(Y_{\mathrm{proét}}, {}^\circ f_! F)$ coincides with $R\Gamma(Y_{\mathrm{proét}}, -) \circ f_! F$, where here ${}^\circ f_! := \mathscr{H}^0 f_!$ is the functor on abelian sheaves. If $f_! = R^\circ f_!$, we would have an isomorphism after applying $\mathrm{R}\Gamma(\mathrm{Y}_{\text{proét}}, -)$. This also implies $\mathrm{H}^0(\mathrm{Y}_{\text{proét}}, \mathrm{R}^{2\circ} f_! \mathrm{F})$ equals H^2 of the derived functor of $\mathrm{F} \mapsto \Gamma(\overline{\mathrm{X}}_{\text{proét}}, j_! \mathrm{F})$.

The formula $\Gamma(\overline{\mathrm{X}}, j_! \mathrm{F}) = \ker(\mathrm{F}(\mathrm{X}) \to \mathrm{F}(\tilde{\eta}))$ arises from applying $\Gamma(\overline{\mathrm{X}}, -)$ to the exact sequence $0 \to j_! \mathrm{F} \to j_* \mathrm{F} \to i_* i^* j_* \mathrm{F} \to 0$.

'$\mathrm{F}(\mathrm{X}) \to \mathrm{F}(\eta)$ is surjective for F injective' ⇝ `093X`.

As $\mathrm{R}\Gamma(\mathbf{A}^1_{\text{ét}}, \mathbf{Z}/n)$ is quasi-isomorphic to the complex $\mathbf{Z}/n$ concentrated in degree zero, the result follows.[66] Recall also that $\mathrm{H}^2_c(\mathbf{A}^1_{\text{ét}}, \mathbf{Z}/n) = \mathbf{Z}/n(-1)$ by Poincaré duality. As for $\mathrm{H}^1(\eta, \mathbf{Z}/n)$, there is an exact sequence (Serre, *Cohomologie Galoisienne*, Annexe – Cohomologie galoisienne des extensions transcendantes pures, §2.4, p. 122):

$$0 \to \mathrm{H}^1(k, \mathbf{Z}/n) \to \mathrm{H}^1(\eta, \mathbf{Z}/n) \to \bigoplus_{x \in |\mathbf{A}^1|} \mathrm{H}^0(k(x), \mathbf{Z}/n(-1)) \to \mathrm{H}^0(k, \mathbf{Z}/n(-1)) \to 0.$$

Here x are closed points of $\mathbf{A}^1$ with residue fields $k(x) = k$. As k is separably closed, $\mathrm{H}^1(k, \mathbf{Z}/n) = 0$ while $\mathrm{H}^0(k, \mathbf{Z}/n(-1))$ has rank 1 as a $\mathbf{Z}/n$-module. In conclusion, $\mathrm{H}^1(\eta, \mathbf{Z}/n)$ is a free $\mathbf{Z}/n$-module of infinite rank.

6.7.9. This is [**SGAA**, XVII 6.1.6].

6.7.10. We have to check g^*_{comp} commutes with $j_!$. This is true as Lemmas 6.2.3 & 6.5.8 allow us to write, for K in $\mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$,

$$\begin{aligned} j_! g^*_{\text{comp}} \mathrm{K} &= j_! \mathrm{R}\lim(g^* \mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) = \mathrm{R}\lim j_!(g^* \mathrm{K} \otimes_{\hat{\mathrm{R}}} j^*(\mathrm{R}/\mathfrak{m}^n)) \\ &= \mathrm{R}\lim(j_! g^* \mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) = \mathrm{R}\lim(g^* j_! \mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) = g^*_{\text{comp}} j_! \mathrm{K}. \end{aligned}$$

6.7.13. As I can't justify Lemma 6.3.13, we proceed along slightly different lines. Derived $\mathfrak{m}$-completeness is local on $\mathrm{X}_{\text{proét}}$, so by filtering K we may assume first that $\mathrm{K} = i_! \hat{\mathrm{M}}$ for M a finite projective R-module and i the immersion of a locally closed constructible subset.[67] Finding a complement N inside a finite free R-module and

[66] Recall that $\mathrm{H}^1(\mathbf{A}^1_{\text{ét}}, \mathbf{Z}/n) = \mathrm{Hom}(\pi_1(\mathbf{A}^1_{\text{ét}}, \overline{\eta}), \mathbf{Z}/n)$, but n was chosen invertible in k and the maximal prime-to-p quotient of $\pi_1(\mathbf{A}^1_{\text{ét}}, \overline{\eta})$ is trivial.

[67] Looking over an affine chart where i becomes a closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$, the fact that henselization is fully faithful for surjective ring maps (Lemma 2.2.12) shows that if $\mathrm{V} \to \mathrm{Z}$ is affine and pro-étale so that $\mathrm{K}|_{\mathrm{V}} = \hat{\mathrm{M}}$ (this exists by Proposition 6.6.11), then $\mathrm{K}|_{\tilde{\mathrm{V}}} = i_! \hat{\mathrm{M}}$.

observing that $\mathrm{T}(\mathrm{M} \oplus \mathrm{N}, x) = \mathrm{T}(\mathrm{M}, x) \oplus \mathrm{T}(\mathrm{N}, x)$, we reduce to $\mathrm{M} = \mathrm{R}$. The adjunction follows from

$$\begin{aligned}\mathrm{Hom}(\mathrm{A}, \mathrm{R}\mathscr{H}om(i_!\mathrm{B}, \mathrm{C})) &= \mathrm{Hom}(\mathrm{A} \otimes i_!\mathrm{B}, \mathrm{C}) = \mathrm{Hom}(i_!(i^*\mathrm{A} \otimes \mathrm{B}), \mathrm{C}) \\ &= \mathrm{Hom}(i^*\mathrm{A} \otimes \mathrm{B}, i^!\mathrm{C}) = \mathrm{Hom}(i^*\mathrm{A}, \mathrm{R}\mathscr{H}om(\mathrm{B}, i^!\mathrm{C})) \\ &= \mathrm{Hom}(\mathrm{A}, i_*\mathrm{R}\mathscr{H}om(\mathrm{B}, i^!\mathrm{C})),\end{aligned}$$

where the second isomorphism is Lemma 6.2.3 3.

Now we prove the second assertion, which is local after we define a map. Let $r : (\mathrm{X}_{\text{proét}}, \mathrm{R}/\mathfrak{m}^n) \to (\mathrm{X}_{\text{proét}}, \hat{\mathrm{R}})$ be the morphism of ringed topos defined by $\hat{\mathrm{R}} \to \mathrm{R}/\mathfrak{m}^n$. Then $\mathrm{A} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = g^*\mathrm{A}$ while $g_*\mathrm{A}$ is given by restriction. As $g_*\mathrm{R}\mathscr{H}om_{\mathrm{R}/\mathfrak{m}^n}(g^*\mathrm{K}, g^*\mathrm{L}) = \mathrm{R}\mathscr{H}om_{\hat{\mathrm{R}}}(\mathrm{K}, g_*g^*\mathrm{L})$,

$$\begin{aligned}&\mathrm{Hom}(g^*\mathrm{R}\mathscr{H}om_{\hat{\mathrm{R}}}(\mathrm{K}, \mathrm{L}), \mathrm{R}\mathscr{H}om_{\mathrm{R}/\mathfrak{m}^n}(g^*\mathrm{K}, g^*\mathrm{L})) \\ =\,&\mathrm{Hom}(\mathrm{R}\mathscr{H}om_{\hat{\mathrm{R}}}(\mathrm{K}, \mathrm{L}), \mathrm{R}\mathscr{H}om_{\hat{\mathrm{R}}}(\mathrm{K}, g_*g^*\mathrm{L})).\end{aligned}$$

There's an evident such morphism given by the unit $\mathrm{L} \to g_*g^*\mathrm{L}$. Checking this map is an isomorphism locally, we're reduced as before to showing

$$(i_*i^!\mathrm{L}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = \mathrm{R}\mathscr{H}om_{\mathrm{R}/\mathfrak{m}^n}(i_!\hat{\mathrm{R}} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n, \mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n),$$

where i is a constructible closed immersion. As $(i_!\hat{\mathrm{R}}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = i_!(\hat{\mathrm{R}} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) = i_!\mathrm{R}/\mathfrak{m}^n$ by Lemma 6.2.3 3, the right-hand side is $i_*i^!(\mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n)$. Then we simply write the morphism of distinguished triangles

$$\begin{tikzcd}
(i_*i^!\mathrm{L}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n \arrow[r] \arrow[d] & \mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n \arrow[r] \arrow[d] & (j_*j^*\mathrm{L}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n \arrow[r] \arrow[d] & {} \\
i_*i^!(\mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) \arrow[r] & \mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n \arrow[r] & j_*j^*(\mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n) \arrow[r] & {}
\end{tikzcd}.$$

As $j^*\hat{\mathrm{R}} = \hat{\mathrm{R}}$ (Lemma 6.5.8), we need only check $j_*j^*\mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = j_*(j^*\mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n)$, which follows by Lemma 6.5.11 in light of Theorem 6.7.1.

To finish the proof that $\mathrm{R}\mathscr{H}om_{\hat{\mathrm{R}}}(\mathrm{K}, \mathrm{L})$ is constructible, we use what we just showed to write

$$\mathrm{R}\mathscr{H}om_{\hat{\mathrm{R}}}(\mathrm{K}, \mathrm{L}) \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m} = \mathrm{R}\mathscr{H}om_{\mathrm{R}/\mathfrak{m}}(\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}, \mathrm{L} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}).$$

Lemma 6.3.16 shows that the right-hand side is ν^* of $\mathrm{R}\mathscr{H}om$ of two objects of $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m})$, and so the conclusion follows from the classical version (in light of $\mathrm{D}_{\mathrm{cons}} = \mathrm{D}^b_{ctf}$ as discussed in the note to Theorem 6.7.1).

6.7.14. We first establish some derived complete versions of standard identities.

Lemma. — Let $(\mathscr{C}, \mathcal{O})$ *be a ringed site and* $\mathscr{I} \subset \mathcal{O}$ *a finite type sheaf of ideals. Let* A *be in* $\mathrm{D}(\mathcal{O})$ *and* B *be in* $\mathrm{D}_{\mathrm{comp}}(\mathcal{O})$. *Then* $\mathrm{R}\mathscr{H}om(\mathrm{A}, \mathrm{B}) = \mathrm{R}\mathscr{H}om(\mathrm{A}^\wedge, \mathrm{B})$.

Proof. — There is a map coming from $\mathrm{A} \to \mathrm{A}^\wedge$, and it suffices to check that it induces an isomorphism after applying $\mathrm{R}\Gamma(\mathrm{X}, -)$ for any X in $\mathscr{C}$. This is an isomorphism as localization commutes with derived completion (`0A0F`), so $\mathrm{R}\,\mathrm{Hom}(j^*\mathrm{A}, j^*\mathrm{B}) = \mathrm{R}\,\mathrm{Hom}(j^*(\mathrm{A}^\wedge), j^*\mathrm{B})$ as derived completion is left adjoint to the inclusion (`099F`), where $j : \mathscr{C}/\mathrm{X} \to \mathscr{C}$ is the localization morphism. □

Corollary. — Let $f : (\tilde{\mathscr{C}}, \mathcal{O}) \to (\tilde{\mathscr{D}}, \mathcal{O}')$ *be a morphism of ringed topos with* $(\mathscr{C}, \mathcal{O})$ *as above. Let* A *be in* $\mathrm{D}(\mathcal{O}')$ *and* B *be in* $\mathrm{D}_{\mathrm{comp}}(\mathcal{O})$. *Then*

$$\mathrm{R}\mathscr{H}om(\mathrm{A}, f_*\mathrm{B}) = f_*\mathrm{R}\mathscr{H}om(f^*_{\mathrm{comp}}\mathrm{A}, \mathrm{B}).$$

Lemma. — Let $f : (\tilde{\mathscr{D}}, \mathcal{O}') \to (\tilde{\mathscr{C}}, \mathcal{O})$ *be a morphism of ringed topos, let* $\mathscr{I} \subset \mathcal{O}$ *and* $\mathscr{I}' \subset \mathcal{O}'$ *be finite type sheaves of ideals so that* $f^\sharp$ *sends* $(f^{-1}\mathscr{I})$ *into* $\mathscr{I}'$. *Then*

$$f^*_{\mathrm{comp}}(\mathrm{A}\hat{\otimes}_{\mathcal{O}}\mathrm{B}) = f^*_{\mathrm{comp}}\mathrm{A}\hat{\otimes}_{\mathcal{O}'} f^*_{\mathrm{comp}}\mathrm{B}$$

whenever A *and* B *are in* $\mathrm{D}_{\mathrm{comp}}(\mathcal{O})$.

Proof. — If C is in $\mathrm{D}_{\mathrm{comp}}(\mathcal{O}')$, then by `099G`, `099K` & `099A` we find

$$\begin{aligned}\mathrm{Hom}(f^*_{\mathrm{comp}}(\mathrm{A}\hat{\otimes}_{\mathcal{O}}\mathrm{B}), \mathrm{C}) &= \mathrm{Hom}(\mathrm{A}\hat{\otimes}_{\mathcal{O}}\mathrm{B}, f_*\mathrm{C}) = \mathrm{Hom}(\mathrm{A}, \mathrm{R}\mathscr{H}om(\mathrm{B}, f_*\mathrm{C}))\\ &= \mathrm{Hom}(\mathrm{A}, f_*\mathrm{R}\mathscr{H}om(f^*_{\mathrm{comp}}\mathrm{B}, \mathrm{C}))\\ &= \mathrm{Hom}(f^*_{\mathrm{comp}}\mathrm{A}, \mathrm{R}\mathscr{H}om(f^*_{\mathrm{comp}}\mathrm{B}, \mathrm{C}))\\ &= \mathrm{Hom}(f^*_{\mathrm{comp}}\mathrm{A}\hat{\otimes}_{\mathcal{O}'} f^*_{\mathrm{comp}}\mathrm{B}, \mathrm{C}).\end{aligned}$$ □

Now we can write down the natural map $f_!\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}}\mathrm{L} \to f_!(\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}} f^*_{\mathrm{comp}}\mathrm{L})$. Suppose f factors as $\mathrm{X} \xhookrightarrow{j} \overline{\mathrm{X}} \xrightarrow{\overline{f}} \mathrm{Y}$. As $f^*_{\mathrm{comp}}\mathrm{L} = j^*\overline{f}^*_{\mathrm{comp}}\mathrm{L}$ (since j^* commutes with completion), the ordinary projection formula of Lemma 6.2.3, the fact that $j_!$ commutes with derived completion (this follows from the projection formula for $j_!$ and the fact that $j_!$ commutes with homotopy-limits) and the above combine to give

$$\begin{aligned}\mathrm{Hom}(f_!\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}}\mathrm{L}, f_!(\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}} f^*_{\mathrm{comp}}\mathrm{L})) &= \mathrm{Hom}(\overline{f}^*_{\mathrm{comp}}(f_!\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}}\mathrm{L}), j_!(\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}} f^*_{\mathrm{comp}}\mathrm{L})) \\ &= \mathrm{Hom}(\overline{f}^*_{\mathrm{comp}} f_!\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}}\overline{f}^*_{\mathrm{comp}}\mathrm{L}), j_!\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}}\overline{f}^*_{\mathrm{comp}}\mathrm{L})).\end{aligned}$$

The map is induced by the counit $\overline{f}^*_{\mathrm{comp}}\overline{f}_* \to \mathrm{id}$.

‘Filtering L’ ⇝ this is discussed in more detail in the note to Lemma 6.7.13.

‘$\mathrm{L} \otimes_{\hat{\mathrm{R}}} i_*\hat{\mathrm{R}} \simeq i_*i^*\mathrm{L}$’ ⇝ j^* of the left hand side vanishes, and i^* of it equals $i^*\mathrm{L}$.

To conclude, $f_!\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}} i_*\hat{\mathrm{R}} = i_*i^*f_!\mathrm{K}$ while

$$f_!(\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}} f^*_{\mathrm{comp}} i_*\hat{\mathrm{R}}) = f_!(\mathrm{K}\hat{\otimes}_{\hat{\mathrm{R}}} i_*\hat{\mathrm{R}}) = f_! i_* i^*\mathrm{K},$$

and we conclude by Lemma 6.7.10. (The first isomorphism above follows from Lemma 6.7.5 and the fact that $f^*_{\mathrm{comp}}\hat{\mathrm{R}} = \hat{\mathrm{R}}$.)[68]

6.7.16. Typo in statement: M is in $\mathrm{D}^b(\mathrm{R}/\mathfrak{m}^n)$.

Assuming first this is a statement in $\mathrm{X}_{\text{proét}}$ rather than $\mathrm{X}_{\text{ét}}$, we can apply Lemma 6.5.11 2 as $\overline{f}_* : \mathrm{Mod}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}) \to \mathrm{Mod}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m})$ has finite cohomological dimension since $\overline{f}$ is proper. The $j_!$ version follows from the projection formula of Lemma 6.2.3 3. To produce the statement in $\mathrm{X}_{\text{proét}}$ from the statement in $\mathrm{X}_{\text{ét}}$, we need to apply ν^* to both sides and invoke Lemmas 5.4.3, 6.2.3 4 & 6.3.4, and `07A4`. Then to go back, both sides of the original equality lie in $\mathrm{D}^b(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$ by Theorem 6.7.1 and the finite tor-dimension of objects in $\mathrm{D}_{\text{cons}}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$ (Lemma 6.3.4), so we're done by `099V`.

6.7.18. The commutation of the squares is evidently up to natural isomorphism. Let $g_{mn} : (?_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n) \to (?_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m)$ by the change of coefficients morphism for ? = X, Y. We're seeking an isomorphism $f_{n!}g^*_{mn} = f_{n!}(\mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n) = f_{m!}\mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n = g^*_{mn}f_{m!}$ in $\mathrm{D}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$. The formula $j_!(\mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n) = j_!\mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n$ reduces us to the case f proper. If $f^*_n g^*_{mn} = g^*_{mn} f^*_m : \mathrm{D}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m) \to \mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$, then the counit gives an arrow $f^*_n g^*_{mn} f_{m*} = g^*_{mn} f^*_m f_{m*} \to g^*_{mn}$ and hence a morphism

[68] $f^*\hat{\mathrm{R}} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = f^{-1}\hat{\mathrm{R}} \otimes_{f^{-1}\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = \mathrm{R}/\mathfrak{m}^n$.

$g_{mn}^* f_{m*} \to f_{n*} g_{mn}^*$, and whether this morphism is an isomorphism can be checked after restricting to $\mathrm{D}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m)$.

To see $f_n^* g_{mn}^* = g_{mn}^* f_m^*$, we have the commutative diagram of ringed topos

$$\begin{array}{ccc} (\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n) & \xrightarrow{g_{mn}} & (\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m) \\ \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_m} \\ (\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n) & \xrightarrow{g_{mn}} & (\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m), \end{array}$$

so $(g_{mn} \circ f_n)_* = (f_m \circ g_{mn})_*$ and in order to show $g_{mn*} f_{n*} = f_{m*} g_{mn*}$ it suffices to show that $g_{mn*} f_{n*} = (g_{mn} \circ f_n)_*$ and $(f_m \circ g_{mn})_* = f_{m*} g_{mn*}$ as functors $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n) \to \mathrm{D}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$, which follows by `072Z`, `0731` & `07K7`.

After restricting to $\mathrm{D}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$, by the above we can write

$$g_{mn*} f_{n*} g_{mn}^* \mathrm{K} = f_{m*} g_{mn*} g_{mn}^* \mathrm{K} = f_{m*}(\mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n) = f_{m*} \mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n = g_{mn*} g_{mn}^* f_{m*}$$

by the projection formula in étale cohomology, since f is proper.

As for the commutativity of the square on the right, fix K in $\mathrm{D}_{\text{cons}}(\mathrm{Y}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m)$. The map $f_m^! \mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n \to f_n^!(\mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n)$ in $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$ comes from adjunction via the counit

$$f_{n!}(f_m^! \mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n) = f_{m!} f_m^! \mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n \to \mathrm{K} \otimes_{\mathrm{R}/\mathfrak{m}^m} \mathrm{R}/\mathfrak{m}^n,$$

where the first isomorphism is what we just established.

When $f^!$ is replaced by g^*, the isomorphism follows since both g_m^* and g_n^* are given simply by g^{-1} on the underlying abelian sheaves. For j_*, Lemma 6.7.17 allows us to conclude an isomorphism in $\mathrm{D}(\mathrm{P}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^m)$, but as restriction is conservative, this allows us to conclude that our original arrow in $\mathrm{D}(\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^n)$ is an isomorphism.

6.7.19. Lemma 3.5.5 and the commutativity of the diagram on the right of Lemma 6.7.18 give that $f^! \mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = f_n^! \mathrm{K}_n$, so $f^! \mathrm{K} = \mathrm{R}\lim f_n^! \mathrm{K}_n$ is actually complete (and constructible). To be clear: actually $f^! \mathrm{K} = \mathrm{R}\lim \nu^* f_n^! \nu_* \mathrm{K}_n$. Adjointness: we will show the existence of an isomorphism

$$f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, f^! \mathrm{L}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(f_! \mathrm{K}, \mathrm{L}).$$

Fix K in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$ and L in $\mathrm{D}_{\mathrm{cons}}(\mathrm{Y}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$. Let $g_n : (?_{\mathrm{pro\acute{e}t}}, \mathrm{R}/\mathfrak{m}^n) \to (?_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$ be the change of coefficients morphism on ? = X, Y, $\mathrm{K}_n = g_n^* \mathrm{K}$, and write

$$\begin{aligned} f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, g_{n*} f^! \mathrm{L}_n) &= f_* g_{n*} \mathrm{R}\mathscr{H}om(\mathrm{K}_n, f^! \mathrm{L}_n) \\ &= g_{n*} f_* \mathrm{R}\mathscr{H}om(\mathrm{K}_n, f^! \mathrm{L}_n) \quad \text{(note to Lemma 6.5.11 1)} \\ &= g_{n*} \mathrm{R}\mathscr{H}om(f_! \mathrm{K}_n, \mathrm{L}_n) \qquad (\mathtt{0GLC}) \\ &= \mathrm{R}\mathscr{H}om(f_! \mathrm{K}, g_{n*} \mathrm{L}_n), \end{aligned}$$

where here $f^! \mathrm{L}_n$ means $\nu^* f^! \nu_* \mathrm{L}_n$, etc. and we use Lemma 6.3.16 implicitly. The last isomorphism follows from the

Lemma. — Let f be a separated finitely-presented map of quasi-excellent ℓ-coprime schemes and K *an object of* $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{pro\acute{e}t}}, \hat{\mathrm{R}})$. *Then* $f_! \mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n = \nu^* f_! (\nu_* (\mathrm{K} \otimes_{\hat{\mathrm{R}}} \mathrm{R}/\mathfrak{m}^n))$ *and* $f_! \mathrm{K} = \mathrm{R} \lim \nu^* f_! \nu_* \mathrm{K}_n$.

Proof. — The second statement follows from the first by Lemma 3.5.5 and the completeness of $f_! \mathrm{K}$. Write $f_!$ as $\overline{f}_* \circ j_!$; the statement for $\overline{f}_*$ follows from Lemma 5.4.3 & Lemma 6.5.11 3, while the statement for $j_!$ follows from the projection formula and $j_! \nu^* = \nu^* j_!$ of Lemma 6.2.3 3 & 4. □

It remains to check that our isomorphism $f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, g_{n*} f^! \mathrm{L}_n) = \mathrm{R}\mathscr{H}om(f_! \mathrm{K}, g_{n*} \mathrm{L}_n)$ is compatible with the transition morphisms $\mathrm{L}_n \to \mathrm{L}_{n-1}$. If $m < n$, let $g_{nm} : (?_\star, \mathrm{R}/\mathfrak{m}^m) \to (?_\star, \mathrm{R}/\mathfrak{m}^n)$ be the change of coefficients morphism for ? = X, Y and $\star$ = ét, proét. The isomorphism is clearly functorial in L_n in $\mathrm{D}^+(\mathrm{Y}_{\mathrm{\acute{e}t}}, \mathrm{R}/\mathfrak{m}^n)$, and the commutativity of the square on the left in Lemma 6.7.18 implies $f_n^! g_{nm*} \mathrm{L}_m = g_{nm*} f_m^! \mathrm{L}_m$, which allows us to write

$$\begin{aligned} f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, g_{n*} f^! g_{nm*} \mathrm{L}_m) &= f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, g_{n*} f^! g_{nm*} \mathrm{L}_m) \\ &= f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, g_{n*} g_{nm*} f^! \mathrm{L}_m) \\ &= f_* \mathrm{R}\mathscr{H}om(\mathrm{K}, g_{m*} f^! \mathrm{L}_m), \quad \text{and} \\ \mathrm{R}\mathscr{H}om(f_! \mathrm{K}, g_{n*} g_{nm*} \mathrm{L}_m) &= \mathrm{R}\mathscr{H}om(f_! \mathrm{K}, g_{m*} \mathrm{L}_m). \end{aligned}$$

Here we've implicitly used the commutation of g_{mn*} with $\nu^* : (\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^?) \to (\mathrm{X}_{\text{ét}}, \mathrm{R}/\mathfrak{m}^?)$, $? = m, n$. This is clear from the explicit description of the latter (0GLZ) since both functors are exact as functors on abelian categories.

The upshot is that we have a morphism of distinguished triangles

$$\begin{tikzcd}
f_*\mathrm{R}\mathscr{H}\!om(\mathrm{K}, f^!\mathrm{L}) \arrow[r] \arrow[d, dashed] & \prod f_*\mathrm{R}\mathscr{H}\!om(\mathrm{K}, f^!\mathrm{L}_n) \arrow[r] \arrow[d, equal] & \prod f_*\mathrm{R}\mathscr{H}\!om(\mathrm{K}, f^!\mathrm{L}_n) \arrow[r] \arrow[d, equal] & {} \\
\mathrm{R}\mathscr{H}\!om(f_!\mathrm{K}, \mathrm{L}) \arrow[r] & \prod \mathrm{R}\mathscr{H}\!om(f_!\mathrm{K}, \mathrm{L}_n) \arrow[r] & \prod \mathrm{R}\mathscr{H}\!om(f_!\mathrm{K}, \mathrm{L}_n) \arrow[r] & {}
\end{tikzcd}$$

which is an isomorphism (here $\mathrm{R}\mathscr{H}\!om(\mathrm{K}, f^!\mathrm{L}_n)$ means $\mathrm{R}\mathscr{H}\!om(\mathrm{K}, g_{n*}\nu^* f^! \nu_*\mathrm{L}_n)$, etc.). We extract the global duality $f_! \dashv f^!$ after applying $\mathrm{R}\Gamma$ and H^0.

6.7.20. Exposé XVII 'Dualité' of *Travaux de Gabber* is edited in somewhat of a strange way, insofar as Théorème 0.2 appears to be stated in a weaker form than what is proved; as far as I understand, when $\Lambda = \mathbf{Z}/n\mathbf{Z}$, the dualizing complex K is unique up to unique isomorphism (Théorèmes 5.1.1 & 6.1.1), and it isn't until we relax our coefficients to be some noetherian Λ-algebra that we get 'uniqueness up to tensor product with invertible objects,' as in Théorème 7.1.2.

We use Théorème 7.1.3 and uniqueness of K_X when $\Lambda = \mathbf{Z}/n\mathbf{Z}$ to obtain unique isomorphisms $\omega_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}} \mathbf{Z}/\ell^n \simeq \omega_n$. The transition morphisms are given by the unique dashed arrow that makes

$$\begin{tikzcd}
\omega_{n+1} \arrow[d, dashed] \arrow[dr] & \\
\omega_n \arrow[r, "\sim"] & \omega_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}} \mathbf{Z}/\ell^n
\end{tikzcd}$$

commute. Then set $\omega_\mathrm{X} := \mathrm{R}\lim \omega_n$. Lemma 3.5.5 implies $\omega_\mathrm{X} \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^n\mathbf{Z} \simeq \omega_n$ once you replace the inverse system $(\omega_n)_{n\in\mathbf{N}^\circ}$ in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \hat{\mathbf{Z}}_\ell)$ with an object of $\mathrm{D}_{\mathrm{comp}}(\mathscr{C}, \mathrm{R}.)$, where $\mathscr{C} = \mathrm{X}^{\mathbf{N}^\circ}_{\text{proét}}$ and $\mathrm{R} = (\mathbf{Z}/\ell^n)_n$. This can always be done (c.f., e.g., 0CQ9). In order to conclude ω_X is ℓ-adically complete, note that

$$\begin{tikzcd}
\omega_X \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^{n+1} \arrow[r, "\sim"] \arrow[d] & \omega_{n+1} \arrow[d] \arrow[dr] & \\
\omega_X \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^{n} \arrow[r, "\sim"] & \omega_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}} \mathbf{Z}/\ell^{n} \arrow[r, "\sim"] & \omega_n
\end{tikzcd}$$

commutes, where the bottom isomorphism of the square is just $- \otimes_{\mathbf{Z}/\ell^{n+1}} \mathbf{Z}/\ell^n$ of the top, and the top one is the isomorphism of Lemma 3.5.5. The composition of the bottom isomorphisms must then be the same isomorphism as the one from Lemma 3.5.5 because ω_n is unique up to unique isomorphism, so $\#\,\mathrm{Isom}(\omega_X \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^n, \omega_n) = 1$. So the isomorphisms of Lemma 3.5.5 give an isomorphism of inverse systems $(\omega_X \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^n)_n \simeq (\omega_n)_n$ and hence an isomorphism of homotopy-limits. So ω_X is ℓ-adically complete.

Now consider the case of general coefficients R. As $R/\mathfrak{m}$ is a finite field of characteristic ℓ, we have a map of inverse systems $(\mathbf{Z}/\ell^n)_n \to (R/\mathfrak{m}^n)_n$, hence maps $\mathbf{Z}_\ell \to R = \lim R/\mathfrak{m}^n$ and $\hat{\mathbf{Z}}_\ell \to \hat{R}$. I claim $\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \hat{R}$ is in $D_{cons}(X_{proét}, \hat{R})$ and is dualizing. By Proposition 6.6.11, as ω_X is in $D_{cons}(X_{proét}, \hat{\mathbf{Z}}_\ell)$, there is a finite (locally closed) stratification $\{X_i \hookrightarrow X\}$ so that $\omega_X|_{X_i}$ is locally isomorphic to $\underline{L} \otimes_{\mathbf{Z}_\ell} \hat{\mathbf{Z}}_\ell$ for some L in $D_{perf}(\mathbf{Z}_\ell)$. By Lemma 6.5.8, $(\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \hat{R})|_{X_i} = \omega_X|_{X_i} \otimes_{\hat{\mathbf{Z}}_\ell} \hat{R}$, so is locally isomorphic to $\underline{L} \otimes_{\mathbf{Z}_\ell} \hat{R} = \underline{L \otimes_{\mathbf{Z}_\ell} R} \otimes_R \hat{R}$, and $L \otimes_{\mathbf{Z}_\ell} R$ is in $D_{perf}(R)$. Therefore $\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \hat{R}$ is in $D_{cons}(X_{proét}, \hat{R})$, and $\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \hat{R} \otimes_R R/\mathfrak{m}^n = \omega_X \otimes_{\hat{\mathbf{Z}}_\ell} R/\mathfrak{m}^n = \omega_n \otimes_{\mathbf{Z}/\ell^n} R_n$ (Lemma 3.5.6 & `06Y6`),[69] which is in $D_{cons}(X_{proét}, R_n)$ and is dualizing by Théorème 7.1.3 of Exp. XVII of *Travaux de Gabber*. Reduction modulo $\mathfrak{m}$, Lemma 6.7.13, and $\mathfrak{m}$-adic completeness then give that $\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \hat{R}$ is dualizing in $D_{cons}(X_{proét}, \hat{R})$.

[69]To check $\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \mathbf{Z}/\ell^n \otimes_{\mathbf{Z}/\ell^n} R_n = \omega_X \otimes_{\hat{\mathbf{Z}}_\ell} R_n$, replacing ω_X by a bounded above complex of flat $\hat{\mathbf{Z}}_\ell$-modules, we may suppose ω_X is a flat $\hat{\mathbf{Z}}_\ell$-module. Then $\omega_X \otimes_{\hat{\mathbf{Z}}_\ell} \mathbf{Z}/\ell^n$ is a flat $\mathbf{Z}/\ell^n$-module (`05V4`), so the statement follows from the un-derived version, which follows from the corresponding statement on presheaves [**SGAA**, IV 12.10], which is just the statement that given a map of rings $R \to S$, M in Mod_R and N in Mod_S, then $M \otimes_R S \otimes_S N = M \otimes_R N$.

6.8.2. This lemma allows us to compute sections of $\mathcal{O}_{E,X}$ and E_X, as discussed below.

$$\begin{aligned}\mathcal{O}_{E,X}(U) &= \mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_E) = \mathrm{colim}_{F\subset E}\,\mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_F)\\ &= \mathrm{colim}_{F\subset E}(\mathcal{O}_F(U)) = (\mathrm{colim}_{F\subset E}\,\mathcal{O}_{F,X})(U)\\ E_X(U) &= \mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_E) = \mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_E)[\ell^{-1}]\\ &= \mathcal{O}_E(U)[\ell^{-1}] = \mathcal{O}_E[\ell^{-1}](U)\end{aligned}$$

when U in $X_{\text{proét}}$ is coherent. Here, the first and third isomorphisms are by definition, but the second and fourth are nontrivial. The ring structure on $\mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_E)$, and what it means to localize this ring at ℓ, is discussed below (bref: the ring structure is given by the continuous product $\mathcal{O}_E \times \mathcal{O}_E \to \mathcal{O}_E$, and localizing at ℓ means inverting powers of the constant map $(\pi_0(U) \to \ell) \in \mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_E)$).

First note that the category of sheaves on $X_{\text{proét}}$ is equivalent to the category of sheaves on the sub-site of $X_{\text{proét}}$ consisting of the affine objects, with the induced topology, so we can restrict to testing on coherent U. Next, the topology on E and $\mathcal{O}_E$ is totally disconnected. For $\overline{\mathbf{Q}}_\ell$, this is true since there is a unique extension of the p-adic absolute value to $\overline{\mathbf{Q}}_\ell$ (there is a unique such extension to any finite extension of $\mathbf{Q}_\ell$, and these extensions are compatible vis à vis successive extensions), which gives a non-Archimedean absolute value, and hence a non-Archimedean metric, on $\overline{\mathbf{Q}}_\ell$. So $\overline{\mathbf{Q}}_\ell$, as a metric space, is Hausdorff, and with this metric is an ultrametric space, hence totally disconnected, and the same is true of any of its subspaces E or $\mathcal{O}_E$. The combination of these observations with Lemma 4.2.12 means that we need only test on spaces of the form $S = \pi_0(U)$ with U in $X_{\text{proét}}$.

The spaces $\mathcal{O}_E$, with E a finite extension of $\mathbf{Q}_\ell$, are profinite, hence compact subspaces of $\overline{\mathbf{Q}}_\ell$; as $\overline{\mathbf{Q}}_\ell$ is Hausdorff, they are closed. Therefore we may apply Lemma 4.3.7 to obtain the formula

$$\mathrm{Map}_{\mathrm{cont}}(S, \mathcal{O}_E) = \mathrm{colim}_{F\subset E}\,\mathrm{Map}_{\mathrm{cont}}(S, \mathcal{O}_F).$$

Sections over (coherent) U of $\mathrm{colim}_{F\subset E}\,\mathcal{O}_{F,X}$ are *a priori* sections of the sheafification of the presheaf $U \mapsto \mathrm{colim}_{F\subset E}\,\mathrm{Map}_{\mathrm{cont}}(\pi_0(U), \mathcal{O}_F)$, but this assignment already describes sections over U of a sheaf.

As for

$$\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathrm{E}) = \mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathcal{O}_{\mathrm{E}})[\ell^{-1}],$$

writing the topological ring E as $\bigcup_n \ell^{-n}\mathcal{O}_{\mathrm{E}}$ (colimit of topological $\mathcal{O}_{\mathrm{E}}$-modules), the same argument gives

$$\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathrm{E}) = \mathrm{colim}_n\,\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\ell^{-n}\mathcal{O}_{\mathrm{E}}),$$

so it suffices to show that

$$\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\ell^{-n}\mathcal{O}_{\mathrm{E}}) = \ell^{-n}\,\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathcal{O}_{\mathrm{E}}),$$

which is obvious once one understands that the product $\mathrm{E}\times\mathrm{E}\to\mathrm{E}$ on the topological ring E induces the product

$$\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathrm{E})\times\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathrm{E})\to\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathrm{E}),$$

and given $a\in\mathrm{E}$ and $\varphi\in\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathrm{E})$, $a\varphi$ is the map obtained under this product thinking of a as the constant map $\mathrm{S}\to\mathrm{E}$ with value a. So,

$$\ell^{-n}:\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\mathcal{O}_{\mathrm{E}})\to\mathrm{Map}_{\mathrm{cont}}(\mathrm{S},\ell^{-n}\mathcal{O}_{\mathrm{E}})$$

is a bijection with inverse ℓ^n.

As before, *a priori* sections over (coherent) U of $\mathcal{O}_{\mathrm{X,E}}[\ell^{-1}]$ are given by the sheafification of $\mathrm{U}\mapsto\mathcal{O}_{\mathrm{X,E}}(\mathrm{U})[\ell^{-1}]$. Since this assignment actually defines a sheaf, however, we know that it coincides with $\mathcal{O}_{\mathrm{X,E}}[\ell^{-1}]$.

6.8.4. 1. Given a partially ordered set $(\mathrm{I},\leq)$, categories $\mathscr{C}_i$ for each $i\in\mathrm{I}$, and functors $\mathrm{F}_{ij}:\mathscr{C}_j\to\mathscr{C}_i$ whenever $i\leq j$, so that $\mathrm{F}_{ii}=\mathrm{id}$ for all $i\in\mathrm{I}$, and $\mathrm{F}_{ij}\mathrm{F}_{jk}=\mathrm{F}_{ik}$ if $i\leq j\leq k$, then $\lim\mathscr{C}_i$ is the category with

- Objects: pairs $\{(\mathrm{C}_i,t_{ij})\}$ where C_i is an object of $\mathscr{C}_i$ and $t_{ij}:\mathrm{F}_{ij}(\mathrm{C}_j)\xrightarrow{\sim}\mathrm{C}_i$ an isomorphism, such that $t_{ii}=\mathrm{id}$ and $t_{ij}\mathrm{F}_{ij}(t_{jk})=t_{ik}$ for all $i\leq j\leq k$.

- Morphisms: a morphism $(C_i, t_{ij}) \to (D_i, s_{ij})$ is a set of arrows $\{f_i : C_i \to D_i\}$ for all i making commutative diagrams (whenever $i < j$)

$$\begin{tikzcd} F_{ij}(C_j) \arrow[r, "t_{ij}"] \arrow[d, "F_{ij}(f_j)"] & C_i \arrow[d, "f_i"] \\ F_{ij}(D_j) \arrow[r, "s_{ij}"] & D_i. \end{tikzcd}$$

Let $\pi : \mathrm{Loc}_X(\mathcal{O}_E) \to \lim_n \mathrm{Loc}_X(\mathcal{O}_E/\varpi^n\mathcal{O}_E)$ be the functor $M \mapsto (M/\varpi^n M)_n$ (so in the above notation, $F_{n-1,n} = - \otimes_{\mathcal{O}_E/\varpi^n\mathcal{O}_E} \mathcal{O}_E/\varpi^{n-1}\mathcal{O}_E)$. The claim is that π is an equivalence of categories with quasi-inverse $\lim_n$. This claim is false in the étale world because there needn't be an étale cover that trivializes the entire system at once, but it is true in the pro-étale world, which is designed to allow the construction of such covers.

In fact, this equivalence is just the restriction of the equivalence of Lemma 3.5.7, where we place the categories $\mathrm{Loc}, \lim_n \mathrm{Loc}_n$ in degree zero. This is because, on the one hand, any $\mathcal{F}$ in $\mathrm{Loc}_X(\mathcal{O}_E)$ is locally free of finite rank over $\mathcal{O}_E$, *a fortiori* flat, and on the other hand, any $\mathcal{F}_n$ in $\mathrm{Loc}_X(\mathcal{O}_E/\varpi^n\mathcal{O}_E)$ is also flat as a sheaf of $\mathcal{O}_E/\varpi^n\mathcal{O}_E$-modules, so an object of $\lim_n \mathrm{Loc}_X(\mathcal{O}_E/\varpi^n\mathcal{O}_E)$ is automatically in $D_{\mathrm{comp}}(\mathscr{C}, \mathcal{O}_{E,\bullet})$, where $\mathscr{C} = \tilde{X}^{\mathbf{N}^\circ}_{\text{proét}}$ and $\mathcal{O}_E = (\mathcal{O}_E/\varpi^n\mathcal{O}_E)_n$ is the sheaf of rings of Definition 3.5.3, and moreover lim computes $\mathrm{R}\lim$ of any object of $\lim_n \mathrm{Loc}_X(\mathcal{O}_E/\varpi^n\mathcal{O}_E)$ by repleteness (Proposition 3.1.10). The subcategories $\mathrm{Loc}_X(\mathcal{O}_E) \subset D_{\mathrm{comp}}(X_{\text{proét}}, \mathcal{O}_E)$ and $\lim_n \mathrm{Loc}_X(\mathcal{O}_E/\varpi^n\mathcal{O}_E) \subset D_{\mathrm{comp}}(\mathscr{C}, \mathcal{O}_{E,\bullet})$ are stable under $\pi_* = \mathrm{R}\lim$, $\pi^* = - \otimes^{\mathbf{L}}_{\mathcal{O}_E} \mathcal{O}_E/\varpi^n\mathcal{O}_E$: for π^* this is clear by what has been said and the fact that $\mathcal{F} \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi^n\mathcal{O}_E$ is locally free of finite rank over $\mathcal{O}_E/\varpi^n\mathcal{O}_E$ if $\mathcal{F}$ is over $\mathcal{O}_E$. For π_* this is also true. After shrinking X, we may assume the $\mathcal{F}_n$ are locally free of a fixed finite rank r and X is affine. Replacing X by a w-contractible cover, $\mathcal{F}_n \simeq (\mathcal{O}_E/\varpi^n)^r$ and $\Gamma(X, \mathcal{F}_{n+1}) \to \Gamma(X, \mathcal{F}_n)$ is surjective for each n. Pick a free basis for $(\mathcal{O}_E/\varpi)^r$, and choose lifts of these global sections of $\mathcal{F}_1$ to r sections of $\Gamma(X, \mathcal{F}) = \lim_n \Gamma(X, \mathcal{F}_n)$, where $\mathcal{F} = \lim_n \mathcal{F}_n$. This defines a morphism $\rho : \hat{\mathcal{O}}_E^r \to \mathcal{F}$. As $\mathcal{F}$ coincides with $\mathrm{R}\lim_n \mathcal{F}_n$ and $(\mathcal{F}_n)_n$ is in $D_{\mathrm{comp}}(\mathscr{C}, \mathcal{O}_{E,\bullet})$, $\mathcal{F}$ is in $D_{\mathrm{comp}}(X_{\text{proét}}, \mathcal{O}_E)$ by Lemma 3.5.7 (c.f. its note), so the reduction of ρ modulo ϖ (in the sense of $- \otimes^{\mathbf{L}}_{\hat{\mathcal{O}}_E} \mathcal{O}_E/\varpi$) is the isomorphism $(\mathcal{O}_E/\varpi^n)^r \xrightarrow{\sim} \mathcal{F}_n$ we started with by Lemma 3.5.5. As both terms are derived ϖ-complete, ρ must be an isomorphism.

2. This is the module version of the fact (already remarked in the note to Lemma 5.4.2) that on any site C, $Y \mapsto \widetilde{(C/Y)}$ for Y in C gives a stack over C, and of course, a sheaf that is locally a local system is a local system.

3. Objects of $\operatorname{colim}_{F\subset E}\operatorname{Loc}_X(\mathcal{O}_F)$ are given by $\operatorname{colim}_{F\subset E}\operatorname{ob}\operatorname{Loc}_X(\mathcal{O}_F)$, and morphisms between $\mathcal{F}_F \in \operatorname{ob}\operatorname{Loc}_X(\mathcal{O}_F)$ and $\mathcal{G}_{F'} \in \operatorname{ob}\operatorname{Loc}_X(\mathcal{O}_{F'})$ are given by

$$\operatorname{colim}_{F,F'\subset K\subset E}\operatorname{Hom}_K(\mathcal{F}_K, \mathcal{G}_K),$$

where the colimit is over finite extensions K of $\mathbf{Q}_\ell$ contained in E and containing F and F′. The functor

$$\operatorname{colim}_{F\subset E}\operatorname{Loc}_X(\mathcal{O}_F) \to \operatorname{Loc}_X(\mathcal{O}_E)$$

takes $\mathcal{F}_F$ to $\operatorname{colim}_{F\subset F'\subset E}\mathcal{F}_{F'} = \mathcal{F}_E$ (this is an isomorphism as tensor product commutes with colimits and $\mathcal{O}_E = \operatorname{colim}_{F\subset E}$ by Lemma 6.8.2 2 and lands in $\operatorname{Loc}_X(\mathcal{O}_E)$ for the same reason), and sends a morphism represented by $\mathcal{F}_K \to \mathcal{G}_K$ to its base change by $-\otimes_{\mathcal{O}_K}\mathcal{O}_E$.

The business about the internal Hom is that if $j : U \to X$ is in $X_{\text{proét}}$, $j^*\mathcal{H}om(\mathcal{F},\mathcal{G}) = \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$, so if $\mathcal{F}|_U \simeq \mathcal{O}_F^r$ and $\mathcal{G}|_U \simeq \mathcal{O}_F^s$, then $j^*\mathcal{H}om(\mathcal{F},\mathcal{G}) \simeq \mathcal{O}_F^{rs}$, so $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is in $\operatorname{Loc}_X(\mathcal{O}_F)$ and $\mathcal{H}om_{\mathcal{O}_F}(\mathcal{F},\mathcal{G})\otimes_{\mathcal{O}_F}\mathcal{O}_{F'} \to \mathcal{H}om_{\mathcal{O}_{F'}}(\mathcal{F}_{F'},\mathcal{G}_{F'})$ is an isomorphism (F′ need not be finite over F).

We've already seen that given $\mathcal{M}_F$ in $\operatorname{Loc}_X(\mathcal{O}_F)$, $\mathcal{M}_E = \operatorname{colim}_{F\subset F'\subset E}\mathcal{M}_{F'}$. Also, the transition maps $\mathcal{M}_F \to \mathcal{M}_{F'}$ are injective, since the maps $\mathcal{O}_F \to \mathcal{O}_{F'}$ are. Taking $\mathcal{M}_F = \mathcal{H}om_F(\mathcal{F}_F, \mathcal{G}_F)$, we have

$$\operatorname{Hom}_E(\mathcal{F}_E, \mathcal{G}_E) = \operatorname{colim}_{F\subset E}\operatorname{Hom}_F(\mathcal{F}_F, \mathcal{G}_F)$$

by `0738`, since a filtered colimit of abelian sheaves can be computed in the category of sheaves of sets.

For essential surjectivity, the point is that the category of descent data (`026B`) consists of an object on Y with some morphisms on $Y\times_X Y$ satisfying the cocycle condition. The triviality of Loc_Y shows that $\operatorname{Loc}_X(\mathbf{Z}_\ell) \to \operatorname{Loc}_X(\mathcal{O}_E)$ is essentially surjective, and as $Y\times_X Y$ is coherent,

$$\operatorname{colim}_{F\subset E}\operatorname{Loc}_{Y\times_X Y}(\mathcal{O}_F) \longrightarrow \operatorname{Loc}_{Y\times_X Y}(\mathcal{O}_E)$$

is fully faithful, so the categories of descent data are equivalent. More explicitly, this means that for any $\mathcal{F}$ in $\mathrm{Loc}_X(\mathcal{O}_E)$, we can find a local system $\mathcal{F}'$ in $\mathrm{Loc}_Y(\mathcal{O}_F)$ for some finite extension $F \supset \mathbf{Q}_\ell$ so that $\mathcal{F}|_Y = \mathcal{F}'_E$ together with an isomorphism $\mathrm{pr}_0^* \mathcal{F}' \to \mathrm{pr}_1^* \mathcal{F}'$ in $\mathrm{Loc}_{Y \times_X Y}(\mathcal{O}_F)$ satisfying the cocycle condition, which, after extending scalars to $\mathcal{O}_E$, gives the isomorphism of descent for $\mathcal{F}$. This data describes an object of $\mathrm{Loc}_X(\mathcal{O}_F)$, which, after extending scalars to $\mathcal{O}_E$, coincides with $\mathcal{F}$.

4. First off, $\mathrm{Loc}_X(\mathcal{O}_E)[\ell^{-1}]$ denotes the category $\mathrm{Loc}_X(\mathcal{O}_E)$ localized at the multiplicative system of arrows $\{\ell^n : \mathcal{M} \to \mathcal{M} \mid n \in \mathbf{Z}_{\geq 0}, \mathcal{M} \in \mathrm{Loc}_X(\mathcal{O}_E)\}$. As this is a multiplicative system, the categories of left fractions and right fractions coincide (`04VL`). If $\mathcal{M}, \mathcal{N}$ are in $\mathrm{Loc}_X(\mathcal{O}_E)$, then $\mathrm{Hom}_{\mathrm{Loc}_X(\mathcal{O}_E)}(\mathcal{M}, \mathcal{N})$ carries the structure of $\mathcal{O}_E$-module.[70] I claim

$$\mathrm{Hom}_{\mathrm{Loc}_X(\mathcal{O}_E)[\ell^{-1}]}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathrm{Loc}_X(\mathcal{O}_E)}(\mathcal{M}, \mathcal{N}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

To see this, morphisms on the right are given by equivalence classes of pairs (ϕ, n) (corresponding to ϕ/ℓ^n) with $\phi \in \mathrm{Hom}_{\mathrm{Loc}_X(\mathcal{O}_E)}(\mathcal{M}, \mathcal{N})$, where two pairs (ϕ, n), (ψ, m) are equivalent if $\ell^b(\ell^m \phi - \ell^n \psi) = 0$ for some $b > 0$. On the other hand, morphisms on the left are given by equivalence classes of pairs (ϕ, n) (corresponding to $\mathcal{M} \xleftarrow{\ell^n} \mathcal{M} \xrightarrow{\phi} \mathcal{N}$) with $\phi \in \mathrm{Hom}_{\mathrm{Loc}_X(\mathcal{O}_E)}(\mathcal{M}, \mathcal{N})$, where two pairs (ϕ, n), (ψ, m) are equivalent if there exists a pair (γ, c), $c \geq n, m$ making a commutative diagram

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\ell^n"'] \arrow[dr, "\phi"] & \\ \mathcal{M} & \mathcal{M} \arrow[l, "\ell^c"] \arrow[r, "\gamma"] \arrow[u, "\ell^{c-n}"'] \arrow[d, "\ell^{c-m}"] & \mathcal{N}. \\ & \mathcal{M} \arrow[ul, "\ell^m"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

In other words, $(\phi, n) \sim (\psi, m)$ if $\ell^{c-n}\phi - \ell^{c-m}\psi = 0$ for $c \gg 0$. Putting $c = b + n + m$, this is the same condition as before.

[70]Given $\phi : \mathcal{M} \to \mathcal{N}$ and $a \in \mathcal{O}_E$, a defines a section of $\mathcal{O}_E$ over any U in $X_{\text{proét}}$ by Lemma 4.2.12 (corresponding to the constant map $U \to a$), and $\mathcal{N}$ is a sheaf of $\mathcal{O}_E$-modules, so $a\phi$ is just ϕ followed by multiplication by a.

Now to show that the stated functor is fully faithful. First of all, it lands in $\mathrm{Loc}_{\mathrm{X}}(\mathrm{E})$ by the definition of lisse $\mathcal{O}_{\mathrm{E}}$-sheaf and Lemma 6.8.2 2. The claim is that the map

$$\mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N})[\ell^{-1}] \longrightarrow \mathcal{H}om_{\mathrm{E}}(\mathcal{M}[\ell^{-1}],\mathcal{N}[\ell^{-1}])$$

in $\mathrm{Loc}_{\mathrm{X}}(\mathrm{E})$ is an isomorphism, where here the localization is as sheaves of modules (`0EMB`). If $g : (\mathrm{X}_{\text{proét}},\mathrm{E}) \to (\mathrm{X}_{\text{proét}},\mathcal{O}_{\mathrm{E}})$ is the change of coefficients morphism, then the above morphism can be written $g^*\mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N}) \to \mathcal{H}om_{\mathrm{E}}(g^*\mathcal{M},g^*\mathcal{N})$. It comes via adjunction from

$$\mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N}) \to \mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},g_*g^*\mathcal{N}) = g_*\mathcal{H}om_{\mathrm{E}}(g^*\mathcal{M},g^*\mathcal{N}),$$

where the first map is given by the unit. As $\mathcal{M}$ and $\mathcal{N}$ are in $\mathrm{Loc}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})$, that the above map in $\mathrm{Loc}_{\mathrm{X}}(\mathrm{E})$ is an isomorphism boils down to the statement $\mathrm{E}^{rs} = \mathcal{O}_{\mathrm{E}}^{rs}[\ell^{-1}] = \mathcal{O}_{\mathrm{E}}^{r}[\ell^{-1}] \otimes_{\mathrm{E}} \mathcal{O}_{\mathrm{E}}^{s}[\ell^{-1}]$. We'll be done upon taking global sections of both sides, provided

$$\mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N})[\ell^{-1}](\mathrm{X}) = \mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N})(\mathrm{X})[\ell^{-1}],$$

as the latter equals $\mathrm{Hom}_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell = \mathrm{Hom}_{\mathrm{Loc}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})[\ell^{-1}]}(\mathcal{M},\mathcal{N})$. Let $\mathcal{H} := \mathcal{H}om_{\mathcal{O}_{\mathrm{E}}}(\mathcal{M},\mathcal{N})$: $\mathcal{H}$ is in $\mathrm{Loc}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})$. As in the note to (3), `0738` gives that

$$\mathcal{H}[\ell^{-1}](\mathrm{X}) = \mathrm{colim}_n(\ell^{-n}\mathcal{H})(\mathrm{X}),$$

since X is quasi-compact, the transition maps $\ell^{-n}\mathcal{H} \to \ell^{-n-1}\mathcal{H}$ are injective,[71] and the colimit can be computed as sheaves of sets. Here $\ell^{-n}\mathcal{H}$ is the image (i.e. sheafification of the presheaf image) of $\ell^{-n} : \mathcal{H} \to \mathcal{H}[\ell^{-1}]$. So in the end we must show that if $\mathcal{H}$ is in $\mathrm{Loc}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})$, $(\ell^{-n}\mathcal{H})(\mathrm{X}) = \ell^{-n}(\mathcal{H}(\mathrm{X}))$; i.e. that the presheaf image of $\ell^{-n} : \mathcal{H} \to \mathcal{H}[\ell^{-1}]$ is a sheaf. This is obvious as the map is injective.

6. That F is a sheaf: explicitly, let F denote the functor $\mathrm{X}^{\circ}_{\text{proét}} \to \mathrm{Set}$ that associates to each $\mathrm{U} \to \mathrm{X}$ in $\mathrm{X}_{\text{proét}}$ the set of subsheaves (of $\mathcal{O}_{\mathrm{E},\mathrm{U}}$-modules) N of $\mathrm{L}|_{\mathrm{U}}$ so that N is an $\mathcal{O}_{\mathrm{E},\mathrm{U}}$-local system and $\mathrm{N} \otimes_{\mathcal{O}_{\mathrm{E},\mathrm{U}}} \mathrm{E}_{\mathrm{U}} = \mathrm{L}|_{\mathrm{U}}$. Out of hand this may seem different from the functor F described in the text, but as any $\mathcal{O}_{\mathrm{E},\mathrm{U}}$-local system embeds in its extension of scalars, the difference is superficial.

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[71]The map $\ell^{-n}\mathcal{H} \hookrightarrow \ell^{-n-1}\mathcal{H}$ is injective since the same is true with $\mathcal{H}$ replaced by $\mathcal{O}_{\mathrm{E}}$, as $\ell^{-n}\mathcal{O}_{\mathrm{E}} \subset \ell^{-n-1}\mathcal{O}_{\mathrm{E}}$ as topological $\mathcal{O}_{\mathrm{E}}$-modules, so the same is true after applying $\mathrm{Map}_{\mathrm{cont}}(\mathrm{U},-)$.

Clearly F is a presheaf, and we may ask if it's a sheaf. We need to check that for each covering $U \to X$ in $X_{\text{proét}}$, subsheaf $N \subset L|_U$ on U so that N belongs to $\mathrm{Loc}_U(\mathcal{O}_{E,U})$, $N \otimes_{\mathcal{O}_{E,U}} E_U = L|_U$, and $\mathrm{pr}_0^* N = \mathrm{pr}_1^* N$ as subsheaves of $L|_{U \times_X U}$, there is a unique subsheaf (of $\mathcal{O}_{E,X}$-modules) N_X of L on X belonging to $\mathrm{Loc}_X(\mathcal{O}_E)$ and restricting to N over U.

This is easily seen to be an example of the following task: given a site $\mathscr{C}$, an object X of $\mathscr{C}$, a sheaf G on $\mathscr{C}$, a covering $(U_i \to X)_{i \in I}$ in $\mathscr{C}$, and for each i a subsheaf $F_i \subset G|_{U_i}$ on $\mathscr{C}/U_i$ so that $\mathrm{pr}_i^* F_i = \mathrm{pr}_j^* F_j$ as subsheaves of $G|_{U_i \times_X U_j}$, there is a unique subsheaf $F \subset G|_X$ on X restricting to F_i on U_i. Replacing $\mathscr{C}$ by $\mathscr{C}/X$, we may assume X is the final object of $\mathscr{C}$. Now the $(U_i \to X)_i$ generate a crible $R \hookrightarrow X$ of the final object, and we can consider the site $\mathscr{C}'$ which as a category consists of the full category of $\mathscr{C}$ consisting of those objects Y of $\mathscr{C}$ so that $R(Y) \neq \emptyset$, with the induced topology. By the comparison lemma [**SGAA**, III 4.1], $\tilde{\mathscr{C}} = \tilde{\mathscr{C}}'$. Define the presheaf F on $\mathscr{C}'$ in the following way. An object of $\mathscr{C}'$ corresponds to an object U of $\mathscr{C}'$ whose unique morphism to X factors through some U_i, say $\alpha : U \to U_i$. So put $F(U) := F_i(\alpha)$. This assignment is well-defined, since given morphisms $\alpha : U \to U_i$, $\beta : U \to U_j$, we get a morphism $\alpha \times \beta : U \to U_i \times_X U_j$, and $F_i(\alpha) = \mathrm{pr}_i^* F_i(\alpha \times \beta) = \mathrm{pr}_j^* F_j(\alpha \times \beta) = F_j(\beta)$. Moreover, F is a sheaf on $\mathscr{C}'$, since given U in $\mathscr{C}'$ and some cover $V_j \to U$ in $\mathscr{C}'$, then $V_j \to U$ is a cover in $\mathscr{C}$ [**SGAA**, III 3.3], and more to the point, it is cover in $\mathscr{C}/U_i$ for any choice of structural arrow $U \to U_i$ [**SGAA**, III 5.2]. So

$$\begin{array}{ccccc} F(U) & \longrightarrow & \prod_j F(V_j) & \rightrightarrows & \prod_{j,j'} F(V_j \times_U V_{j'}) \\ \| & & \| & & \| \\ F_i(U) & \longrightarrow & \prod F_i(V_j) & \rightrightarrows & \prod_{j,j'} F_i(V_j \times_U V_{j'}) \end{array}$$

shows that F is a sheaf on $\tilde{C}'$. It is evidently the unique presheaf on $\mathscr{C}'$ with $F|_{U_i} = F_i$, and it's evidently a subsheaf of $u_s G = G \circ u$, where $u : \mathscr{C}' \to \mathscr{C}$ is the continuous inclusion, hence corresponds to a subsheaf of G on $\mathscr{C}$ by $\tilde{\mathscr{C}}' \simeq \tilde{\mathscr{C}}$.

Now proceeding with the proof, we must ask how $GL_n(E)$ acts on sections of E_X^n. Given $f \in E_X^n(X)$, corresponding to $(f_1, \ldots, f_n)$ with $f_i : X \to E$ a continuous map, and $A \in GL_n(E)$, $Af \in E_X^n(X)$ is given by $(g_1, \ldots, g_n)$, where $g_i = \sum_{j=1}^n A_{ij} f_j$. (We know how to multiply functions $X \to E$ by elements of E.) Now that we know how

$\mathrm{GL}_n(\mathrm{E})$ acts on sections of $\mathrm{E}^n_{\mathrm{X}}$, we know how it acts on the free $\mathcal{O}_{\mathrm{E,X}}$-module generated by a certain global section of $\mathrm{E}^n_{\mathrm{X}}$, since the action of $\mathrm{GL}_n(\mathrm{E})$ is $\mathcal{O}_{\mathrm{E,X}}$-linear.

To check that the map $\mathrm{S} \to \mathrm{F}$ is surjective, it suffices to pick U in $\mathrm{X}_{\text{proét}}$ and $\mathrm{M} \in \mathrm{F}(\mathrm{U})$, and to find a pro-étale cover so that the restriction of M to this cover is in the image of S. As M is in $\mathrm{Loc}_{\mathrm{U}}(\mathcal{O}_{\mathrm{E,X}})$, over some pro-étale cover M becomes isomorphic to a finite free $\mathcal{O}_{\mathrm{E,X}}$-module, so, replacing X by any one of the maps in this cover, we may assume M is finite free of rank n. By looking at where the n global sections corresponding to $1 \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{X})$ go under the injection $\mathcal{O}^n_{\mathrm{E,X}} = \mathrm{M} \hookrightarrow \mathrm{M} \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}} = \mathrm{E}^n_{\mathrm{X}}$, we get n global sections $m_1, \ldots, m_n \in \mathrm{E}^n_{\mathrm{X}}(\mathrm{X})$, and these global sections generate a free $\mathcal{O}_{\mathrm{E,X}}$-submodule of $\mathrm{E}^n_{\mathrm{X}}$ of rank n, namely, the image of M under the above injection. Each section m_i corresponds to n continuous maps $m_{ij} : \mathrm{X} \to \mathrm{E}$.

It's equivalent to show that $g\mathcal{O}^n_{\mathrm{E,X}} = \mathrm{M}$ for some $g \in \mathrm{GL}_n(\mathrm{E})$ or that $g\mathrm{M} = \mathcal{O}^n_{\mathrm{E,X}}$, so we'll show the latter. Fix a geometric point $\overline{x}$ centered on a point $x \in \mathrm{X}$. Then by `0993`, $(\mathcal{O}_{\mathrm{E,X}})_{\overline{x}} = \mathcal{O}_{\mathrm{E}}$, $(\mathrm{E}_{\mathrm{X}})_{\overline{x}} = \mathrm{E}$, and $\mathrm{M}_{\overline{x}} = (m_1(x), \ldots, m_n(x))$, for the simple reason that a continuous map from the spectrum of a local ring to a Hausdorff space is determined by where the unique closed point goes. By hypothesis, $\mathrm{M}_{\overline{x}}$ is of rank n, so we can find a $g \in \mathrm{GL}_n(\mathrm{E})$ so that $g\mathrm{M}_{\overline{x}} = (1_1, 1_2, \ldots, 1_n)$, where 1_i is the vector $(\delta_{ij})^n_{j=1}$. In this way, replacing M by $g\mathrm{M}$, we may assume $m_{ij}(x) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function, and our task is to show that there is a Zariski open neighborhood of x restricted to which $\mathrm{M} = \mathcal{O}^n_{\mathrm{E,X}}$.

Given a continuous map $f : \mathrm{X} \to \mathrm{E}$ so that $f(x) \in \mathcal{O}_{\mathrm{E}}$, $f^{-1}(\mathcal{O}_{\mathrm{E}})$ is a (Zariski) open subset of X. This is because $\mathcal{O}_{\mathrm{E}}$ is open in E: $\mathcal{O}_{\mathrm{E}}$ is the closed ball of radius 1 about $0 \in \mathrm{E}$, and E is an ultrametric space, so closed balls in E are open.

Therefore there's a Zariski open neighborhood U of x restricted to which the n^2 maps $\mathrm{X} \to \mathrm{E}$ corresponding to the sections $m_1, \ldots, m_n$ land in $\mathcal{O}_{\mathrm{E}} \subset \mathrm{E}$. Over U, therefore, $\mathrm{M} \subset \mathcal{O}^n_{\mathrm{E,X}}$. On the other hand, proceeding inductively, fix $1 \leq k \leq n$ and suppose we've found an open neighborhood U of x and sections $b_{ij} \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$ for $1 \leq i \leq k$, $1 \leq j \leq i$ such that for each $i \leq k$, $f_i := \sum_{j=1}^{i} b_{ij} m_j = (f_{i1}, \ldots, f_{in}) \in \mathrm{M}(\mathrm{U})$ has $f_{ij} = \delta_{ij} \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$ for $1 \leq j \leq i$. Then since $m_{k+1,j}$, as a continuous map $\mathrm{U} \to \mathrm{E}$, lands in $\mathcal{O}_{\mathrm{E}}$, for all $1 \leq j \leq n$, $m_{k+1} - m_{k+1,1} f_1 = (0, ?, \ldots, ?)$, where the $? \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$, and proceeding in this way we can find $a_i \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$ for $1 \le i \le k$ so that, putting $g := \sum_{i=1}^{k} a_i f_i \in \mathrm{M}(\mathrm{U})$, $g_j = m_{k+1,j}$ for $1 \le j \le k$. As $m_{i,k+1}(x) = 0$ for $1 \le i \le k$ by hypothesis, $f_{i,k+1}(x) = 0$ for $1 \le i \le k$, as f_{ij} is an $\mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$-linear combination of the m_{ij}, so $m_{k+1} - g = (0, \dots, 0, ?_{k+1}, \dots, ?_n)$, where $?_i \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$ and $?_{k+1}(x) = 1$. As $?_{k+1}^{-1}(\mathrm{E} \smallsetminus \{0\})$ is an open neighborhood of x, we shrink U and assume $?_{k+1}(y) \neq 0$ for $y \in \mathrm{U}$. Let a_{k+1} denote the composition $\mathrm{U} \xrightarrow{?_{k+1}} \mathrm{E}^\times \xrightarrow{\mathrm{inv}} \mathrm{E}^\times$, where the second map is the map $a \mapsto a^{-1}$. Shrinking U further, we may assume $a_{k+1} \in \mathcal{O}_{\mathrm{E}}(\mathrm{U})$. Then $f_{k+1} := a_{k+1}(m_{k+1} - g) = (0, \dots, 0, 1, a_{k+1}?_{k+2}, \dots, a_{k+1}?_n) \in \mathrm{M}(\mathrm{U})$, completing the inductive step. In conclusion, we find $f_1, \dots, f_n \in \mathrm{M}(\mathrm{U})$ which are $\mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$-linear combinations of the m_i, with the property that $f_{ij} = \delta_{ij}$ for $1 \le j \le i$. Now it's easy to write $1_i \in \mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$ as a $\mathcal{O}_{\mathrm{E,X}}(\mathrm{U})$-linear combination of the f_i, where $1_i = (\delta_{ij})_{j=1}^n$. In conclusion, $\mathcal{O}_{\mathrm{E,U}}^n \subset \mathrm{M}|_{\mathrm{U}}$, and $\mathcal{O}_{\mathrm{E,U}}^n = \mathrm{M}|_{\mathrm{U}}$. In conclusion, we've found a Zariski open cover of X over which the restriction of M is in the image of our map $\mathrm{S} \to \mathrm{F}$, so $\mathrm{S} \to \mathrm{F}$ is surjective.

6.8.5. '$\mathrm{Loc}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})$ is equivalent to the category of (continuous) representations of $\pi_1(\mathrm{X}, \overline{x})$ on finite free $\mathcal{O}_{\mathrm{E}}$-modules' $\rightsquigarrow$ This follows formally from Proposition 6.8.4 and the fact that $\mathrm{Loc}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}}/\varpi^n\mathcal{O}_{\mathrm{E}})$ is equivalent to the category of finite free $\mathcal{O}_{\mathrm{E}}/\varpi^n\mathcal{O}_{\mathrm{E}}$-modules endowed with a continuous action of $\pi_1 := \pi_1(\mathrm{X}, \overline{x})$ whenever E is a finite extension of $\mathbf{Q}_\ell$. Indeed, when $\mathrm{E} = \mathbf{Q}_\ell$, this is Lemme 1.2.4.2 of Exposé VI, SGA 5. Namely, the category $(\ell\mathbf{Z})\text{−ad}(\mathrm{Abf}(\pi_1))$ that appears in that lemma is, by Définition 3.1.1 of Exposé V of SGA 5, the full subcategory of $\mathrm{Abf}(\pi_1)^{\mathbf{N}^\circ}$ ($\mathrm{Abf}(\pi_1)$ denotes the category of finite abelian groups endowed with continuous action of π_1) consisting of those inverse systems (F_n) with $\ell^{n+1}\mathrm{F}_n = 0$ and so that whenever $m \ge n \ge 0$, the morphism $\mathbf{Z}/\ell^{n+1} \otimes_{\mathbf{Z}} \mathrm{F}_m \to \mathrm{F}_n$ induced by the transition morphism $\mathrm{F}_m \to \mathrm{F}_n$ is an isomorphism. On the other hand, the category $\lim_n \mathrm{Loc}_{\mathrm{X}}(\mathbf{Z}/\ell^n)$ is equivalent to the category with

- Objects: $(\mathrm{F}_n, t_n)_{n \in \mathbf{Z}_{>0}}$ where F_n is a finite free $\mathbf{Z}/\ell^n$-module endowed with continuous action of π_1 and t_n an isomorphism $\mathrm{F}_n \otimes \mathbf{Z}/\ell^{n-1} \xrightarrow{\sim} \mathrm{F}_{n-1}$.

- Morphisms: arrows $(\mathrm{F}_n, t_n) \to (\mathrm{G}_n, s_n)$ are given by π_1-equivariant maps $(f_n : \mathrm{F}_n \to \mathrm{G}_n)_n$ making commutative diagrams

$$\begin{tikzcd} \mathrm{F}_n \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^{n-1} \arrow[r, "t_n"] \arrow[d, "f_n \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^{n-1}"] & \mathrm{F}_{n-1} \arrow[d, "f_{n-1}"] \\ \mathrm{G}_n \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^{n-1} \arrow[r, "s_n"] & \mathrm{G}_{n-1}. \end{tikzcd}$$

This diagram commutes iff the diagram

$$\begin{tikzcd} \mathrm{F}_n \arrow[r] \arrow[d, "f_n"] & \mathrm{F}_n \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^{n-1} \arrow[r, "t_n"] \arrow[d, "f_n \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^{n-1}"] & \mathrm{F}_{n-1} \arrow[d, "f_{n-1}"] \\ \mathrm{G}_n \arrow[r] & \mathrm{G}_n \otimes_{\mathbf{Z}} \mathbf{Z}/\ell^{n-1} \arrow[r, "s_n"] & \mathrm{G}_{n-1}. \end{tikzcd}$$

commutes (the left square always commutes as it's given by a unit of adjunction, while the square to the right is the (restriction of the) adjoint of the outer rectangle). Therefore, this category is equivalent to the category $(\ell\mathbf{Z})\text{–ad}(\mathrm{Abf}(\pi_1))$.

The same argument works for $\mathbf{Z}_\ell$ replaced by $\mathcal{O}_\mathrm{F}$ for F a finite extension of $\mathbf{Q}_\ell$, and shows that $\mathrm{Loc}_\mathrm{X}(\mathcal{O}_\mathrm{F})$ is equivalent to the category of continuous representations of π_1 on finite free $\mathcal{O}_\mathrm{F}$-modules. We know from Lemma 6.8.4 3 that for E any algebraic extension of $\mathbf{Q}_\ell$, $\mathrm{Loc}_\mathrm{X}(\mathcal{O}_\mathrm{E})$ is equivalent to $\mathrm{colim}_{\mathrm{F}\subset\mathrm{E}} \mathrm{Loc}_\mathrm{X}(\mathcal{O}_\mathrm{F})$, and it remains to be seen that the colimit of the corresponding categories of continuous representations of π_1 on finite free $\mathcal{O}_\mathrm{F}$-modules gives the category of continuous representations of π_1 on finite free $\mathcal{O}_\mathrm{E}$-modules. Denote these categories by $\mathrm{Loc}_?(\pi_1)$ with $? = \mathcal{O}_\mathrm{F}, \mathcal{O}_\mathrm{E}, \mathrm{F}, \mathrm{E}$. There is a functor

$$\mathrm{colim}_{\mathrm{F}\subset\mathrm{E}} \mathrm{Loc}_{\mathcal{O}_\mathrm{F}}(\pi_1) \to \mathrm{Loc}_{\mathcal{O}_\mathrm{E}}(\pi_1)$$

given by $- \otimes_{\mathcal{O}_\mathrm{F}} \mathcal{O}_\mathrm{E}$, and extending the action of π_1 linearly.

As a topological ring, $\mathcal{O}_\mathrm{E} = \bigcup_{\mathrm{F}\subset\mathrm{E}} \mathcal{O}_\mathrm{F}$, and the $\mathcal{O}_\mathrm{F}$ are compact. Suppose the action of π_1 on $\mathrm{N} \simeq \mathcal{O}_\mathrm{E}^r$ is continuous. Then $\pi_1 \times \mathbf{Z}_\ell^r$ is compact, so its image in $\mathcal{O}_\mathrm{E}^r$ is contained in some $\mathcal{O}_\mathrm{F}^r$ by Lemma 4.3.7. Let L denote the $\mathcal{O}_\mathrm{F}$-span of $\pi_1 \mathbf{Z}_\ell^r \subset \mathcal{O}_\mathrm{F}^n$. This is a finitely-generated submodule of $\mathcal{O}_\mathrm{F}^n$ since $\mathcal{O}_\mathrm{F}$ is noetherian. It's finite free as $\mathcal{O}_\mathrm{F}$ is a PID. It has rank r since $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{F}} \mathcal{O}_\mathrm{E} = \mathcal{O}_\mathrm{E}^n$. Therefore $\mathrm{L} \simeq \mathcal{O}_\mathrm{F}^r$ is endowed with a continuous action of π_1, and $L \otimes_{\mathcal{O}_F} \mathcal{O}_E \simeq N$ compatibly with the action of π_1. This shows that our functor is essentially surjective.

Full faithfulness is easy: given M and N in $\mathrm{Loc}_{\mathcal{O}_F}(\pi_1)$ and $\mathrm{Loc}_{\mathcal{O}_{F'}}(\pi_1)$, morphisms $M \to N$ in our colimit category are given by

$$\mathrm{colim}_{F,F' \subset K \subset E} \mathrm{Hom}_{\mathcal{O}_K[\pi_1]}(M_K, N_K),$$

where the colimit is over finite extensions K of $\mathbf{Q}_\ell$ containing F and F′, $M_K := M \otimes_{\mathcal{O}_F} \mathcal{O}_K$ and similarly for N, and the action of π_1 on M_K is the linear extension of its action on M. Now let $\varphi : M_E \to N_E$ be any π_1-equivariant map of $\mathcal{O}_E$-modules, and choose a free basis for M as $\mathcal{O}_F$-module. There is some finite extension $K \supset F, F'$ of $\mathbf{Q}_\ell$ so that N_K contains the image of this (finite) free basis. Extending $\mathcal{O}_K$-linearly, in this way φ defines a π_1-equivariant map $M_K \to N_K$ inducing φ after extending scalars, and our functor is seen to be full. It is evidently faithful, as different morphisms $M_K \to N_K$ remain different after extending scalars. We have proved the following

Proposition. — Let E *be an algebraic extension of* $\mathbf{Q}_\ell$. *The category of finite free* $\mathcal{O}_E$*-modules endowed with continuous* $\mathcal{O}_E$*-linear action of* π_1 *is equivalent to the colimit over finite sub-extensions* F *of* $\mathbf{Q}_\ell$ *of the corresponding categories over* $\mathcal{O}_F$*; i.e.*

$$\mathrm{colim}_{F \subset E} \mathrm{Loc}_{\mathcal{O}_F}(\pi_1) \to \mathrm{Loc}_{\mathcal{O}_E}(\pi_1)$$

induced by extension of scalars is an equivalence.

By Proposition 6.8.4 (3) and the aforementioned equivalence of $\mathrm{Loc}_X(\mathcal{O}_F)$ with $\mathrm{Loc}_{\mathcal{O}_F}(\pi_1)$ when F is a finite extension of $\mathbf{Q}_\ell$, this proves that $\mathrm{Loc}_X(\mathcal{O}_E)$ is equivalent to $\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)$ when E is any algebraic extension of $\mathbf{Q}_\ell$.

Proposition. — The functor $\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)[\ell^{-1}] \longrightarrow \mathrm{Loc}_E(\pi_1)$ *given by* $M \mapsto M[\ell^{-1}]$ *is an equivalence.*

(The category on the left is described in the note to Proposition 6.8.4 (4).)

Proof. — Essential surjectivity: let $V \simeq E^n$ be in $\mathrm{Loc}_E(\pi_1)$ of rank n; then $\pi_1 \times \mathbf{Z}_\ell^n$ is compact and its image $\pi_1 \mathbf{Z}_\ell^n$ in V is contained in $\ell^{-r} \mathcal{O}_F^n$ for some r and finite extension F of $\mathbf{Q}_\ell$ contained in E by Lemma 4.3.7 (you can take $r = [F : \mathbf{Q}_\ell]$ to get a countable tower with colimit E). As the action of π_1 is E-linear, this means that $\pi_1 \mathcal{O}_F^n \subset \ell^{-r}\mathcal{O}_F^n$. Let L denote the $\mathcal{O}_F$-span of $\pi_1\mathcal{O}_F^n$. Then (as $\mathcal{O}_F$ is a d.v.r. and $\ell^{-r}\mathcal{O}_F^n$ is a finite torsion-free $\mathcal{O}_F$-module), L is a free $\mathcal{O}_F$-module of rank n stable under π_1 with $L \otimes_{\mathcal{O}_F} E = V$.

Fullness: by essential surjectivity and the above Proposition, we may assume we're given representations of π_1 on finite free $\mathcal{O}_F$-modules M and N (F a finite extension of $\mathbf{Q}_\ell$ contained in E) and a π_1-equivariant morphism $M \otimes_{\mathcal{O}_F} E \to N \otimes_{\mathcal{O}_F} E$. Letting $M_K := M \otimes_{\mathcal{O}_F} \mathcal{O}_K$ for $F \subset K \subset E$, since $N_E[\ell^{-1}] = \bigcup_{r,K} \ell^{-r} N_K$ (sum over K/F finite), a free basis for M as $\mathcal{O}_F$-module lands in $\ell^{-r}N_K$ for some $K \supset F$ finite, inducing a map $M_K \to \ell^{-r}N_K$ (since our map $M_E[\ell^{-1}] \to N_E[\ell^{-1}]$ is of course E-linear, and M_K and $\ell^{-r}N_K$ are sitting inside). Our original map $M_E[\ell^{-1}] \to N_E[\ell^{-1}]$ is obtained from the map $M_K \to \ell^{-r}N_K$ by extending scalars $- \otimes_{\mathcal{O}_K} E$, so we've found free $\mathcal{O}_E$-submodules M_E and $\ell^{-r}N_E$ of $M_E[\ell^{-1}]$ and $N_E[\ell^{-1}]$, respectively, and a map $M_E \to \ell^{-r}N_E$ inducing the map we started with after inverting ℓ.

Faithfulness: Given representations of π_1 on free $\mathcal{O}_E$-modules M and N,

$$\mathrm{Hom}_{\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)[\ell^{-1}]}(M, N) = \mathrm{Hom}_{\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)}(M, N)[\ell^{-1}],$$

as discussed in the note to Proposition 6.8.4 (4). In particular, as our functor induces an E-linear map of Homs, it suffices to show that the composite functor

$$\mathrm{Loc}_{\mathcal{O}_E}(\pi_1) \to \mathrm{Loc}_{\mathcal{O}_E}(\pi_1)[\ell^{-1}] \to \mathrm{Loc}_E(\pi_1)$$

is faithful. This is clear: given a nonzero map $M \to N$ in $\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)$, the map $M \otimes_{\mathcal{O}_E} E \to N \otimes_{\mathcal{O}_E} E$ is nonzero since M and N are finite free. □

Now to show that $\mathrm{Loc}_X(E)$ admits kernels and cokernels. By Proposition 6.8.4, $\mathrm{Loc}_E(\pi_1) \simeq \mathrm{Loc}_X(\mathcal{O}_E)[\ell^{-1}] \to \mathrm{Loc}_X(E)$ is fully faithful, and by assumption L and L′ are in the essential image of this functor. The kernel and cokernel of the corresponding map in $\mathrm{Loc}_E(\pi_1)$ give rise to maps in $\mathrm{Loc}_X(E)$ that we must check are the kernel and cokernel for f. So suppose $k : N \to L$ is the map in $\mathrm{Loc}_X(E)$ corresponding to the kernel of f in $\mathrm{Loc}_E(\pi_1)$ and suppose $K \to L$ composes with f to give the zero morphism. Then, over an étale cover $(U_i \to X)$, $K|_{U_i}$ is in the essential image of $\mathrm{Loc}_E(\pi_1)$, giving rise to unique factorizations $K|_{U_i} \to N|_{U_i} \xrightarrow{k} L|_{U_i}$ for each i. By uniqueness and étale descent, we get a unique factorization $K \to N \xrightarrow{k} L$, so N is indeed the kernel of f in $\mathrm{Loc}_X(E)$. (Restriction along a connected étale neighborhood $U_i \to X$ of $\overline{x}$ of an object of $\mathrm{Loc}_X(E)$ in the image of $\mathrm{Loc}_E(\pi_1(X,\overline{x}))$ corresponds to restriction along $\pi_1(U_i,\overline{x}) \to \pi_1(X,\overline{x})$, and this restriction commutes with the formation of kernels and cokernels.)

It remains to show that $k : N \to L$ is the kernel of f in $\mathrm{Mod}_X(E)$. *A priori,* it might be smaller, but there's always an injective map $N \to \ker f$ (injective since the injection $N \hookrightarrow L$ factors through $\ker f$). Localizing, let's assume we have a $s \in L(X)$ that goes to zero under $L(X) \to L'(X)$. Then s defines a morphism $E \to L$ in $\mathrm{Loc}_X(E)$ that factors via k. In particular, $s \in N(X)$. Therefore $N \to \ker f$ is surjective, and an isomorphism.

6.8.6. 'The construction of E_X is compatible with pullback under locally closed immersions' $\rightsquigarrow$ by Lemma 6.8.2, $E_X = \mathrm{colim}(\mathcal{O}_{X,E} \xrightarrow{\ell} \mathcal{O}_{X,E} \xrightarrow{\ell} \dots)$, $\mathcal{O}_{X,E} = \mathrm{colim}_{F\subset E}\,\mathcal{O}_{F,X}$, and $\mathcal{O}_{F,X} = \hat{\mathcal{O}}_E$. So the result follows from the fact that if j locally closed immersion, j^* commutes with limits and colimits (Corollary 6.1.5 for j closed immersion).

6.8.9. To see $\mathrm{Loc}_X(E)$ (or $\mathrm{Loc}_X(\mathcal{O}_E)$) is closed under extensions: suppose

$$0 \longrightarrow \mathcal{O}_E^r \longrightarrow F \longrightarrow \mathcal{O}_E^s \longrightarrow 0$$

is exact in $\mathrm{Mod}(X_{\text{proét}}, \mathcal{O}_E)$. Localizing, we may assume $F(X) \to \mathcal{O}_E^s(X)$ is surjective, corresponding to s global sections of F. Similarly, $\mathcal{O}_E^r \to F$ is the data of r global sections of F. In this way we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_E^r \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{O}_E^{r+s} \arrow[r] \arrow[d] & \mathcal{O}_E^s \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathcal{O}_E^r \arrow[r] & F \arrow[r] & \mathcal{O}_E^s \arrow[r] & 0. \end{tikzcd}$$

which must be an isomorphism in $\mathrm{Mod}(X_{\text{proét}}, \mathcal{O}_E)$.

6.8.10. The sheaves $\mathcal{O}_E$ and $\mathcal{O}_{E,X}$ both appear in $\Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}$: $\mathcal{O}_E$ is the constant sheaf; i.e. the sheaf of Lemma 4.2.12 where $\mathcal{O}_E$ has the discrete topology. Recall that $\mathcal{O}_{F,X} = \hat{\mathcal{O}}_F$ when F is a finite extension of $\mathbf{Q}_\ell$ by Lemma 6.8.2. We already saw in the note to Corollary 6.5.7 that when F is a finite extension of $\mathbf{Q}_\ell$ and M is a finite $\mathcal{O}_F$-module, $\underline{\hat{M}} = \underline{M} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X}$. Note that even if you don't assume *a priori* that Λ depends only on X_i, after possibly refining the partition, you can secure this (c.f. first paragraph of `099Y`). The following results work to establish that $\mathrm{Cons}_X(\mathcal{O}_E)$ agrees with the classical definition of constructible $\mathcal{O}_E$-sheaf (at least where that classical definition exists; i.e. for $E/\mathbf{Q}_\ell$ finite). To this end, see Propositions ⊛ and ⊙ below.

For any scheme X, let $\mathrm{Lisse}_X(\mathcal{O}_E)$ denote the full subcategory of $\mathrm{Mod}_X(\mathcal{O}_{E,X})$ generated by those objects $\mathcal{F}$ locally isomorphic to $\Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}$ for Λ an $\mathcal{O}_E$-module of finite presentation.[72] (In light of what has just been said, the module Λ may vary by connected component.) We can define $\mathrm{Lisse}_X(\mathcal{O}_F/\varpi^n\mathcal{O}_F)$ in the obvious way, and it's equivalent to the category of locally constant constructible étale sheaves of $\mathcal{O}_F/\varpi^n$-modules (`099Y`).

Proposition (⊛). — *Let* $F \supset \mathbf{Q}_\ell$ *be a finite extension and* X *a topologically noetherian scheme. The functors*

$$\begin{aligned} \mathrm{Lisse}_X(\mathcal{O}_F) &\leftrightarrows \lim_n \mathrm{Lisse}_X(\mathcal{O}_F/\varpi^n\mathcal{O}_F) \\ A : \mathcal{F} &\mapsto (\mathcal{F}/\varpi^n\mathcal{F})_n \\ B : \lim_n \mathcal{F}_n &\leftarrow\!\shortmid (\mathcal{F}_n)_n \end{aligned}$$

define an equivalence. Therefore if X *is moreover connected with geometric point* x*,* $\mathrm{Lisse}_X(\mathcal{O}_F)$ *is equivalent to the category of finite* $\mathcal{O}_F$*-modules endowed with a continuous action of* $\pi_1(X, x)$.

Proof. — If we start with $\mathcal{F}$ in $\mathrm{Lisse}_X(\mathcal{O}_F)$, then, localizing, we may assume $\mathcal{F} = \underline{\Lambda} \otimes_{\mathcal{O}_F} \hat{\mathcal{O}}_F = \underline{\hat{\Lambda}}$, and as $\hat{\mathcal{O}}_F \otimes_{\mathcal{O}_F} \mathcal{O}_F/\varpi^n = \mathcal{O}_F/\varpi^n$ (`093M`), $BA\mathcal{F} = \mathcal{F}$. Now starting with an inverse system $(\mathcal{F}_n)_n$, to show $AB(\mathcal{F}_n)_n = (\mathcal{F}_n)_n$ we may assume X connected. For each n there is a connected finite étale $f_n : Y_n \to X$ factoring via f_{n-1} so that $f_n^*\mathcal{F}_n = \underline{\Lambda}_n$ for some finite $\mathcal{O}_F/\varpi^n$-module Λ_n. The morphisms $\mathcal{F}_{n+1} \to \mathcal{F}_n$ give rise (upon applying $\Gamma(Y_{n+1,\text{ét}}, -)$) to morphisms $a_{n+1} : \Lambda_{n+1} \to \Lambda_n$ inducing $f_{n+1}^*\mathcal{F}_{n+1} \to f_{n+1}^*\mathcal{F}_n$. Let $f : \lim_n Y_n \to X$ be the integral, pro-étale cover. On Y we have an isomorphism of inverse systems $(\underline{\Lambda}_n)_n \to (f^*\mathcal{F}_n)_n$ where the transition maps of the system on the left are given by $\underline{a}_n$. Therefore if $\Lambda := \lim_n \Lambda_n$, $\underline{\hat{\Lambda}} \to f^* \lim_n \mathcal{F}_n$ is an isomorphism, and we find that the functor B indeed lands in $\mathrm{Lisse}_X(\mathcal{O}_F)$. To see that

[72]The choice of notation reflects that classically, lisse sheaves of $\mathcal{O}_E$-modules needn't be locally free.

$AB(\mathcal{F}_n)_n = (\mathcal{F}_n)_n$, note that the projections $\lim_n \mathcal{F}_n \to \mathcal{F}_n$ induce a morphism of inverse systems $((\lim_n \mathcal{F}_n) \otimes_{\hat{\mathcal{O}}_F} \mathcal{O}_F/\varpi^n)_n \to (\mathcal{F}_n)_n$, which, upon applying f^*, is isomorphic to $(\underline{\Lambda}_n)_n \to (f^*\mathcal{F}_n)_n$ by `087W` (since $\hat{\underline{\Lambda}} = \Lambda \otimes_{\mathcal{O}_F} \hat{\mathcal{O}}_F$), which we know is an isomorphism. As in the note to Lemma 6.8.5, the last sentence follows from Lemma 1.2.4.2 of SGA 5, Exposé VI. □

Let π_1 be a profinite group and let $\mathrm{Lisse}_{\mathcal{O}_E}(\pi_1)$ denote the category of $\mathcal{O}_E$-modules of finite presentation with continuous action of π_1.

Proposition (※). — *The functor*

$$\mathrm{colim}_{F\subset E}\, \mathrm{Lisse}_{\mathcal{O}_F}(\pi_1) \longrightarrow \mathrm{Lisse}_{\mathcal{O}_E}(\pi_1),$$

given by $M \mapsto M \otimes_{\mathcal{O}_F} \mathcal{O}_E$ *for* $F \supset \mathbf{Q}_\ell$ *finite and* M *in* $\mathrm{Lisse}_{\mathcal{O}_F}(\pi_1)$*, is an equivalence.*

Proof. — Let M be an object of $\mathrm{Lisse}_{\mathcal{O}_E}(\pi_1)$. Then we can find a finite extension $F \supset \mathbf{Q}_\ell$ and a finite $\mathcal{O}_F$-module M_F so that $M_F \otimes_{\mathcal{O}_F} \mathcal{O}_E = M$ as $\mathcal{O}_E$-modules (`05N7`). Then $M_F \to M_F \otimes_{\mathcal{O}_F} \mathcal{O}_E$ is injective by `00HL` since the quotient $\mathcal{O}_E/\mathcal{O}_F$ is flat over $\mathcal{O}_F$.[73] In fact, for the same reason, we have that $M_{F'} := M_F \otimes_{\mathcal{O}_F} \mathcal{O}_{F'} \hookrightarrow M$ and $M_F \hookrightarrow M_{F'}$ inject for each finite extension F'/F, and $M = \bigcup_{F\subset F'} M_{F'}$. By Lemma 4.3.7, $\pi_1(M_F) \subset M_{F'}$ for some $F' \supset F$ finite,[74] and as the action of π_1 is $\mathcal{O}_E$-linear, it's certainly $\mathcal{O}_{F'}$-linear, so $\pi_1(M_{F'}) \subset M_{F'}$. Therefore $M_{F'}$, with this action of π_1, defines after extending scalars $M = M_{F'} \otimes_{\mathcal{O}_{F'}} \mathcal{O}_E$ with its action of π_1.

The functor $\mathrm{colim}_{F\subset E}\, \mathrm{Lisse}_{\mathcal{O}_F}(\pi_1) \to \mathrm{Lisse}_{\mathcal{O}_E}(\pi_1)$ is faithful because $M_F \to M_F \otimes_{\mathcal{O}_F} \mathcal{O}_E$ is a monomorphism when M_F is in $\mathrm{Lisse}_{\mathcal{O}_F}(\pi_1)$. Suppose M and N are objects of $\mathrm{Lisse}_{\mathcal{O}_E}(\pi_1)$ and we have a morphism $\varphi : M \to N$. Then we can find a finite extension F of $\mathbf{Q}_\ell$ and objects M_F, N_F of $\mathrm{Lisse}_{\mathcal{O}_F}(\pi_1)$ so that $M = M_F \otimes_{\mathcal{O}_F} \mathcal{O}_E$, $N = N_F \otimes_{\mathcal{O}_F} \mathcal{O}_E$ as above, and we can regard M_F as a compact $\mathcal{O}_F$-submodule of M, so $\varphi(M_F) \subset N_{F'}$ for some $F' \supset F$ finite. As φ is $\mathcal{O}_E$-linear, it's $\mathcal{O}_{F'}$-linear, so $\varphi(M_{F'}) \subset N_{F'}$, and we've found

[73]To see that $\mathcal{O}_E/\mathcal{O}_F$ is flat over $\mathcal{O}_F$, well, it's a filtered colimit of $\mathcal{O}_{F'}/\mathcal{O}_F$ where F'/F is a finite extension, so it suffices to see that $\mathcal{O}_{F'}/\mathcal{O}_F$ is flat as an $\mathcal{O}_F$-module (`05UT`). In fact it is free of rank $n-1$ as $\mathcal{O}_{F'}$ admits an integral basis $\{1, x, x^2, \ldots, x^{n-1}\}$ over $\mathcal{O}_F$, where $x \in \mathcal{O}_{F'}$ and $n = [F' : F]$ (Serre, *Corps Locaux* III §6 Proposition 12).

[74]The finitely-generated $\mathcal{O}_{F'}$-module M'_F is compact Hausdorff since it's complete as $\mathcal{O}_{F'}$-module.

a finite extension $F' \supset F$ and a map $M_{F'} \to N_{F'}$ inducing φ after extending scalars to $\mathcal{O}_E$. In conclusion, our functor is full. $\square$

Next, we would like to show that $\operatorname{colim}_{F\subset E} \operatorname{Lisse}_X(\mathcal{O}_F) \longrightarrow \operatorname{Lisse}_X(\mathcal{O}_E)$ is an equivalence.

Lemma (◎). — *Fix an algebraic extension* E *of* $\mathbf{Q}_\ell$.

(1) If $\mathscr{F}$ *and* $\mathscr{G}$ *are in* $\operatorname{Lisse}_X(\mathcal{O}_E)$*, then* $\mathscr{H}om_{\mathcal{O}_{E,X}}(\mathscr{F},\mathscr{G})$ *is in* $\operatorname{Lisse}_X(\mathcal{O}_E)$*. More precisely, if* M *and* N *are* $\mathcal{O}_E$*-modules of finite presentation, the natural map*

$$\underline{\operatorname{Hom}_{\mathcal{O}_E}(M,N)} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \longrightarrow \mathscr{H}om_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X})$$

is an isomorphism.

(2) If F *is a finite extension of* $\mathbf{Q}_\ell$ *contained in* E *and* $\mathscr{F}$ *and* $\mathscr{G}$ *are in* $\operatorname{Lisse}_X(\mathcal{O}_F)$*, then the natural map*

$$\mathscr{H}om_{\mathcal{O}_{F,X}}(\mathscr{F},\mathscr{G}) \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X} \longrightarrow \mathscr{H}om_{\mathcal{O}_{E,X}}(\mathscr{F} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}, \mathscr{G} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X})$$

is an isomorphism.

(Both natural maps described above are examples of the map

$$f^*\mathscr{H}om(M,N) \to \mathscr{H}om(f^*M, f^*N)$$

obtained by adjunction from $\mathscr{H}om(M,N) \to \mathscr{H}om(M, f_*f^*N) = f_*\mathscr{H}om(f^*M, f^*N)$ when f is a morphism of ringed topos: take $f : (X_{\text{proét}}, \mathcal{O}_{E,X}) \to (*, \mathcal{O}_E)$ or $f : (X_{\text{proét}}, \mathcal{O}_{E,X}) \to (X_{\text{proét}}, \mathcal{O}_{F,X})$.) When F is a finite extension of $\mathbf{Q}_\ell$ and M is a finite $\mathcal{O}_F$-module, recall that we have $\mathcal{O}_{F,X} = \hat{\mathcal{O}}_F$ and $M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X} = \underline{\hat{M}} = \lim_n \underline{M}/\varpi^n\underline{M}$, where ϖ is a uniformizer for $\mathcal{O}_F$.

Proof. — First consider the case where E = F is a finite extension of $\mathbf{Q}_\ell$. The claims are local, so suppose $\mathscr{F} = \underline{\hat{M}}$ and $\mathscr{G} = \underline{\hat{N}}$ for finite $\mathcal{O}_F$-modules M and N. Then $\mathscr{H}om(\mathscr{F},\mathscr{G})$ is locally isomorphic to $\mathscr{H}om(\underline{\hat{M}}, \underline{\hat{N}})$ and we must show

$$\mathscr{H}om_{\hat{\mathcal{O}}_F}(\underline{\hat{M}}, \underline{\hat{N}}) = \underline{\operatorname{Hom}_{\mathcal{O}_F}(M,N)} \otimes_{\mathcal{O}_F} \hat{\mathcal{O}}_F.$$

Recall the structure of an $\mathcal{O}_F$-module Λ of finite type: Λ decomposes as a direct sum of a finite free $\mathcal{O}_F$-module and a finite torsion $\mathcal{O}_F$-module. The finite torsion piece is of the form of a finite direct sum $\bigoplus_i \mathcal{O}_F/(\varpi^{n_i})$. When $M, N = \mathcal{O}_F$, the assertion is clear. When $M = \mathcal{O}_F/(\varpi^m)$, $\underline{\hat{M}} = \underline{M}$ (`093M`), so if $N = \mathcal{O}_F$, both sides are zero, since $\hat{\mathcal{O}}_F$ is torsion-free as $\mathcal{O}_F$-module.[75] When $N = \mathcal{O}_F/(\varpi^n)$ and M is still $\mathcal{O}_F/(\varpi^m)$, then the right-hand side is given simply by $\underline{\mathrm{Hom}_{\mathcal{O}_F}(M, N)}$, while the left-hand side is $\mathcal{H}om_{\mathcal{O}_F}(\underline{M}, \underline{N})$, and we conclude by `093T`. When N is the same but $M = \mathcal{O}_F$, both sides are isomorphic to $\underline{N}$.

Now replace F by E no longer finite but still algebraic over $\mathbf{Q}_\ell$. Suppose given $\mathcal{F}, \mathcal{G}$ in $\mathrm{Lisse}_X(\mathcal{O}_E)$ and $\mathcal{O}_E$-modules of finite presentation so that $\mathcal{F} = M \otimes_{\mathcal{O}_E} \hat{\mathcal{O}}_E$ and $\mathcal{G} = N \otimes_{\mathcal{O}_E} \hat{\mathcal{O}}_E$. Then we can find a finite algebraic extension F of $\mathbf{Q}_\ell$ and $\mathcal{O}_F$-modules M_F, N_F so that $M = M_F \otimes_{\mathcal{O}_F} \mathcal{O}_E$ and $N = N_F \otimes_{\mathcal{O}_F} \mathcal{O}_E$. We have to show

$$\mathcal{H}om_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) = \underline{\mathrm{Hom}_{\mathcal{O}_E}(M, N)} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}.$$

Again we break up M_F and N_F into pieces: when $M_F = \mathcal{O}_F$, the statement is clearly true. When $M_F = \mathcal{O}_F/(\varpi^m)$ and $N_F = \mathcal{O}_F$, both sides are zero.[76] When $M_F = \mathcal{O}_F/(\varpi^m)$ and $N_F = \mathcal{O}_F/(\varpi^n)$, $\mathrm{Hom}_{\mathcal{O}_E}(M, N)$ is again M or N depending on whether $m \le n$ or $m \ge n$, respectively, while $\mathrm{Hom}_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X})$ is all of $(N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X})(X)$ when $m \ge n$ and when $m \le n$ it is the $\mathcal{O}_F$-submodule of $(N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X})(X)$ annihilated by ϖ^m. A section $f \in (\mathcal{O}_{E,X}/\varpi^n\mathcal{O}_{E,X})(X)$ can be represented by functions $f_i : U_i \to \mathcal{O}_E$ over some pro-étale cover $U_i \to X$. To be annihilated by ϖ^m means that $\varpi^m f_i$ takes values in $\varpi^n\mathcal{O}_E$. Therefore f_i takes values in $\varpi^{n-m}\mathcal{O}_E$. In other words, f_i goes to zero under the map $\mathcal{O}_{E,X}(U_i) \to (\mathcal{O}_{E,X}/\varpi^{n-m}\mathcal{O}_{E,X})(U_i)$, and f goes to zero under $(\mathcal{O}_{E,X}/\varpi^n\mathcal{O}_{E,X})(X) \to (\mathcal{O}_{E,X}/\varpi^{n-m}\mathcal{O}_{E,X})(X)$. This proves that the last map in the exact sequence

$$\begin{aligned} 0 \longrightarrow M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \longrightarrow \mathcal{H}om_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) \longrightarrow \\ \longrightarrow \mathcal{H}om_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, \mathcal{O}_F/(\varpi^{n-m}) \otimes_{\mathcal{O}_F} \mathcal{O}_{E,X}) \end{aligned}$$

is null. (The sequence comes from

$$0 \longrightarrow \varpi^{n-m}\mathcal{O}_F/\varpi^n\mathcal{O}_F = M \longrightarrow \mathcal{O}_F/(\varpi^n) = N \longrightarrow \mathcal{O}_F/(\varpi^{n-m}) \longrightarrow 0,$$

[75] Any section of $\mathcal{H}om_{\hat{\mathcal{O}}_F}(\underline{M}, \hat{\mathcal{O}}_F)$ determines a map from continous maps to M to continuous maps to $\mathcal{O}_F$, but the constraint that this be $\mathcal{O}_F$-linear forces this map to be null since the only continuous map to $\mathcal{O}_F$ of torsion is the null map.

[76] A function to $\mathcal{O}_E$ annihilated by a power of $\varpi \in \mathcal{O}_F \subset \mathcal{O}_F$ is everywhere zero.

since $\mathcal{O}_{E,X}$ is flat over $\mathcal{O}_F$.) In conclusion, when $M_F = \mathcal{O}_F/(\varpi^m)$ and $N_F = \mathcal{O}_F/(\varpi^n)$,

$$\mathrm{Hom}_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) = \underline{\mathrm{Hom}_{\mathcal{O}_E}(M, N)} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} = \begin{cases} M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} & m \le n \\ N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} & m \ge n. \end{cases}$$

Moreover, as we've seen, $\mathrm{Hom}_{\mathcal{O}_E}(M, N)$ is a finite direct sum of pieces of the form $\mathcal{O}_E$, 0, or $\mathcal{O}_E/\varpi^n\mathcal{O}_E$, all of which are $\mathcal{O}_E$-modules of finite presentation.

It remains to show that if $\mathcal{F}, \mathcal{G}$ are in $\mathrm{Lisse}_X(\mathcal{O}_F)$ for $F \supset \mathbf{Q}_\ell$ a finite extension, and $E \supset F$ is any algebraic extension, then

$$\mathcal{H}om_{\mathcal{O}_{F,X}}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{E,X}}(\mathcal{F} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}, \mathcal{G} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}).$$

We may assume $\mathcal{F} = M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X}$ and $\mathcal{G} = N \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X}$ for $\mathcal{O}_F$-modules M and N of finite type. Let $M_E := M \otimes_{\mathcal{O}_F} \mathcal{O}_E$, etc. Breaking into the same four cases as above, we see $\mathrm{Hom}_{\mathcal{O}_F}(M, N) \otimes_{\mathcal{O}_F} \mathcal{O}_E = \mathrm{Hom}_{\mathcal{O}_E}(M_E, N_E)$. (This is a simplification of what we saw above. The interesting case is when $M = \mathcal{O}_F/(\varpi^m)$ and $N = \mathcal{O}_F/(\varpi^n)$. Then if $m \le n$, $\mathrm{Hom}_{\mathcal{O}_F}(M, N) = M$, $\mathrm{Hom}_{\mathcal{O}_E}(M_E, N_E) = \mathrm{Hom}_{\mathcal{O}_E}(M_E, \varpi^{n-m}\mathcal{O}_E/\varpi^n\mathcal{O}_E)$, and $M_E \simeq \varpi^{n-m}\mathcal{O}_E/\varpi^n\mathcal{O}_E$, while if $m \ge n$, $\mathrm{Hom}_{\mathcal{O}_F}(M, N) = N$.) So

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_{F,X}}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X} &= \underline{\mathrm{Hom}_{\mathcal{O}_F}(M, N)} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X} \\
&= \underline{\mathrm{Hom}_{\mathcal{O}_F}(M, N)} \otimes_{\mathcal{O}_F} \mathcal{O}_E \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \\
&= \underline{\mathrm{Hom}_{\mathcal{O}_F}(M, N) \otimes_{\mathcal{O}_F} \mathcal{O}_E} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \\
&= \underline{\mathrm{Hom}_{\mathcal{O}_E}(M \otimes_{\mathcal{O}_F} \mathcal{O}_E, N \otimes_{\mathcal{O}_F} \mathcal{O}_E)} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \\
&= \mathcal{H}om_{\mathcal{O}_{E,X}}(M \otimes_{\mathcal{O}_F} \mathcal{O}_E \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}, N \otimes_{\mathcal{O}_F} \mathcal{O}_E \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) \\
&= \mathcal{H}om_{\mathcal{O}_{E,X}}(\mathcal{F} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}, \mathcal{G} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}).
\end{aligned}$$
□

Proposition (⊙). — *Let* X *be a coherent scheme and* E *an algebraic extension of* $\mathbf{Q}_\ell$. *The functor*

$$\mathrm{colim}_{F \subset E}\, \mathrm{Lisse}_X(\mathcal{O}_F) \longrightarrow \mathrm{Lisse}_X(\mathcal{O}_E)$$

given by extension of scalars is an equivalence, where the colimit is over finite extensions of $\mathbf{Q}_\ell$ *contained in* E. *In particular, if* X *is connected with geometric point* x, *then*

$\mathrm{Lisse}_X(\mathcal{O}_E)$ *is equivalent to the category* $\mathrm{Lisse}_{\mathcal{O}_E}(\pi_1(X,x))$*, the category of* $\mathcal{O}_E$*-modules of finite presentation with continuous action of* $\pi_1(X,x)$.

Proof. — That the last statement is implied by the previous is an immediate consequence of Propositions ⊛ and ※. The proof of the first statement goes like that of Proposition 6.8.4 (3).

Full faithfulness: let $\mathscr{M}$ be in $\mathrm{Lisse}_X(\mathcal{O}_F)$ with F a finite extension of $\mathbf{Q}_\ell$. Then $\mathscr{M}_E := \mathscr{M} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X} = \mathrm{colim}_{F\subset F'\subset E}\, \mathscr{M} \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{F',X}$ by Lemma 6.8.2 (2) and the fact that tensor product commutes with colimits. I claim that the transition maps of this colimit are injective; i.e. that if Λ is a finite $\mathcal{O}_F$-module and $F' \supset F$ is a finite extension, then $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X} \to \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F',X}$ is injective.

$$\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F',X} = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F'} \otimes_{\mathcal{O}_{F'}} \mathcal{O}_{F',X}$$

We have $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X} = \underline{\hat{\Lambda}} := \lim_n \Lambda/\varpi^n\Lambda$, where $\varpi \in \mathcal{O}_F$ is a uniformizer, and $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F',X} = \lim_n(\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}) \otimes_{\mathcal{O}_{F'}} \mathcal{O}_{F'}/\tilde{\varpi}^n$, where $\tilde{\varpi} \in \mathcal{O}_{F'}$ is a uniformizer. As $\tilde{\varpi}$ can be chosen so that $\tilde{\varpi}^q = \varpi$ for some q,

$$\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F',X} = \lim_n (\Lambda/\varpi^n\Lambda \otimes_{\mathcal{O}_F/(\varpi^n)} \mathcal{O}_{F'}/(\tilde{\varpi}^{qn})),^{77}$$

so it suffices to show that any $\mathcal{O}_F/(\varpi^n)$-module M injects into $M \otimes_{\mathcal{O}_F/(\varpi^n)} \mathcal{O}_{F'}/(\tilde{\varpi}^{qn})$. This is clear, since as we saw in the proof of Proposition ※ above, $\mathcal{O}_{F'}$ admits an integral basis as an $\mathcal{O}_F$-module, so is is free of rank $n = [F' : F]$ as an $\mathcal{O}_F$-module, and $\mathcal{O}_{F'}/(\tilde{\varpi}^{qn}) = \mathcal{O}_{F'} \otimes_{\mathcal{O}_F} \mathcal{O}_F/(\varpi^n)$, so this morphism is just the inclusion of M into one of the factors of $M^{\oplus n}$.

Set $\mathscr{M} := \mathscr{H}om_{\mathcal{O}_{F,X}}(\mathscr{F},\mathscr{G})$ for $\mathscr{F},\mathscr{G}$ in $\mathrm{Lisse}_X(\mathcal{O}_F)$. Then we've just seen in Lemma ⊚ that $\mathscr{M}$ is also in $\mathrm{Lisse}_X(\mathcal{O}_F)$ and $\mathscr{M}_E = \mathscr{H}om_{\mathcal{O}_{E,X}}(\mathscr{F}_E,\mathscr{G}_E)$. As the colimit of abelian sheaves can be computed in underlying sheaves of sets, we have by `0738` that

$$\mathrm{Hom}_{\mathcal{O}_{E,X}}(\mathscr{F}_E,\mathscr{G}_E) = \mathscr{M}_E(X) = \mathrm{colim}_{F\subset F'\subset E}\, \mathscr{M}_{F'}(X) = \mathrm{colim}_{F\subset F'\subset E}\, \mathrm{Hom}_{\mathcal{O}_{F',X}}(\mathscr{F}_{F'},\mathscr{G}_{F'}).$$

It remains only to show essential surjectivity. Just as $\mathrm{Loc}_X(\mathcal{O}_F)$ satisfies pro-étale descent, so too does $\mathrm{Lisse}_X(\mathcal{O}_F)$, so fix a coherent w-contractible pro-étale cover $Y \to X$ and describe $\mathrm{Lisse}_X(\mathcal{O}_E)$ in terms of descent data for $Y \to X$. Given any $\mathscr{F}$ in $\mathrm{Lisse}_Y(\mathcal{O}_E)$, there is a finite partition $\mathrm{Y} = \coprod_i \mathrm{Y}_i$ into clopen subsets and $\mathcal{O}_\mathrm{E}$-modules Λ_i of finite presentation so that $\mathcal{F}_i = \Lambda_i \otimes_{\mathcal{O}_\mathrm{E}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$. There is a finite extension F of $\mathbf{Q}_\ell$ and $\mathcal{O}_\mathrm{F}$-modules Λ_i' so that $\Lambda_i = (\Lambda_i')_\mathrm{E}$ (05N7). The sheaves $\mathcal{F}_i' = \Lambda_i' \otimes_{\mathcal{O}_\mathrm{F}} \mathcal{O}_{\mathrm{F},\mathrm{X}}$ define a sheaf $\mathcal{F}'$ in $\mathrm{Lisse}_\mathrm{Y}(\mathcal{O}_\mathrm{F})$ with the property that $\mathcal{F}_\mathrm{E}' = \mathcal{F}$. By full faithfulness, the categories of descent data are the same, and we conclude that our functor is essentially surjective.[78] □

Proposition (⋇). — *Let* X *be a topologically noetherian scheme and* E *an algebraic extension of* $\mathbf{Q}_\ell$. *Then* $\mathrm{Lisse}_\mathrm{X}(\mathcal{O}_\mathrm{E})$ *is a weak Serre subcategory of* $\mathrm{Mod}_\mathrm{X}(\mathcal{O}_\mathrm{E})$.

Proof. — We must check that $\mathrm{Lisse}_\mathrm{X}(\mathcal{O}_\mathrm{E})$ is closed under kernels, cokernels, and extensions. As X is topologically noetherian, we may assume that X is connected with geometric point x. As the category of $\mathcal{O}_\mathrm{E}$-modules of finite presentation endowed with continuous action of $\pi_1(\mathrm{X}, x)$ has kernels and cokernels, so does $\mathrm{Lisse}_\mathrm{X}(\mathcal{O}_\mathrm{E})$ by Proposition ⊙. As in the proof of Corollary 6.8.5, one might worry that if $0 \to \mathcal{K} \xrightarrow{k} \mathcal{F} \xrightarrow{f} \mathcal{G}$ is exact in $\mathrm{Lisse}_\mathrm{X}(\mathcal{O}_\mathrm{E})$, $\mathcal{K}$ may not be all of $\ker f$ (computed in $\mathrm{Mod}_\mathrm{X}(\mathcal{O}_\mathrm{E})$), but that's not the case: if $s \in \mathcal{F}(\mathrm{U})$ goes to zero under $\mathcal{F}(\mathrm{U}) \to \mathcal{G}(\mathrm{U})$ ($\mathrm{U} \to \mathrm{X}$ in $\mathrm{X}_{\text{proét}}$), s defines a morphism $\mathcal{O}_{\mathrm{E},\mathrm{U}} \to \mathcal{F}|_\mathrm{U}$ that factors through $k|_\mathrm{U}$ (the formation of kernels in $\mathrm{Lisse}_\mathrm{X}(\mathcal{O}_\mathrm{E})$ commutes with restriction, because the same is true in $\mathrm{Lisse}_{\mathcal{O}_\mathrm{E}}(\pi_1(\mathrm{X}, x))$).

It remains to check that $\mathrm{Lisse}_\mathrm{X}(\mathcal{O}_\mathrm{E})$ is closed under extensions. Suppose

$$0 \longrightarrow \mathrm{M} \otimes_{\mathcal{O}_\mathrm{E}} \mathcal{O}_{\mathrm{E},\mathrm{X}} \longrightarrow \mathcal{F} \longrightarrow \mathrm{N} \otimes_{\mathcal{O}_\mathrm{E}} \mathcal{O}_{\mathrm{E},\mathrm{X}} \longrightarrow 0$$

is exact in $\mathrm{Mod}_\mathrm{X}(\mathcal{O}_\mathrm{E})$, with M and N $\mathcal{O}_\mathrm{E}$-modules of finite presentation. We have an exact sequence

$$0 \longrightarrow \mathrm{K} \longrightarrow \mathcal{O}_\mathrm{E}^r \longrightarrow \mathrm{N} \longrightarrow 0$$

with K a finite $\mathcal{O}_\mathrm{E}$-module, as N is of finite presentation as $\mathcal{O}_\mathrm{E}$-module. Localizing, we may assume the map $\mathcal{O}_{\mathrm{E},\mathrm{X}}^r \to \mathrm{N} \otimes_{\mathcal{O}_\mathrm{E}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$ factors through the map $\mathcal{F} \to \mathrm{N} \otimes_{\mathcal{O}_\mathrm{E}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$. We

[78] For more details, see the note to Proposition 6.8.4 (3).

obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] \arrow[d] & \mathcal{O}_{E,X}^r \arrow[r] \arrow[d] & N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] & \mathcal{F} \arrow[r] & N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] & 0
\end{tikzcd}$$

realizing $\mathcal{F}$ as a pushout in $\mathrm{Mod}_X(\mathcal{O}_E)$ (more details below). The category $\mathrm{Lisse}_X(\mathcal{O}_E)$ has pushouts since $\mathrm{Lisse}_{\mathcal{O}_E}(\pi_1(X, x))$ does. Therefore, as discussed below, and using the fact that kernels and cokernels in $\mathrm{Lisse}_X(\mathcal{O}_F)$ coincide with those computed in $\mathrm{Mod}_X(\mathcal{O}_E)$, if $\mathcal{P}$ denotes the pushout of $M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \leftarrow K \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \to \mathcal{O}_{E,X}^r$ in $\mathrm{Lisse}_X(\mathcal{O}_E)$, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] \arrow[d] & \mathcal{O}_{E,X}^r \arrow[r] \arrow[d] & N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] & \mathcal{P} \arrow[r] & N \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \arrow[r] & 0.
\end{tikzcd}$$

The general remarks below then give that $\mathcal{P}$ is a pushout in $\mathrm{Mod}_X(\mathcal{O}_E)$, so $\mathcal{F} = \mathcal{P}$. □

In any abelian category, suppose we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

with exact rows and so that f is an isomorphism. Then E is a pushout of $D \leftarrow A \to B$. This is because there is automatically a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & D \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & B \coprod_A D \arrow[r] \arrow[d, "e"] & G \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

with exact rows (d is injective by `08N4`). The objects C and G have the same universal property, giving rise to the isomorphism g. The morphism e comes from the universal property of the pushout. We get the morphism h from the universal property of G applied to the map $\mathrm{B} \coprod_{\mathrm{A}} \mathrm{D} \xrightarrow{e} \mathrm{E} \to \mathrm{F}$. Since the right half of the diagram commutes and the composite $\mathrm{B} \to \mathrm{B} \coprod_{\mathrm{A}} \mathrm{D} \xrightarrow{e} \mathrm{E}$ is the middle vertical map in the first diagram, by the uniqueness of the universal property of C applied to $\mathrm{B} \to \mathrm{E} \to \mathrm{F}$ we find that $h \circ g = f$. As g and f are isomorphisms, h is an isomorphism. We conclude that e is an isomorphism by the five-lemma.

6.8.11. 1. This is immediate from Proposition ⋇ in the note to Definition 6.8.10.

2. I don't attempt to show that $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is constructible when $\mathscr{F}, \mathscr{G}$ are in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$. (See, however, Lemma 6.7.13.) Rather, if $\mathscr{F}, \mathscr{G}$ are in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$ for $\mathrm{F}/\mathbf{Q}_\ell$ finite and E/F is algebraic, I would like to show that

$$\mathscr{H}om_{\mathcal{O}_{\mathrm{F,X}}}(\mathscr{F}, \mathscr{G}) \otimes_{\mathcal{O}_{\mathrm{F,X}}} \mathcal{O}_{\mathrm{E,X}} = \mathscr{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathscr{F} \otimes_{\mathcal{O}_{\mathrm{F,X}}} \mathcal{O}_{\mathrm{E,X}}, \mathscr{G} \otimes_{\mathcal{O}_{\mathrm{F,X}}} \mathcal{O}_{\mathrm{E,X}}). \qquad (*)$$

The right-hand side is equal (as $\mathcal{O}_{\mathrm{F,X}}$-module) to $\mathscr{H}om_{\mathcal{O}_{\mathrm{F,X}}}(\mathscr{F}, \mathscr{G} \otimes_{\mathcal{O}_{\mathrm{F,X}}} \mathcal{O}_{\mathrm{E,X}})$. If $\mathbf{Q}_\ell \subset \mathrm{F} \subset \mathrm{E}$ are algebraic extensions with $\mathrm{F}/\mathbf{Q}_\ell$ finite, $\mathcal{O}_{\mathrm{E}}$ is free as an $\mathcal{O}_{\mathrm{F}}$-module (of rank $[\mathrm{E} : \mathrm{F}]$ if this number is finite, and free on a countable set of generators otherwise).[79] The same is true of $\mathcal{O}_{\mathrm{E,X}}$ as $\mathcal{O}_{\mathrm{F,X}}$-module by Lemma 6.8.2. Indeed, if E/F is finite and ϖ is a uniformizer for F, then $\hat{\mathcal{O}}_{\mathrm{E}} = \lim_n \mathcal{O}_{\mathrm{E}}/\varpi^n\mathcal{O}_{\mathrm{E}}$ is free as $\hat{\mathcal{O}}_{\mathrm{F}}$-module of rank $[\mathrm{E} : \mathrm{F}]$, since the same is true of $\mathcal{O}_{\mathrm{E}}$ as $\mathcal{O}_{\mathrm{F}}$-module, and in general, $\mathcal{O}_{\mathrm{E,X}} = \operatorname{colim}_{\mathrm{F} \subset \mathrm{F}' \subset \mathrm{E}} \mathcal{O}_{\mathrm{F}',\mathrm{X}}$ with F'/F finite. Therefore, it suffices to show that $\mathscr{H}om_{\mathcal{O}_{\mathrm{F,X}}}(\mathscr{F}, -)$ commutes with direct sums in $\mathrm{Mod}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$. As direct sums are exact in $\mathrm{Mod}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E,X}})$, it suffices to show that $\mathrm{R}\mathscr{H}om_{\mathcal{O}_{\mathrm{F,X}}}(\mathscr{F}, -)$ commutes with direct sums in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{F,X}})$. Moreover, we would like to conclude something about $\mathrm{R}\,\mathrm{Hom}$ as well. To this end, denote by $\mathscr{C}_{\mathrm{X}} \subset \mathrm{D}^b(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{F,X}})$ the full (triangulated) subcategory spanned by those $\mathscr{F}$ for which $\mathrm{R}\mathscr{H}om_{\mathcal{O}_{\mathrm{F,X}}}(\mathscr{F}, -)$ commutes with all direct sums in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{F,X}})$. We'll show $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}}) \subset \mathscr{C}_{\mathrm{X}}$ in four steps and the statement about $\mathrm{R}\,\mathrm{Hom}$ in a fifth. Although the definition of $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$ is made only for topologically noetherian X, the points below hold for any coherent X.

(1) For any M in $\mathrm{D}_{\mathrm{perf}}(\mathcal{O}_{\mathrm{F}})$, one has $\mathrm{M} \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{F,X}} \in \operatorname{ob} \mathscr{C}_{\mathrm{X}}$.

(2) The property of lying in $\mathscr{C}_{\mathrm{X}}$ is pro-étale local on X.

[79] This is easily seen from the fact that when E/F is finite, $\mathcal{O}_{\mathrm{E}}$ admits an integral basis as an $\mathcal{O}_{\mathrm{F}}$-module, and in general, $\mathcal{O}_{\mathrm{E}} = \operatorname{colim}_{\mathrm{F} \subset \mathrm{F}'} \mathcal{O}_{\mathrm{F}'}$ for $\mathrm{F} \subset \mathrm{F}' \subset \mathrm{E}$ and F'/F finite.

(3) If $j : \mathrm{U} \hookrightarrow \mathrm{X}$ is a qcqs map[80] in $\mathrm{X}_{\text{proét}}$ and K belongs to $\mathscr{C}_{\mathrm{U}}$, then $j_!\mathrm{K} \in \mathrm{ob}\,\mathscr{C}_{\mathrm{X}}$.
(4) If $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is a closed immersion and $\mathscr{E}$ is in $\mathrm{Lisse}_{\mathrm{X}}(\mathscr{O}_{\mathrm{E}})$, then $i_*\mathscr{E} \in \mathrm{ob}\,\mathscr{C}_{\mathrm{X}}$.
(5) Any M in $\mathrm{D}^b(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{F,X}})$ lies in $\mathscr{C}_{\mathrm{X}}$ if and only if $\mathrm{R}\,\mathrm{Hom}(\mathrm{M}|_{\mathrm{U}}, -)$ commutes with direct sums in $\mathrm{D}^{\geq 0}(\mathrm{U}_{\text{proét}}, \mathscr{O}_{\mathrm{F,X}})$ for each coherent U in $\mathrm{X}_{\text{proét}}$.

It's clear that points (1–4) together imply that any $\mathscr{F}$ in $\mathrm{Cons}_{\mathrm{X}}(\mathscr{O}_{\mathrm{F}})$ belongs to $\mathscr{C}_{\mathrm{X}}$, since one may assume $\mathscr{F} = k_!\mathscr{E}$ for $\mathscr{E}$ lisse on a locally closed $k : \mathrm{Y} \hookrightarrow \mathrm{X}$. By (1) & (2), $\mathscr{E} \in \mathrm{ob}\,\mathscr{C}_{\mathrm{Y}}$. Factoring k as $\mathrm{Y} \xrightarrow{i} \mathrm{U} \xrightarrow{j} \mathrm{X}$, (4) gives that $i_*\mathscr{E}$ is in $\mathscr{C}_{\mathrm{U}}$, while (3) allows us to conclude that $k_!\mathscr{E} = j_!i_*\mathscr{E}$ is in $\mathscr{C}_{\mathrm{X}}$.

It will be useful to know in the following that $\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, -)$ commutes with direct sums in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{F,X}})$. To check this, the spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(\mathrm{X}_{\text{proét}}, \mathscr{H}^q\mathrm{K}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{X}_{\text{proét}}, \mathrm{K}),$$

the fact that direct sums are exact in $\mathrm{Mod}_{\mathrm{X}}(\mathscr{O}_{\mathrm{F,X}})$ and direct sums in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{F,X}})$ are taken termwise (`07D9`) reduces to showing that $\mathrm{R}\Gamma(\mathrm{X}_{\text{proét}}, -)$ commutes with direct sums in $\mathrm{Mod}_{\mathrm{X}}(\mathscr{O}_{\mathrm{F,X}})$, where the statement follows from `09A3`, since cohomology can be computed on the underlying abelian sheaf (`03FD`).

Now we prove the four points when X is a coherent scheme.

(1) By dévissage one reduces to $\mathrm{M} = \mathscr{O}_{\mathrm{F}}$ in which case the statement is trivial.
(2) This is clear since $\mathscr{H}om$ localizes [**SGAA**, IV 12.3] and the acyclicity of the cone can be checked locally.
(3) Let K be an object of $\mathscr{C}_{\mathrm{U}}$. Then

$$\mathrm{R}\mathscr{H}om_{\mathscr{O}_{\mathrm{F,X}}}(j_!\mathrm{K}, \mathrm{M}) = j_*\mathrm{R}\mathscr{H}om_{\mathscr{O}_{\mathrm{F,U}}}(\mathrm{K}, j^*\mathrm{M})$$

for any M in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{F,X}})$, so it will suffice to show that j_* commutes with direct sums in $\mathrm{D}^{\geq 0}$, knowing that $\mathrm{R}\Gamma(?_{\text{proét}}, -)$ does for ? a coherent scheme. Let $\{\mathrm{M}_i\}_{i\in\mathrm{I}}$ be a set of objects of $\mathrm{D}^{\geq 0}(\mathrm{U}_{\text{proét}}, \mathscr{O}_{\mathrm{F,U}})$, let V be a coherent object

[80] As X is coherent (qcqs), j is qcqs iff U is coherent.

of $X_{\text{proét}}$, and let $W := V \times_X U$. Then W is also coherent, so

$$\begin{aligned}R\Gamma(V_{\text{proét}}, \oplus_{i\in I} j_* M_i) &= \oplus_{i\in I} R\Gamma(V_{\text{proét}}, j_* M_i) = \oplus_{i\in I} R\Gamma(W_{\text{proét}}, M_i)\\ &= R\Gamma(W_{\text{proét}}, \oplus_{i\in I} M_i) = R\Gamma(V_{\text{proét}}, j_*(\oplus_{i\in I} M_i)).\end{aligned}$$

As this is true for all coherent V in $X_{\text{proét}}$, we conclude that $\oplus_{i\in I} j_* M_i = j_*(\oplus_{i\in I} M_i)$.

(4) Localizing using (2), we may assume X is affine. There is a pro-étale covering $(V_i \to Z)_{i\in I}$ in $Z^{\text{aff}}_{\text{proét}}$ over which $\mathscr{E}$ trivializes, and we may assume I finite. Then $(\tilde{V}_i \to X)_{i\in I}$ together with $j : X \smallsetminus Z \hookrightarrow X$ is a pro-étale covering of X, where $\tilde{V}_i$ denotes the henselization of §6.1. Localizing again, as $j^* i_* \mathscr{E} = 0$, it suffices to show the claim when $\mathscr{E} = M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X}$ with M a finite $\mathcal{O}_F$-module, at the cost of replacing X by one of the $\tilde{V}_i$. Recalling Lemma 6.5.8 (1),[81] we have the distinguished triangle

$$j_!(M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,U}) \longrightarrow M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X} \longrightarrow i_*(M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,Z}) \longrightarrow \quad ,$$

and we conclude by (1) & (3) that $i_*(M \otimes_{\mathcal{O}_F} \mathcal{O}_{F,X})$ is in $\mathscr{C}_X$.

(5) On the one hand, if $M \in \operatorname{ob} \mathscr{C}_X$ and $j : U \to X$ is qcqs in $X_{\text{proét}}$,

$$R\operatorname{Hom}(M|_U, -) = R\Gamma(X_{\text{proét}}, j_* R\mathscr{H}om(j^* M, -)) = R\Gamma(U_{\text{proét}}, R\mathscr{H}om(M, j_*(-))),$$

and all three functors have already been seen to commute with direct sums in $D^{\geq 0}(U_{\text{proét}}, \mathcal{O}_{F,X})$. On the other hand, if K_i belong to $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_{F,X})$, in order to show $R\mathscr{H}om(M, \oplus_i K_i) = \oplus_i R\mathscr{H}om(M, K_i)$ it suffices to check after applying $R\Gamma(U_{\text{proét}}, -)$ for U coherent in $X_{\text{proét}}$; i.e. it suffices to check $R\operatorname{Hom}(M|_U, \oplus_i K_i|_U) = \oplus_i R\operatorname{Hom}(M|_U, K_i|_U)$.

In conclusion, we've shown the following

Lemma. — If $\mathscr{F}$ is in $\operatorname{Cons}_X(\mathcal{O}_F)$*, then* $R\mathscr{H}om_{\mathcal{O}_F}(\mathscr{F}, -)$ *commutes with direct sums in* $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_{F,X})$*; in particular,* $\mathscr{H}om_{\mathcal{O}_F}(\mathscr{F}, -)$ *commutes with direct sums in* $\operatorname{Mod}_X(\mathcal{O}_{F,X})$*. The same is true of* $R\operatorname{Hom}_{\mathcal{O}_F}(\mathscr{F}, -)$ *and* $\operatorname{Hom}_{\mathcal{O}_F}(\mathscr{F}, -)$.

[81] Since j is quasi- compact, its base change is, and $Z \times_X \tilde{V}_i \hookrightarrow \tilde{V}_i$ is a constructible closed subset.

In particular, the map $(*)$ is an isomorphism for E an algebraic extension of F. In conclusion, if $\mathscr{F}$ is in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$ and K is in $\mathrm{D}^{\geq 0}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{F},\mathrm{X}})$, then

$$\begin{aligned} \mathrm{R}\operatorname{Hom}_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathscr{F}, \mathrm{K}) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{E}} &\xrightarrow{\sim} \mathrm{R}\operatorname{Hom}_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathscr{F}, \mathrm{K} \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}) \\ &= \mathrm{R}\operatorname{Hom}_{\mathcal{O}_{\mathrm{E},\mathrm{X}}}(\mathscr{F} \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}, \mathrm{K} \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}). \end{aligned}$$

As $\mathrm{R}\operatorname{Hom}_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathscr{F}, \mathrm{K}) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{E}} = \operatorname{colim}_{\mathrm{F} \subset \mathrm{F}' \subset \mathrm{E}} \mathrm{R}\operatorname{Hom}_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathscr{F}, \mathrm{K}) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{F}'}$, where the colimit is over finite extensions F' of F contained in E, we conclude from the exactness of filtered colimits that

$$\mathrm{C} : \operatorname{colim}_{\mathrm{F} \subset \mathrm{E}} \mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}}) \longrightarrow \mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})$$

is fully faithful. For the same reasons, we also have that if $\mathscr{F}$ and $\mathscr{G}$ are in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$,

$$\begin{aligned} &\operatorname{Ext}^1_{\mathcal{O}_{\mathrm{E},\mathrm{X}}}(\mathscr{F}_{\mathrm{E}}, \mathscr{G}_{\mathrm{E}}) = \operatorname{Ext}^1_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathscr{F}, \mathscr{G}) \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{E}}, \quad \text{and} \\ &\operatorname{Ext}^1_{\mathcal{O}_{\mathrm{E},\mathrm{X}}}(\mathscr{F}_{\mathrm{E}}, \mathscr{G}_{\mathrm{E}}) = \operatorname{colim}_{\mathrm{F} \subset \mathrm{F}' \subset \mathrm{E}} \operatorname{Ext}^1_{\mathcal{O}_{\mathrm{F}',\mathrm{X}}}(\mathscr{F}_{\mathrm{F}'}, \mathscr{G}_{\mathrm{F}'}), \end{aligned} \tag{†}$$

where $\mathscr{F}_{\mathrm{E}} := \mathscr{F} \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$, etc. and the colimit is over finite extensions F' of F contained in E. Any $\mathscr{F}$ in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{E}})$ can be filtered by constructible $\mathcal{O}_{\mathrm{E}}$-sheaves of the form $j_!\mathscr{E}$ with $j : \mathrm{Z} \hookrightarrow \mathrm{X}$ a locally closed immersion and $\mathscr{E}$ in $\mathrm{Lisse}_{\mathrm{Z}}(\mathcal{O}_{\mathrm{E}})$. By Proposition ⊙ of the note to Definition 6.8.10, we can write $\mathscr{E}$ as $\mathscr{E}' \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{Z}}} \mathcal{O}_{\mathrm{E},\mathrm{Z}}$ for some $\mathscr{E}'$ in $\mathrm{Lisse}_{\mathrm{Z}}(\mathcal{O}_{\mathrm{F}})$, where F is some finite extension of $\mathbf{Q}_\ell$. As j^* commutes with limits and colimits by Corollary 6.1.5, $j^*\mathcal{O}_{\mathrm{E},\mathrm{X}} = \mathcal{O}_{\mathrm{E},\mathrm{Z}}$ by Lemma 6.8.2, and $j_!(\mathscr{E}') \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}} = j_!\mathscr{E}$ by the projection formula Lemma 6.2.3 (3), so $j_!\mathscr{E}$ is in the essential image of our functor C. By induction on strata, we may assume our $\mathscr{F}$ is an extension of sheaves in the essential image of C. By (†), we can find a finite extension $\mathrm{F}' \supset \mathrm{F}$ and an extension of sheaves in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}'})$ coinciding with the given one after extending scalars. By the five lemma, we conclude $\mathscr{F}$ is in the essential image of C.

3. The first paragraph concludes via Proposition 6.6.11. For the second paragraph, if K is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{E}})$, then there is a finite stratification $\{\mathrm{X}_i \hookrightarrow \mathrm{X}\}$ so that $\mathrm{K}|_{\mathrm{X}_i}$ is locally isomorphic to $\mathrm{L} \otimes_{\mathcal{O}_{\mathrm{E}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$ for some L in $\mathrm{D}_{\mathrm{perf}}(\mathcal{O}_{\mathrm{E}})$. Then $\mathscr{H}^q\mathrm{K}|_{\mathrm{X}_i}$ is locally isomorphic to $(\mathrm{H}^q\mathrm{L}) \otimes_{\mathcal{O}_{\mathrm{E}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$, as $\mathcal{O}_{\mathrm{E},\mathrm{X}}$ is flat as an $\mathcal{O}_{\mathrm{E}}$-module (`093M`). As $\mathcal{O}_{\mathrm{E}}$ is noetherian, $\mathrm{H}^q\mathrm{L}$ is a finite $\mathcal{O}_{\mathrm{E}}$-module. I don't see why induction is needed.

I will include here a little lemma that allows us to bridge the gap between the pro-étale and classical definitions of constructible sheaf.

Lemma. — If X *is a topologically noetherian scheme and* F *is a finite extension of* $\mathbf{Q}_\ell$*, then the functors*

$$\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}}) \leftrightarrows \lim_n \mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}}/\varpi^n)$$
$$\mathrm{A} : \mathcal{F} \mapsto (\mathcal{F}/\varpi^n\mathcal{F})_n$$
$$\mathrm{B} : \lim_n \mathcal{F}_n \leftarrow\!\shortmid (\mathcal{F}_n)_n$$

define an equivalence of categories.

Proof. — We must first check that B actually lands in $\mathrm{Cons}_{\mathrm{X}}(\mathcal{O}_{\mathrm{F}})$. Given an inverse system $(\mathcal{F}_n)_n$ so that the transition morphisms $\mathcal{F}_{n+1} \to \mathcal{F}_n$ factor via an isomorphism $\mathcal{F}_{n+1} \otimes_{\mathcal{O}_{\mathrm{F}}/\varpi^{n+1}} \mathcal{O}_{\mathrm{F}}/\varpi^n \to \mathcal{F}_n$, there is a partition of X into locally closed subsets X_i so that $\mathcal{F}_n|_{\mathrm{X}_i}$ is locally constant for all n (SGA $4\frac{1}{2}$, *Rapport*, Proposition 2.5). If $j_i : \mathrm{X}_i \hookrightarrow \mathrm{X}$ denotes the immersion, then $j^* : \mathrm{Mod}_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathrm{X}_{\text{proét}}) \to \mathrm{Mod}_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathrm{X}_{i,\text{proét}})$ commutes with limits by Corollary 6.1.5, so $\lim_n \mathcal{F}_n$ is constructible with respect to the stratification $\coprod \mathrm{X}_i$ by Proposition ⊛ of the note to Definition 6.8.10. That the functors A and B are quasi-inverses can then be checked after restricting to a stratification, putting us in the setting of Proposition ⊛. □

6.8.12. We already went through the argument in the note to Proposition 6.8.11, but in that case E was a finite extension of $\mathbf{Q}_\ell$. Let's discuss why the same argument holds when E is no longer supposed finite, but merely algebraic.

First, let Λ be an $\mathcal{O}_{\mathrm{E}}$-module of finite presentation given as

$$\mathcal{O}_{\mathrm{E}}^n \xrightarrow{a} \mathcal{O}_{\mathrm{E}}^m \to \Lambda \longrightarrow 0.$$

Then $\mathbf{Z}_\ell^n \subset \mathcal{O}_{\mathrm{E}}^n$ lands in $\mathcal{O}_{\mathrm{F}}^m$ for some finite extension F of $\mathbf{Q}_\ell$,[82] defining a morphism $a_{\mathrm{F}} : \mathcal{O}_{\mathrm{F}}^n \to \mathcal{O}_{\mathrm{F}}^m$ that coincides with the one defining Λ after extending scalars. Therefore putting $\Lambda_{\mathrm{F}} := \operatorname{coker} a_{\mathrm{F}}$, we have $\Lambda_{\mathrm{F}} \otimes_{\mathcal{O}_{\mathrm{F}}} \mathcal{O}_{\mathrm{E}} = \Lambda$. Now Λ_{F} is a finite $\mathcal{O}_{\mathrm{F}}$-module, so it has

[82] When $\mathbf{Q}_\ell \subset \mathrm{F} \subset \mathrm{E}$ with $\mathrm{F}/\mathbf{Q}_\ell$ finite, $\mathcal{O}_{\mathrm{F}} \subset \mathcal{O}_{\mathrm{E}}$ is not open, but $\mathcal{O}_{\mathrm{F}}$ is compact Hausdorff (it's a profinite metric space), so we conclude by Lemma 4.3.7.

finite projective dimension. Let

$$0 \longrightarrow \mathcal{O}_F^{r_k} \longrightarrow \mathcal{O}_F^{r_{k-1}} \longrightarrow \dots \longrightarrow \mathcal{O}_F^{r_0} \longrightarrow \Lambda_F \longrightarrow 0$$

be a finite free resolution of Λ_F. Tensoring with $\mathcal{O}_E$ gives a finite resolution of Λ in finite, free $\mathcal{O}_E$-modules. In particular, an $\mathcal{O}_E$-module of finite presentation is perfect.

This allows us to reprise the five points of the note to Proposition 6.8.11. Nothing changes: in (1), dévissage along stupid truncations reduces us to the case of M an $\mathcal{O}_E$-module of finite presentation, and by the above, to the trivial case $M = \mathcal{O}_E$. In (4), now $\mathscr{E} = M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}$ with M an $\mathcal{O}_E$-module of finite presentation. We continue to have $k^*\mathcal{O}_{E,X} = \mathcal{O}_{E,Z}$ when $k : Z \hookrightarrow X$ is a locally closed immersion by Lemma 6.8.2, as k^* commutes with small limits and colimits by Corollary 6.1.5.

This shows that for any $\mathscr{F}$ in $\mathrm{Cons}_X(\mathcal{O}_E)$, the functor $\mathrm{R\,Hom}(\mathscr{F}, -)$ commutes with direct sums in $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_E)$. It's true for K in $D_{\text{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ as any such K is bounded with cohomology sheaves in $\mathrm{Cons}_X(\mathcal{O}_E)$, and the category $\mathscr{C}_X$ of the note to Proposition 6.8.11 is triangulated. We've proved the following

Lemma. — If E *is an algebraic extension of* $\mathbf{Q}_\ell$ *and* K *is in* $D_{\text{cons}}(X_{\text{proét}}, \mathcal{O}_E)$*, then* $\mathrm{R}\mathscr{H}\!om_{\mathcal{O}_{E,X}}(K, -)$ *and* $\mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(K, -)$ *commute with direct sums in* $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_{E,X})$.

6.8.13. The point is that any étale cover of the topologically noetherian X can be refined by one where Y is an affine scheme. Then `03S0` gives the desired stratification.

'$L|_Y$ corresponds to a continuous representation of the profinite fundamental group $\pi_1(Y, \bar{x})$ on a finite-dimensional E-vector space' $\rightsquigarrow$ we may assume Y is connected, so, letting $\pi_1 := \pi_1(Y, \bar{x})$, in the note to Corollary 6.8.5 we showed that $\mathrm{Loc}_Y(\mathcal{O}_E)$ is equivalent to $\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)$, the category of finite free $\mathcal{O}_E$-modules endowed with continuous $\mathcal{O}_E$-linear action of π_1, and evidently $\mathrm{Loc}_E(\pi_1)$, which has the same description with $\mathcal{O}_E$ replaced by E, is equivalent to $\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)[\ell^{-1}] \simeq \mathrm{Loc}_X(\mathcal{O}_E)[\ell^{-1}]$. By hypothesis, $L|_Y$ lies in the essential image of the functor $\mathrm{Loc}_X(\mathcal{O}_E)[\ell^{-1}] \to \mathrm{Loc}_X(E)$ of Proposition 6.8.4 (4).

Let $f : Y \to X$ denote our finite étale morphism and $g = f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2 : Y \times_X Y \to X$. Since L is a sheaf, the diagram

$$L \to f_* f^* L \rightrightarrows g_* g^* L$$

is exact, where here the two arrows are induced by $f^*L \to \mathrm{pr}_{?*} \mathrm{pr}_?^* f^* L$ for $? = 1, 2$. As $f^*L = L|_Y$ is in the essential image of the functor $\mathrm{Loc}_E(\pi_1) \to \mathrm{Loc}_Y(E)$, the same is true of $f_* f^* L$ and $g_* g^* L = f_* \mathrm{pr}_{1*} \mathrm{pr}_1^* L|_Y$.[83] As $\mathrm{Loc}_E(\pi_1) \to \mathrm{Loc}_X(E)$ is fully faithful, L corresponds to the equalizer of this pair of arrows in $\mathrm{Loc}_E(\pi_1)$,[84] hence to a representation of $\pi_1(X, x)$ on a finite-dimensional E-vector space, and the representation corresponding to $L|_Y$ is obtained from this one by restriction to an open subgroup. In this way, the representation of $\pi_1(Y, x)$ on a finite-dimensional E-vector space extends to one of $\pi_1(X, x)$ on the same vector space.

'Any such representation admits an invariant $\mathcal{O}_E$-lattice' $\rightsquigarrow$ this is the essential surjectivity of the equivalence $\mathrm{Loc}_{\mathcal{O}_E}(\pi_1)[\ell^{-1}] \simeq \mathrm{Loc}_E(\pi_1)$ discussed in the note to Corollary 6.8.5.

To wrap up, let $f : (X_{\text{proét}}, E_X) \to (X_{\text{proét}}, \mathcal{O}_{E,X})$ be the morphism of topos associated to the morphism $\mathcal{O}_{E,X} \to \mathcal{O}_{E,X}[\ell^{-1}] = E_X$, which is flat (`0EMB` & `03ET`). The claim is that if $\mathcal{F}$ is in $\mathrm{Cons}_X(\mathcal{O}_{E,X})$ and K is in $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_{E,X})$, then $\mathrm{R\,Hom}_{E_X}(\mathcal{F}[\ell^{-1}], K[\ell^{-1}]) = \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, K)[\ell^{-1}]$. As $\mathrm{R\,Hom}_{E_X}(f^*\mathcal{F}, f^*K) = \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, f_* f^* K)$, the claim follows from the fact that $\mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, -)$ commutes with direct sums in $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_{E,X})$ and $E_X = \mathcal{O}_{E,X}[\ell^{-1}] = \mathrm{hocolim}(\mathcal{O}_{E,X} \xrightarrow{\ell} \mathcal{O}_{E,X} \xrightarrow{\ell} \ldots) =: \mathrm{hocolim}_\ell\, \mathcal{O}_{E,X}$. Namely,

$$\begin{aligned}
\mathrm{R\,Hom}_{E_X}(\mathcal{F}[\ell^{-1}], K[\ell^{-1}]) &= \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, K \otimes^{\mathbf{L}}_{\mathcal{O}_{E,X}} \mathrm{hocolim}_\ell\, \mathcal{O}_{E,X}) \\
&= \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, \mathrm{hocolim}_\ell\, K) \\
&= \mathrm{hocolim}_\ell\, \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, K) \\
&= \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, K) \otimes^{\mathbf{L}}_{\mathcal{O}_E} \mathrm{hocolim}_\ell\, \mathcal{O}_E \\
&= \mathrm{R\,Hom}_{\mathcal{O}_{E,X}}(\mathcal{F}, K)[\ell^{-1}].
\end{aligned}$$

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[83] If f corresponds to the open subgroup $H \subset \pi_1$ and $L|_Y$ corresponds to the representation V of H (V is a finite-dimensional E-vector space), then $f_* L|_Y$ corresponds to the representation $\mathrm{Ind}_H^{\pi_1} V$ of π_1.

[84] The argument goes like the one in the note to Corollary 6.8.5 to show $\mathrm{Loc}_X(E)$ admits kernels.

To see $E_X = \mathrm{hocolim}_\ell \mathcal{O}_{E,X}$, or equivalently $E_X = \cup_n \ell^{-n}\mathcal{O}_{E,X}$, it suffices by `0738` & Lemma 4.2.12 to show $\mathrm{Map}_{\mathrm{cont}}(S, E) = \cup_n \mathrm{Map}_{\mathrm{cont}}(S, \ell^{-n}\mathcal{O}_E)$ for S compact Hausdorff, which follows from Lemma 4.3.7.

In more words, the sequence

$$0 \to \bigoplus \mathcal{O}_{E,X} \to \bigoplus \mathcal{O}_{E,X} \to \mathrm{colim}(\mathcal{O}_{E,X} \xrightarrow{\ell} \mathcal{O}_{E,X} \xrightarrow{\ell} \dots) = \cup_n \ell^{-n}\mathcal{O}_{E,X} \to 0$$

is exact in $\mathrm{Mod}_X(\mathcal{O}_E)$, where the map $\oplus\mathcal{O}_{E,X} \to \oplus\mathcal{O}_{E,X}$ is given by $1-\ell$, where ℓ sends the n^{th} term to the $(n+1)^{\text{th}}$ by multiplication by ℓ. To check exactness: as sheafification commutes with colimits, it suffices to show that sections over coherent Y in $X_{\text{proét}}$ of the presheaf cokernel $\mathcal{K}$ of the map $\oplus\mathcal{O}_{E,X} \to \oplus\mathcal{O}_{E,X}$ are given by the stated colimit of sections. As $(\oplus\mathcal{O}_{E,X})(Y) = \oplus\mathcal{O}_{E,X}(Y)$ (`0935`), every $s \in \mathcal{K}(Y)$ can be represented by an integer n and a continuous map $f : Y \to \mathcal{O}_E$, two sections (n, f), (m, g) agreeing if there exists a $k \geq n, m$ so that $\ell^{k-n} f = \ell^{k-m} g$. This is a description of $\mathrm{colim}(\mathcal{O}_{E,X}(Y) \xrightarrow{\ell} \mathcal{O}_{E,X}(Y) \xrightarrow{\ell} \dots)$.

6.8.14. 1. The functor is given by $- \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$. As discussed in the note to Proposition 6.8.11, $\mathcal{O}_{E,X}$ is free as an $\mathcal{O}_{F,X}$-module, where F is a finite extension of $\mathbf{Q}_\ell$ and E is an algebraic extension of F. If K and L are in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_F)$ and $K_E := K \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$ etc., then $\mathrm{Hom}_{\mathcal{O}_{E,X}}(K_E, L_E) = \mathrm{Hom}_{\mathcal{O}_{F,X}}(K, L_E)$, and in order to show $\mathrm{Hom}_{\mathcal{O}_{E,X}}(K_E, L_E) = \mathrm{Hom}_{\mathcal{O}_{F,X}}(K, L) \otimes_{\mathcal{O}_F} \mathcal{O}_E$, which would imply

$$\mathrm{Hom}_{\mathcal{O}_{E,X}}(K_E, L_E) = \mathrm{colim}_{F\subset F'\subset E} \mathrm{Hom}_{\mathcal{O}_{F,X}}(K, L)\otimes_{\mathcal{O}_F}\mathcal{O}_{F'} = \mathrm{colim}_{F\subset F'\subset E} \mathrm{Hom}_{\mathcal{O}_{F',X}}(K_{F'}, L_{F'}),$$

it suffices to show that $R\mathrm{Hom}_{\mathcal{O}_{F,X}}(K, L_E) = R\mathrm{Hom}_{\mathcal{O}_{F,X}}(K, L) \otimes_{\mathcal{O}_F} \mathcal{O}_E$, which is true since $R\mathrm{Hom}_{\mathcal{O}_{F,X}}(K, -)$ commutes with direct sums in $D^{\geq 0}(X_{\text{proét}}, \mathcal{O}_F)$ by Lemma 6.8.12. Therefore our functor

$$\mathrm{colim}_{F\subset E} D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_F) \longrightarrow D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$$

is fully faithful. Denote this functor by $F : D' \to D$.

Having established full faithfulness, one proceeds by induction on amplitude to show essential surjectivity, the base case being established in Proposition 6.8.11 (2). Given a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow$$

with A and C in the essential image, say $A = F(A')$ and $C = F(C')$, the map $C[-1] \to A$ is the image of some map $C'[-1] \to A'$, and F applied to the cone on that morphism gives an object of D isomorphic to B.

2. The functor is $M \mapsto M[\ell^{-1}] = M \otimes_{\mathcal{O}_E} E$. Full faithfulness is the statement

$$\mathrm{Hom}_E(M[\ell^{-1}], N[\ell^{-1}]) = \mathrm{Hom}_{\mathcal{O}_E}(M, N)[\ell^{-1}]$$

for M, N in $D_{cons}(X_{proét}, \mathcal{O}_E)$ (Homs in the localized category $D_{cons}(X_{proét}, \mathcal{O}_E)[\ell^{-1}]$ are discussed in the note to Proposition 6.8.4 (4)). As discussed in the note to Lemma 6.8.13, $E_X = \mathrm{hocolim}(\mathcal{O}_E \xrightarrow{\ell} \mathcal{O}_E \xrightarrow{\ell} \dots)$; let's denote this $\mathrm{hocolim}_\ell\, \mathcal{O}_E$ as before. Then

$$\begin{aligned}
R\,\mathrm{Hom}_{E,X}(M \otimes_{\mathcal{O}_{E,X}} E_X, N \otimes_{\mathcal{O}_{E,X}} E_X) &= R\,\mathrm{Hom}_{\mathcal{O}_{E,X}}(M, N \otimes_{\mathcal{O}_{E,X}} E_X) \\
&= R\,\mathrm{Hom}_{\mathcal{O}_{E,X}}(M, N \otimes^{\mathbf{L}}_{\mathcal{O}_{E,X}} \mathrm{hocolim}_\ell\, \mathcal{O}_{E,X}) \\
&= R\,\mathrm{Hom}_{\mathcal{O}_{E,X}}(M, \mathrm{hocolim}_\ell\, N) \\
&= \mathrm{hocolim}_\ell\, R\,\mathrm{Hom}_{\mathcal{O}_{E,X}}(M, N) \\
&= R\,\mathrm{Hom}_{\mathcal{O}_{E,X}}(M, N) \otimes^{\mathbf{L}}_{\mathcal{O}_E} \mathrm{hocolim}_\ell\, \mathcal{O}_E \\
&= R\,\mathrm{Hom}_{\mathcal{O}_{E,X}}(M, N) \otimes_{\mathcal{O}_E} E
\end{aligned}$$

by Lemma 6.8.12. The result is obtained upon applying H^0.

Essential surjectivity: let K be in $D_{cons}(X_{proét}, E)$. As in part (1), we can filter K so that it's first concentrated in degree zero, then a sheaf of the form $j_!\mathcal{E}$ where $j : Y \to X$ is a locally closed immersion and $\mathcal{E}$ is in $\mathrm{Loc}_Y(E)$. By Corollary 6.8.13, we can find a constructible $\mathcal{O}_E$-sheaf $\mathcal{F}$ so that $\mathcal{E} = \mathcal{F} \otimes_{\mathcal{O}_{E,Y}} E_Y$. Then $j_!\mathcal{F} \otimes_{\mathcal{O}_{E,X}} E_X = j_!\mathcal{E}$ by the projection formula (Lemma 6.2.3 (3)), as $j^*E_X = E_Y$ (this follows from Lemma 6.8.2 as j^* commutes with all limits and colimits by Corollary 6.1.5).

6.8.15. There are two claims: first, that many of the six functors can be defined on $D(X_{proét}, \Lambda)$ and computed in the setting of the ringed topos $(X_{proét}, \Lambda)$ in terms of injective sheaves, etc., and second, that all six functors can be defined on $D_{cons}(X_{proét}, \Lambda)$

(via restriction of the functor on all of $D(X_{\text{proét}}, \Lambda)$ if it is defined) and preserve constructibility. Let $f : Y \to X$ be a morphism of topologically noetherian schemes and fix an infinite algebraic extension E of $\mathbf{Q}_\ell$.

The functor f_*. The morphism f induces a map $u : X_{\text{proét}} \to Y_{\text{proét}}$ via pullback, which is a continuous functor between sites [**SGAA**, III 1.6], defining adjoint functors $u_s : \tilde{X}_{\text{proét}} \to \tilde{Y}_{\text{proét}}$ and $u^s : \tilde{Y}_{\text{proét}} \to \tilde{X}_{\text{proét}}$, and $f^* = u^s \dashv u_s = f_*$. The functor u is obviously left-exact, so the same is true of $u^s = f^*$ [**SGAA**, I 5.4 (4), III 1.3 (5)]. In other words, we have a morphism of sites $X_{\text{proét}} \to Y_{\text{proét}}$. On modules, we have the given description of f^* as $f^{-1}(-) \otimes_{f^{-1}\Lambda} \Lambda_Y$. One would like to use the results of §6.7 to conclude that f_* preserves constructibility.

Let K be in $D_{\text{cons}}(Y_{\text{proét}}, \mathcal{O}_E)$. If E is finite over $\mathbf{Q}_\ell$, then we're in the situation of §6.7, so suppose instead E is an algebraic extension of $\mathbf{Q}_\ell$ of infinite degree. Then $K \simeq K' \otimes_{\mathcal{O}_{F,Y}} \mathcal{O}_{E,Y}$ for some finite extension F of $\mathbf{Q}_\ell$ and some K' in $D_{\text{cons}}(Y_{\text{proét}}, \mathcal{O}_F)$. Let $\upsilon : (?_{\text{proét}}, \mathcal{O}_{E,?}) \to (?_{\text{proét}}, \mathcal{O}_{F,?})$ denote the change of coefficients morphism for $? = X, Y$, so $K = \upsilon^* K'$ (recall that υ is a flat morphism of topos as discussed in the note to Proposition 6.8.11). If $K \to I^\bullet$ is a quasi-isomorphism into a bounded-below complex of injectives in $\mathrm{Mod}(Y_{\text{proét}}, \mathcal{O}_{E,Y})$, then $\nu_* I^\bullet$ is a bounded-below complex of injectives in $\mathrm{Mod}(Y_{\text{proét}}, \mathcal{O}_{F,Y})$ (`0730`), so if f_*, f_{F*} denote the functors $D(Y_{\text{proét}}, \mathcal{O}_E) \to D(X_{\text{proét}}, \mathcal{O}_E)$ and $D(Y_{\text{proét}}, \mathcal{O}_F) \to D(X_{\text{proét}}, \mathcal{O}_F)$, respectively, then $\upsilon_* f_* K = f_{F*} \upsilon_* K$. As f_{F*} commutes with direct sums in $D^{\geq 0}(Y_{\text{proét}})$[85] and $K \simeq \oplus K'$ (on countably many generators) as $\mathcal{O}_{F,Y}$-modules, we find

$$\upsilon_* f_* K = f_{F*} \upsilon_* K = (f_{F*} K') \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$$

in $D(X_{\text{proét}}, \mathcal{O}_{F,X})$. We deduce from §6.7 that $f_{F*}K'$ is in $D_{\text{cons}}(X_{\text{proét}}, \mathcal{O}_F)$ under suitable assumptions on Y, X, and f, and the map $(f_{F*}K') \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X} \to f_* K$ obtained by adjunction from $f_{F*}K' \to \upsilon_* f_* K = f_{F*} \upsilon_* K$ is an isomorphism as this can be checked after applying υ_*.[86] In conclusion, $f_* K \xleftarrow{\sim} (f_{F*}K') \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$ is an isomorphism and recognizes $f_* K$ as an object of $D_{\text{cons}}(X_{\text{proét}}, \mathcal{O}_E)$.

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[85]See, for example, the lemma in the note to Lemma 6.5.11 (2).

[86]The map $K' \to \upsilon_*(K' \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X})$ is induced by the inclusion $\mathcal{O}_{F,X} \to \mathcal{O}_{E,X}$.

Now suppose K is in $D_{cons}(Y_{proét}, E)$, so $K \simeq K' \otimes_{\mathcal{O}_{E,Y}} E_Y$ for some K' in $D_{cons}(Y_{proét}, \mathcal{O}_E)$. Let $\upsilon : (?_{proét}, E_?) \to (?_{proét}, \mathcal{O}_{E,?})$ be the change of coefficients morphism for ? = X, Y; υ is a flat morphism since (dropping subscripts) $E = \mathcal{O}_E[\ell^{-1}] = \mathrm{colim}(\mathcal{O}_E \xrightarrow{\ell} \mathcal{O}_E \xrightarrow{\ell} \ldots)$ as discussed in the note to Lemma 6.8.13 (`03EU`). (Note this colimit also coincides with $\mathrm{hocolim}(\mathcal{O}_E \xrightarrow{\ell} \mathcal{O}_E \xrightarrow{\ell} \ldots) =: \mathrm{hocolim}_\ell\, \mathcal{O}_E$.) So if $f_*, f_{\mathcal{O}*}$ denote the functors $D(Y_{proét}, E) \to D(X_{proét}, E)$ and $D(Y_{proét}, \mathcal{O}_E) \to D(X_{proét}, \mathcal{O}_E)$, respectively, $\upsilon_* f_* = f_{\mathcal{O}*}\upsilon_*$, so

$$\upsilon_* f_* K = f_{\mathcal{O}*}\upsilon_* K = f_{\mathcal{O}*}(\mathrm{hocolim}_\ell\, K') = \mathrm{hocolim}_\ell\, f_{\mathcal{O}*}K' = (f_{\mathcal{O}*}K') \otimes_{\mathcal{O}_{E,X}} E_X.$$

Therefore the map $(f_{\mathcal{O}*}K') \otimes_{\mathcal{O}_{E,X}} E_X \to f_* K$ obtained via adjunction from $f_{\mathcal{O}*}K' \to \upsilon_* f_* K = f_{\mathcal{O}*}\upsilon_* K$[87] is an isomorphism, as can be checked upon applying υ_*. The previous paragraph tells us that $f_{\mathcal{O}*}K'$ is in $D_{cons}(X_{proét}, \mathcal{O}_E)$, so f_*K is in $D_{cons}(X_{proét}, E)$.

The functor f^*. Let Λ_* denote the sheaf on $*_{proét}$ which assigns to each profinite set S the ring $\mathrm{Map}_{cont}(S, \Lambda)$. (It is a sheaf by Lemma 4.2.12.) Let $g : X_{proét} \to *_{proét}$ denote the canonical morphism of sites.

Lemma. — The sheaf Λ_X *on* X *coincides with* $g^{-1}\Lambda_*$.

Proof. — The adjunction between the category of affine X-schemes and $\pi_0(X)_{proét}$

$$\mathrm{Hom}_X(V, X \times_{\pi_0 X} S) = \mathrm{Hom}_{\pi_0 X}(\pi_0 V, S)$$

for S a profinite set shows that if $V \to X$ in $X_{proét}$ is affine, every arrow $V \to X \times_{\pi_0 X} S$ factors canonically via the unit $V \to X \times_{\pi_0 X} \pi_0 V$, so $g^{-1}\Lambda_*$ is the sheafification of the presheaf which on such V takes the value $\Lambda_*(\pi_0(V)) = \mathrm{Map}_{cont}(\pi_0 V, \Lambda) = \Lambda_X(V)$. □

Corollary. — We have $f^{-1}\Lambda_X = \Lambda_Y$*, so* $f^* : \mathrm{Mod}_X(\Lambda_X) \to \mathrm{Mod}_Y(\Lambda_Y)$ *coincides with* f^{-1} *and* $f : (Y_{proét}, \Lambda_Y) \to (X_{proét}, \Lambda_X)$ *is a flat morphism of ringed topos.*

If M is a Λ-module of finite presentation, we have $f^*(M \otimes_\Lambda \Lambda_X) = f^{-1}(M \otimes_\Lambda \Lambda_X) = M \otimes_\Lambda \Lambda_Y$ in $\mathrm{Mod}_Y(\mathcal{O}_{E,Y})$. That $f^* : D(X_{proét}, \Lambda_X) \to D(Y_{proét}, \Lambda_Y)$ preserves constructibility is a simple corollary of this since $f^* : \mathrm{Mod}_X(\Lambda_X) \to \mathrm{Mod}_Y(\Lambda_Y)$ is exact.

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[87] Induced from the map $K' \to \upsilon_*(K' \otimes_{\mathcal{O}_{E,Y}} E_Y)$ induced by the inclusion $\mathcal{O}_{E,Y} \to E_Y$.

The functor $\otimes$. To check that $\otimes_{\mathcal{O}_{E,X}}$ preserves constructibility, one must show that $j_!\mathcal{F} \otimes_{\mathcal{O}_{E,X}} k_!\mathcal{G}$ is in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ when $j : Z \hookrightarrow X$ and $k : W \hookrightarrow X$ are locally closed immersions and $\mathcal{F}$ and $\mathcal{G}$ are in $\mathrm{Lisse}_Z(\mathcal{O}_E)$ and $\mathrm{Lisse}_W(\mathcal{O}_E)$, respectively. This tensor product is zero on the complement of $Z \cap W$ (`07A4`). If $i : Z \cap W \hookrightarrow X$ is the immersion, then i^* sends $\mathrm{Lisse}_X(\mathcal{O}_E)$ to $\mathrm{Lisse}_{Z\cap W}(\mathcal{O}_E)$, since $i^{-1}\mathcal{O}_{E,X} = \mathcal{O}_{E,Z\cap W}$, so $i^* = i^{-1}$ and $i^*(\Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) = i^{-1}(\Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) = \Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{E,Z\cap W}$. Since $j_!\mathcal{F} \otimes_{\mathcal{O}_{E,X}} k_!\mathcal{G} \xleftarrow{\sim} i_!i^*(j_!\mathcal{F} \otimes_{\mathcal{O}_{E,X}} k_!\mathcal{G}) = i_!(\mathcal{F}|_{Z\cap W} \otimes_{\mathcal{O}_{E,Z\cap W}} \mathcal{G}|_{Z\cap W})$ (the first isomorphism by Lemma ✠ of the note to Definition 6.5.1), replacing X by $Z \cap W$, we just have to check that $\mathcal{F} \otimes_{\mathcal{O}_{E,X}} \mathcal{G}$ is in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ when $\mathcal{F}$ and $\mathcal{G}$ are in $\mathrm{Lisse}_X(\mathcal{O}_E)$. As $\Lambda_1 \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} \otimes_{\mathcal{O}_{E,X}} \Lambda_2 \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X} = \Lambda_1 \otimes_{\mathcal{O}_E} \Lambda_2 \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}$ with Λ_1, Λ_2 $\mathcal{O}_E$-modules of finite presentation, this is immediate from the fact that both Λ_1 and Λ_2 admit finite resolutions by finite free $\mathcal{O}_E$-modules as mentioned in the note to Lemma 6.8.12, so the same is true of $\Lambda_1 \otimes_{\mathcal{O}_E} \Lambda_2$. Given a finite complex M of finite free $\mathcal{O}_E$-modules, there is a finite extension F of $\mathbf{Q}_\ell$ and a finite complex M' of finite free $\mathcal{O}_F$-modules with $M = M' \otimes_{\mathcal{O}_F} \mathcal{O}_E$. As $- \otimes_{\mathcal{O}_F} \mathcal{O}_E$ is exact and $\mathcal{O}_F$ noetherian, $H^iM = H^iM' \otimes_{\mathcal{O}_F} \mathcal{O}_E$ are $\mathcal{O}_E$-modules of finite presentation, so $H^i(\Lambda_1 \otimes_{\mathcal{O}_E} \Lambda_2 \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}) = H^i(\Lambda_1 \otimes_{\mathcal{O}_E} \Lambda_2) \otimes_{\mathcal{O}_E} \mathcal{O}_{E,X}$ is in $\mathrm{Lisse}_X(\mathcal{O}_E)$ (since $\mathcal{O}_{E,X}$ is flat over $\mathcal{O}_E$) and is *a fortiori* constructible.

Checking that $\otimes_{E_X}$ preserves constructibility is the same but easier. Now $\mathcal{F}$ and $\mathcal{G}$ are in $\mathrm{Loc}_Z(E)$ and $\mathrm{Loc}_W(E)$, respectively, and $i^{-1}E_X = E_{Z\cap W}$. As $\mathcal{F}|_{Z\cap W}$ and $\mathcal{G}|_{Z\cap W}$ are both locally free of finite rank as sheaves of E-modules, the same is true of $\mathcal{F}|_{Z\cap W} \otimes_{E_{Z\cap W}} \mathcal{G}|_{Z\cap W}$.

The functor $\mathcal{H}om$. Let's check that $R\mathcal{H}om$ preserves constructibility. Let K and L be in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ and write $K \simeq K' \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$ and $L \simeq L' \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$ for K', L' in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_F)$ where $F \subset E$ is a finite extension of $\mathbf{Q}_\ell$. Then by the lemma in the note to Lemma 6.8.12 and the fact that $\mathcal{O}_{E,X}$ is a free $\mathcal{O}_{F,X}$-module as discussed in the note to Proposition 6.8.11 (2),

$$R\mathcal{H}om_{\mathcal{O}_{E,X}}(K, L) = R\mathcal{H}om_{\mathcal{O}_{F,X}}(K', L' \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}) = R\mathcal{H}om_{\mathcal{O}_{F,X}}(K', L') \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}$$

via the natural map obtained by adjunction from

$$R\mathcal{H}om_{\mathcal{O}_{F,X}}(K', L') \longrightarrow R\mathcal{H}om_{\mathcal{O}_{F,X}}(K', L' \otimes_{\mathcal{O}_{F,X}} \mathcal{O}_{E,X}).$$

If X is a quasi-excellent ℓ-coprime scheme, then Lemma 6.7.13 gives the constructibility of $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{F,X}}}(\mathrm{K}', \mathrm{L}')$, and hence of $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathrm{K}, \mathrm{L})$.

Now suppose K and L are in $\mathrm{D_{cons}}(\mathrm{X_{proét}}, \mathrm{E})$ with $\mathrm{K} \simeq \mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E_X}$ and $\mathrm{L} \simeq \mathrm{L}' \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E_X}$, where K′ and L′ are in $\mathrm{D_{cons}}(\mathrm{X_{proét}}, \mathcal{O}_\mathrm{E})$. By the same lemma as above, since $\mathrm{E_X} = \mathrm{hocolim}(\mathcal{O}_{\mathrm{E,X}} \xrightarrow{\ell} \mathcal{O}_{\mathrm{E,X}} \xrightarrow{\ell} \dots) =: \mathrm{hocolim}_\ell\, \mathcal{O}_{\mathrm{E,X}}$, we write

$$\begin{aligned}\mathrm{R}\mathcal{H}om_{\mathrm{E_X}}(\mathrm{K}, \mathrm{L}) &= \mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathrm{K}', \mathrm{L}' \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E_X})\\ &= \mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathrm{K}', \mathrm{L}' \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{hocolim}_\ell\, \mathcal{O}_{\mathrm{E,X}})\\ &= \mathrm{hocolim}_\ell\, \mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathrm{K}', \mathrm{L}')\\ &= \mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathrm{K}', \mathrm{L}') \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E_X}.\end{aligned}$$

As $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathrm{E,X}}}(\mathrm{K}', \mathrm{L}')$ is in $\mathrm{D_{cons}}(\mathrm{X_{proét}}, \mathcal{O}_\mathrm{E})$, $\mathrm{R}\mathcal{H}om_{\mathrm{E_X}}(\mathrm{K}, \mathrm{L})$ is in $\mathrm{D_{cons}}(\mathrm{X_{proét}}, \mathrm{E})$.

The functor $f_!$. If $j : \mathrm{U} \hookrightarrow \mathrm{X}$ is an open immersion, the definition of $j_!$ is purely topos-theoretic, and from what has been shown about f_* we know that if $j : \mathrm{Y} \hookrightarrow \overline{\mathrm{Y}}$ is a compactification of topologically noetherian schemes and $\overline{f} : \overline{\mathrm{Y}} \to \mathrm{X}$ a proper morphism so that $f = \overline{f} \circ j$, $f_! = \overline{f}_* j_!$ is defined as a functor $\mathrm{D}(\mathrm{Y_{proét}}, \mathcal{O}_\mathrm{E}) \to \mathrm{D}(\mathrm{X_{proét}}, \mathcal{O}_\mathrm{E})$. The way I know to show the independence of $f_!$ of the compactification, however, uses proper base change, which shouldn't be available on all of $\mathrm{D}(\mathrm{Y_{proét}}, \mathcal{O}_\mathrm{E})$, so let's now turn our attention to the constructible subcategory.

If K is in $\mathrm{D_{cons}}(\mathrm{Y_{proét}}, \mathcal{O}_\mathrm{E})$, and K′ in $\mathrm{D_{cons}}(\mathrm{Y_{proét}}, \mathcal{O}_\mathrm{F})$ ($\mathrm{F} \subset \mathrm{E}$ a finite extension of $\mathbf{Q}_\ell$) is found so that $\mathrm{K} \simeq \mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F,Y}}} \mathcal{O}_{\mathrm{E,Y}}$, then

$$f_!\mathrm{K} = \overline{f}_* j_!(\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F,Y}}} \mathcal{O}_{\mathrm{E,Y}}) = \overline{f}_*(j_!\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F},\overline{\mathrm{Y}}}} \mathcal{O}_{\mathrm{E},\overline{\mathrm{Y}}}) = \overline{f}_* j_!\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F,X}}} \mathcal{O}_{\mathrm{E,X}} = f_!\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F,X}}} \mathcal{O}_{\mathrm{E,X}}$$

by the projection formula Lemma 6.2.3 (3) and the above. We immediately deduce that $f_! : \mathrm{D_{cons}}(\mathrm{Y_{proét}}, \mathcal{O}_\mathrm{E}) \to \mathrm{D_{cons}}(\mathrm{X_{proét}}, \mathcal{O}_\mathrm{E})$ is independent of the compactification since the same is true for $\mathcal{O}_\mathrm{E}$ replaced by $\mathcal{O}_\mathrm{F}$.

The story is identical when $\mathcal{O}_\mathrm{E}$ is replaced by E: given K in $\mathrm{D_{cons}}(\mathrm{Y_{proét}}, \mathrm{E})$, there is a K′ in $\mathrm{D_{cons}}(\mathrm{Y_{proét}}, \mathcal{O}_\mathrm{E})$ so that $\mathrm{K} = \mathrm{K}' \otimes_{\mathcal{O}_\mathrm{E}} \mathrm{E}$, and the analysis of the functor f_* plus the projection formula allow us to write

$$f_!\mathrm{K} = \overline{f}_* j_!(\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{E,Y}}} \mathrm{E_Y}) = \overline{f}_*(j_!\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{E},\overline{\mathrm{Y}}}} \mathrm{E}_{\overline{\mathrm{Y}}}) = \overline{f}_* j_!\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E_X} = f_!\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{E,X}}} \mathrm{E_X}.$$

The functor $f^!$. When $f = i : \mathrm{Y} \hookrightarrow \mathrm{X}$ is a closed immersion, Lemma 6.1.16 gives the desired right adjoint to $i_! = i_*$, but the existence of this adjoint is standard topos theory: i_* admits a right adjoint $i^!$ as functors $\mathrm{Mod}(\mathrm{X}_{\text{proét}}, \Lambda_{\mathrm{X}}) \leftrightarrows \mathrm{Mod}(\mathrm{Y}_{\text{proét}}, \Lambda_{\mathrm{Y}})$ defined by $i^!\mathscr{F} = \mathscr{H}\!om_{\Lambda_{\mathrm{X}}}(i_*\Lambda_{\mathrm{Y}}, \mathscr{F})$; here $\Lambda_?$ is $\mathscr{O}_{\mathrm{E},?}$ or $\mathrm{E}_?$ [**SGAA**, IV §14]. Therefore the right derived functor of $i^!$ is the right adjoint to i_* as functors $\mathrm{D}(\mathrm{X}_{\text{proét}}, \Lambda_{\mathrm{X}}) \leftrightarrows \mathrm{D}(\mathrm{Y}_{\text{proét}}, \Lambda_{\mathrm{Y}})$ (`09T5`). The distinguished triangle

$$i^!\mathrm{K} \longrightarrow i^*\mathrm{K} \longrightarrow i^*j_*j^*\mathrm{K} \longrightarrow$$

shows that $i^!$ defines a functor $\mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \Lambda_{\mathrm{X}}) \to \mathrm{D}_{\text{cons}}(\mathrm{Y}_{\text{proét}}, \Lambda_{\mathrm{Y}})$.

Now we construct $f^! : \mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \Lambda_{\mathrm{X}}) \to \mathrm{D}_{\text{cons}}(\mathrm{Y}_{\text{proét}}, \Lambda_{\mathrm{X}})$ for $f : \mathrm{Y} \to \mathrm{X}$ a separated finitely-presented map of quasi-excellent ℓ-coprime schemes. First suppose $\Lambda = \mathscr{O}_{\mathrm{E}}$ and K in $\mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{E}})$ is isomorphic to $\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}$ for some K' in $\mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{F}})$. Lemma 6.7.19 allows us to write $f^!_{\mathrm{F}}\mathrm{K}' := f^!\mathrm{K}'$, and we define $f^!\mathrm{K} := f^!_{\mathrm{F}}\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}$. Our first task is to show that this assignement is well-defined.

Let $\upsilon : (?_{\text{proét}}, \mathscr{O}_{\mathrm{E},?}) \to (?_{\text{proét}}, \mathscr{O}_{\mathrm{F},?})$ be the change of coefficients morphism for E an algebraic extension of F and ? = X, Y. I claim $\upsilon_* f^!_{\mathrm{E}} = f^!_{\mathrm{F}}\upsilon_*$ on $\mathrm{D}_{\text{cons}}(\mathrm{X}_{\text{proét}}, \mathscr{O}_{\mathrm{E}})$. To check this it suffices to show that $f_{\mathrm{E}!}\upsilon^* \xleftarrow{\sim} \upsilon^* f_{\mathrm{F}!}$ on $\mathrm{D}_{\text{cons}}(\mathrm{Y}_{\text{proét}}, \mathscr{O}_{\mathrm{F}})$, where the map is the adjoint of $f_{\mathrm{F}!} \to \upsilon_* f_{\mathrm{E}!}\upsilon^* = f_{\mathrm{F}!}\upsilon_*\upsilon^*$. (Here we've used that, as in the discussion of the functor f_* above, $\overline{f}_{\mathrm{F}*}\upsilon_* = \upsilon_*\overline{f}_{\mathrm{E}*}$, and obviously $\upsilon_* j_! = j_!\upsilon_*$.) Of course, υ_* is conservative, and as we've just seen, $\upsilon_* f_{\mathrm{E}!} = f_{\mathrm{F}!}\upsilon_*$, so we just need to know that $f_{\mathrm{F}!}$ commutes with direct sums in $\mathrm{D}^{\geq 0}(\mathrm{Y}_{\text{proét}}, \mathscr{O}_{\mathrm{F}})$, which follows from the fact that the same is true for $\overline{f}_*$ and $j_!$, separately.

Specializing this discussion to the case that $\mathrm{E} = \mathrm{F}'$ is a finite extension of F, we have $f^!_{\mathrm{F}'}(\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{F}',\mathrm{X}}) = f^!_{\mathrm{F}}\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{F}',\mathrm{X}}$ via the natural map obtained as the adjoint of the map $f^!_{\mathrm{F}}\mathrm{K}' \to \upsilon_* f^!_{\mathrm{F}'}(\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{F}',\mathrm{X}}) = f^!_{\mathrm{F}}\upsilon_*(\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{F}',\mathrm{X}})$ corresponding to $\mathscr{O}_{\mathrm{F,X}} \to \mathscr{O}_{\mathrm{F}',\mathrm{X}}$, since we may check this map is an isomorphism after applying υ_*, after which, using that $\upsilon_* f^!_{\mathrm{F}'} = f^!_{\mathrm{F}}\upsilon_*$, it follows from the fact that $f^!$ commutes with finite (bi)products. This shows that $f^!$ is well-defined (i.e. does not depend on our choice of F).

Now we must show that $f_! \dashv f^!$. To this end, let K be in $\mathrm{D}_{\mathrm{cons}}(\mathrm{Y}_{\mathrm{proét}}, \mathscr{O}_{\mathrm{E}})$ and L be in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{proét}}, \mathscr{O}_{\mathrm{E}})$, isomorphic to $\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}$ and $\mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{F',X}}} \mathscr{O}_{\mathrm{E,X}}$ for F, F' finite extensions of $\mathbf{Q}_\ell$. If F'' is some finite extension of both, then

$$f^!\mathrm{K} = f^!\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}} = f^!(\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{F'',X}}) \otimes_{\mathscr{O}_{\mathrm{F'',X}}} \mathscr{O}_{\mathrm{E,X}}$$

as discussed, and the same for L, so we may replace F by F'' and suppose $\mathrm{F} = \mathrm{F}'$. As

$$\begin{aligned}
\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,X}}}(f_!\mathrm{K}, \mathrm{L}) &= \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,X}}}(f_!\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}, \mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}) \\
&= \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{F,X}}}(f_!\mathrm{K}', \mathrm{L}') \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}} \\
&= f_*\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{F,Y}}}(\mathrm{K}', f^!\mathrm{L}') \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}} \\
&= f_*\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,Y}}}(\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}, f^!\mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}}) \\
&= f_*\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,Y}}}(\mathrm{K}, f^!\mathrm{L}).
\end{aligned}$$

Here we've used both the local adjunction proved in the note to Lemma 6.7.19 and the fact, proved above, that $f_*\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}} = f_*(\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{F,X}}} \mathscr{O}_{\mathrm{E,X}})$ when K' is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{Y}_{\mathrm{proét}}, \mathscr{O}_{\mathrm{F}})$. The global statement is evidently recovered by taking global sections.

Now suppose $\Lambda = \mathrm{E}$. If K is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{proét}}, \mathrm{E})$, we can write $\mathrm{L} = \mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}}$ for L' in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{proét}}, \mathscr{O}_{\mathrm{E}})$, and pose $f^!\mathrm{L} := f^!\mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{E,Y}}} \mathrm{E}_{\mathrm{Y}}$. If $\mathrm{K} = \mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{E,Y}}} \mathrm{E}_{\mathrm{Y}}$ is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{Y}_{\mathrm{proét}}, \mathrm{E})$, then by the discussion of f_*, $\mathscr{H}\!om$ and $f_!$ above,

$$\begin{aligned}
\mathrm{R}\mathscr{H}\!om_{\mathrm{E}_{\mathrm{X}}}(f_!\mathrm{K}, \mathrm{L}) &= \mathrm{R}\mathscr{H}\!om_{\mathrm{E}_{\mathrm{X}}}(f_!\mathrm{K}' \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}}, \mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}}) \\
&= \mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,X}}}(f_!\mathrm{K}', \mathrm{L}') \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}} \\
&= f_*\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,Y}}}(\mathrm{K}', f^!\mathrm{L}') \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}} \\
&= f_*(\mathrm{R}\mathscr{H}\!om_{\mathscr{O}_{\mathrm{E,Y}}}(\mathrm{K}', f^!\mathrm{L}') \otimes_{\mathscr{O}_{\mathrm{E,Y}}} \mathrm{E}_{\mathrm{Y}}) \\
&= f_*\mathrm{R}\mathscr{H}\!om_{\mathrm{E}_{\mathrm{Y}}}(\mathrm{K}, f^!\mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{E,Y}}} \mathrm{E}_{\mathrm{Y}}) \\
&=: f_*\mathrm{R}\mathscr{H}\!om_{\mathrm{E}_{\mathrm{Y}}}(\mathrm{K}, f^!\mathrm{L}).
\end{aligned}$$

Taking global sections recovers the global adjunction $f_! \dashv f^!$.

It remains to show that our definition of $f^!\mathrm{L}$ is well-defined; i.e. if we have $\alpha : \mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}} \xrightarrow{\sim} \mathrm{L}'' \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}}$ with $\mathrm{L}', \mathrm{L}''$ in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\mathrm{proét}}, \mathscr{O}_{\mathrm{E}})$, are $f^!\mathrm{L}' \otimes_{\mathscr{O}_{\mathrm{E,X}}} \mathrm{E}_{\mathrm{X}}$ and

$f^! L'' \otimes_{\mathcal{O}_{E,X}} E_X$ naturally isomorphic? As

$$\mathrm{Hom}_{E_X}(L' \otimes_{\mathcal{O}_{E,X}} E_X, L'' \otimes_{\mathcal{O}_{E,X}} E_X) \simeq \mathrm{Hom}_{\mathcal{O}_{E,X}}(L', L'') \otimes_{\mathcal{O}_E} E,$$

$\ell^n \alpha \in \mathrm{Hom}_{\mathcal{O}_{E,X}}(L', L'')$ for some $n > 0$. As $\ell^n : L'' \otimes_{\mathcal{O}_{E,X}} E_X \to L'' \otimes_{\mathcal{O}_{E,X}} E_X$ is an isomorphism, $\ell^n \alpha : L' \to L''$ is a morphism in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ with the property that $\ell^n \alpha : L' \otimes_{\mathcal{O}_{E,X}} E_X \to L'' \otimes_{\mathcal{O}_{E,X}} E_X$ is an isomorphism. In other words, the cone C on $\ell^n \alpha$

$$L' \xrightarrow{\ell^n \alpha} L'' \longrightarrow C \longrightarrow$$

has the property that $C \otimes_{\mathcal{O}_{E,X}} E_X = 0$. In order to conclude that $\ell^n \alpha : f^! L' \otimes_{\mathcal{O}_{E,Y}} E_Y \to f^! L'' \otimes_{\mathcal{O}_{E,Y}} E_Y$ is an isomorphism, which would mean that $f^! : D_{\mathrm{cons}}(X_{\text{proét}}, E) \to D_{\mathrm{cons}}(Y_{\text{proét}}, \mathcal{O}_E)$ is well-defined, it suffices to check that $f^!$ carries objects 'of torsion' in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$; i.e. with $K[\ell^{-1}] = K \otimes_{\mathcal{O}_{E,X}} E_X = 0$,[88] to objects 'of torsion' in $D_{\mathrm{cons}}(Y_{\text{proét}}, \mathcal{O}_E)$.

Lemma. — Suppose K *in* $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ *has* $K[\ell^{-1}] = 0$*. Then* $(f^! K)[\ell^{-1}] = 0$*.*

Proof. — Suppose K in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$ is of torsion. We must show that the same is true of $f^! K$. As $\mathcal{O}_{E,X} \to E_X$ is flat (03EU), $H^i K \otimes_{\mathcal{O}_{E,X}} E_X = H^i(K \otimes_{\mathcal{O}_{E,X}} E_X) = 0$. The triangles $(\tau_{<n} K, K, \tau_{\geq n} K)$ allow us to replace K by $H^i K$ and assume K is acyclic off degree zero; i.e. in $\mathrm{Cons}_X(\mathcal{O}_E)$. As K admits a finite filtration (in the abelian category $\mathrm{Cons}_X(\mathcal{O}_E)$) with successive quotients of the form $j_! \mathcal{E}$ for $j : Y \hookrightarrow X$ and $\mathcal{E}$ in $\mathrm{Lisse}_Y(\mathcal{O}_E)$, and each of these subquotients is of torsion since K is, we may replace K by one of them and assume $K = j_! \mathcal{E}$. The projection formula gives $K[\ell^{-1}] = j_!(\mathcal{E} \otimes_{\mathcal{O}_{E,X}} E_X)$, so $\mathcal{E}[\ell^{-1}] = 0$. As X is topologically noetherian we may assume Y is connected and pointed by a geometric point y. Then $\mathcal{E}$ corresponds to an $\mathcal{O}_E$-module Λ of finite presentation endowed with $\mathcal{O}_E$-linear action of $\pi_1(Y, y)$, and $\Lambda[\ell^{-1}] = 0$ as $\mathcal{E}[\ell^{-1}] = \mathrm{colim}(\mathcal{E} \xrightarrow{\ell} \mathcal{E} \xrightarrow{\ell} \ldots) = 0$. Therefore $\ell^n \Lambda = 0$ for some n, as the same is true separately for each of the finitely many

[88] Recall $K[\ell^{-1}] = K \otimes_{\mathcal{O}_{E,X}} E_X$ since both are given by $\mathrm{hocolim}(K \xrightarrow{\ell} K \xrightarrow{\ell} K \to \ldots)$. To see this for $K[\ell^{-1}]$: as direct sums in $D(X_{\text{proét}})$ are taken termwise and localization (in the sense of inverting ℓ) is exact, it suffices to show that $0 \to \oplus K \to \oplus K \to K[\ell^{-1}] \to 0$ is exact when K is a sheaf of $\mathcal{O}_{E,X}$)-modules. As sheafification is exact, it suffices to show the same, i.e. that $K[\ell^{-1}] = \mathrm{colim}(K \xrightarrow{\ell} K \xrightarrow{\ell} K \ldots)$, when K is a module over the ring $\mathbf{Z}_\ell$. Given a module M over any ring R and $f \in R$, $M_f = \mathrm{colim}(M \xrightarrow{f} M \xrightarrow{f} \ldots)$. This is seen from the universal property of localization (07K0), which says that localization is left adjoint to restriction as functors $\mathrm{Mod}_R \leftrightarrows \mathrm{Mod}_{R_f}$.

generators of Λ; in other words, $\mathrm{id} : \Lambda \to \Lambda$ is an ℓ-torsion element of the $\mathcal{O}_{\mathrm{E}}$-module $\mathrm{Hom}_{\mathcal{O}_{\mathrm{E}}}(\Lambda, \Lambda)$ as $\ell^n\,\mathrm{id} = 0$. This property is evidently preserved by any additive functor, so it's true for $\mathscr{E}$, $j_!\mathscr{E}$, and $f^! j_!\mathscr{E}$, so $0 = \ell^n\,\mathrm{id} =: \ell^n : f^!\mathrm{K} \to f^!\mathrm{K}$. Then

$$
\begin{aligned}
(f^!\mathrm{K})[\ell^{-1}] &= \mathrm{hocolim}(f^!\mathrm{K} \xrightarrow{\ell} f^!\mathrm{K} \xrightarrow{\ell} \dots) \\
&= \mathrm{hocolim}(f^!\mathrm{K} \xrightarrow{\ell^n} f^!\mathrm{K} \xrightarrow{\ell^n} \dots) \\
&= \mathrm{hocolim}(f^!\mathrm{K} \xrightarrow{0} f^!\mathrm{K} \xrightarrow{0} \dots) = 0
\end{aligned}
$$

(`0CRJ`), so indeed $f^!\mathrm{K}$ is of torsion. □

Duality. Assume X is noetherian and $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathbf{Z}_\ell)$ admits a dualizing complex ω_{X} as in Lemma 6.7.20 (so that $\omega_n := \omega_{\mathrm{X}} \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^n$ is dualizing in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathbf{Z}/\ell^n)$). We would like to find dualizing complexes for $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{E}})$ and $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathrm{E})$.

Lemma. — If E *is an algebraic extension of* $\mathbf{Q}_\ell$*, then* $\omega_{\mathrm{X}} \otimes_{\mathbf{Z}_{\ell,\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$ *in* $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{E}})$ *is dualizing, and* $\omega_{\mathrm{X}} \otimes_{\mathbf{Z}_{\ell,\mathrm{X}}} \mathrm{E}_{\mathrm{X}}$ *in* $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathrm{E})$ *is dualizing.*

Proof. — When $\mathrm{E}/\mathbf{Q}_\ell$ is finite, the first statement was proved in the note to Lemma 6.7.20.[89] So suppose E is infinite and K is in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{E}})$, so $\mathrm{K} = \mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}$ for F a finite extension of $\mathbf{Q}_\ell$ and K′ in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathcal{O}_{\mathrm{F}})$. Then our discussion of $\mathrm{R}\mathscr{H}\!om$ above gives

$$
\begin{aligned}
\mathrm{R}\mathscr{H}\!om_{\mathcal{O}_{\mathrm{E},\mathrm{X}}}(\mathrm{K}, \omega_{\mathrm{X}} \otimes_{\mathbf{Z}_{\ell,\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}) &= \mathrm{R}\mathscr{H}\!om_{\mathcal{O}_{\mathrm{E},\mathrm{X}}}(\mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}, \omega_{\mathrm{X}} \otimes_{\mathbf{Z}_{\ell,\mathrm{X}}} \mathcal{O}_{\mathrm{F},\mathrm{X}} \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}) \\
&= \mathrm{R}\mathscr{H}\!om_{\mathcal{O}_{\mathrm{F},\mathrm{X}}}(\mathrm{K}', \omega_{\mathrm{X}} \otimes_{\mathbf{Z}_{\ell,\mathrm{X}}} \mathcal{O}_{\mathrm{F},\mathrm{X}}) \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}, \quad \text{so} \\
\mathrm{D}_{\mathrm{X}}^2\mathrm{K} &= \mathrm{D}_{\mathrm{X}}\mathrm{R}\mathscr{H}\!om_{\mathcal{O}_{\mathrm{E},\mathrm{X}}}(\mathrm{K}, \omega_{\mathrm{X}} \otimes_{\mathbf{Z}_{\ell,\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}}) \\
&= (\mathrm{D}_{\mathrm{X}}^2\mathrm{K}') \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}} = \mathrm{K}' \otimes_{\mathcal{O}_{\mathrm{F},\mathrm{X}}} \mathcal{O}_{\mathrm{E},\mathrm{X}} = \mathrm{K}.
\end{aligned}
$$

[89]Théorème 7.1.3 of Exp. XVII of *Travaux de Gabber*, which is what is used in Lemma 6.7.20 to conclude $\omega_n \otimes_{\mathbf{Z}/\ell^n} \mathrm{R}_n$ is dualizing in $\mathrm{D}_{\mathrm{cons}}(\mathrm{X}_{\text{proét}}, \mathrm{R}_n)$, merely requires X be noetherian and ω_n dualizing.

Suppose now K is in $D_{\mathrm{cons}}(X_{\text{proét}}, E)$ and $K = K' \otimes_{\mathcal{O}_{E,X}} E_X$ with K' in $D_{\mathrm{cons}}(X_{\text{proét}}, \mathcal{O}_E)$. Then what we've just seen and our discussion of $R\mathscr{H}\!om$ above shows that

$$\begin{aligned} R\mathscr{H}\!om_{E_X}(K, \omega_X \otimes_{\mathbf{Z}_{\ell,X}} E_X) &= R\mathscr{H}\!om_{E_X}(K' \otimes_{\mathcal{O}_{E,X}} E_X, \omega_X \otimes_{\mathbf{Z}_{\ell,X}} \mathcal{O}_{E,X} \otimes_{\mathcal{O}_{E,X}} E_X) \\ &= R\mathscr{H}\!om_{\mathcal{O}_{E,X}}(K', \omega_X \otimes_{\mathbf{Z}_{\ell,X}} \mathcal{O}_{E,X}) \otimes_{\mathcal{O}_{E,X}} E_X, \quad \text{so} \\ D_X^2 K &= D_X R\mathscr{H}\!om_{E_X}(K, \omega_X \otimes_{\mathbf{Z}_{\ell,X}} E_X) \\ &= (D_X^2 K') \otimes_{\mathcal{O}_{E,X}} E_X = K' \otimes_{\mathcal{O}_{E,X}} E_X = K. \end{aligned}$$

□

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