Theories of Exponential and Logarithmic Functions: Part 3

In this concluding post on the theories of exponential and logarithmic functions we will present the most intuitive and obvious approach to define the expression a^b directly without going to the number e and the function $\log x$. This approach stems from the fact that an irrational number can be approximated by rational numbers and we can find as good approximations as we want. The idea is that if b is irrational and $a > 0$ then we have many rational approximations $b', b'', \ldots$ to b and the numbers $a^{b'}, a^{b''}, \ldots$ would be the approximations to the number a^b being defined. Inherent in such a procedure is the belief that we can find as good approximations to a^b as we want by choosing sufficiently good rational approximations to b. Thus we can see that the numbers

$$2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, \ldots$$

are approximations to the number $2^{\sqrt{2}}$.

Intuitive Definition of a^b

In order to make the definition described in previous paragraph more precise we start with the assumption that $a > 0$ and b is any real number. If $\{b_n\}$ is a sequence of rational numbers such that $\lim_{n\to\infty} b_n = b$ then we define

$$a^b = \lim_{n\to\infty} a^{b_n} \tag{1}$$

While this seems to be the most obvious approach to the theory of exponential and logarithmic functions it has two stumbling blocks:

- We need to show that the limit in definition (1) exists
- If there are two sequences $\{b_n\}, \{c_n\}$ both tending to b then we must show that $\lim_{n\to\infty} a^{b_n} = \lim_{n\to\infty} a^{c_n}$

Both these statements are obvious if $a = 1$. In the further development of the theory we can assume that $a > 1$ and the case $0 < a < 1$ can be treated by noting that $a^b = 1/(1/a)^b$ where b is rational and $(1/a) > 1$.

We first show that there is a sequence b_n tending to b such that $\lim_{n\to\infty} a^{b_n}$ exists. Clearly we can choose any monotonically increasing sequence b_n which tends to b. By any theory of real numbers (Dedekind cuts or Cantor's constructions of reals via Cauchy sequences of rationals) it is possible to find such a sequence b_n of rationals. In practice we may choose b_n to consist of decimal approximations to b adding each extra digit for each term of the sequence. Note that such a sequence b_n is bounded above (because it is increasing and reaches a limit b). This means that there is a rational number B such that $b_n \le B$ for all n. Now we can see that the sequence $x_n = a^{b_n}$ is increasing (because b_n is increasing and $a > 1$) and at the same time it is bounded above by a^B. It follows that x_n tends to a positive limit.

Next we show that if there are two sequences b_n, c_n of rationals each tending to b then $\lim_{n\to\infty} a^{b_n} = \lim_{n\to\infty} a^{c_n}$. Clearly if we put $d_n = b_n - c_n$ then d_n is a sequence of rationals tending to 0. We will first show that $a^{d_n} \to 1$ as $n \to \infty$. Let $\epsilon > 0$ be an arbitrary number. We know from previous post that $a^{1/n} \to 1$ and $a^{-1/n} \to 1$ as $n \to \infty$ hence there is an integer m such that

$$|a^{-1/n} - 1| < \epsilon,\ |a^{1/n} - 1| < \epsilon \tag{2}$$

whenever $n \geq m$. Since $d_n \to 0$ it follows that there is a positive integer N such that $|d_n| < 1/m$ for all $n \geq N$. This means that

$$a^{-1/m} < a^{d_n} < a^{1/m} \tag{3}$$

for all $n \geq N$. From equations (2) and (3) we can see that

$$|a^{d_n} - 1| < \epsilon$$

whenever $n \geq N$. It follows that $a^{d_n} \to 1$ as $n \to \infty$. Now we can see that

$$\lim_{n\to\infty} a^{b_n} = \lim_{n\to\infty} a^{d_n + c_n} = \lim_{n\to\infty} a^{d_n} \lim_{n\to\infty} a^{c_n} = \lim_{n\to\infty} a^{c_n}$$

Thus both the stumbling blocks in definition (1) are taken care of and we can proceed further.

We next prove that if $b > 0$ then $a^b > 1$. Since $b > 0$ there must be an increasing sequence b_n of positive rationals tending to b. Clearly we have

$$b_n > b_1 > 0$$

for all $n > 1$ and hence it follows that

$$a^{b_n} > a^{b_1} > 1$$

for all $n > 1$. Taking limits as $n \to \infty$ we see that $a^b \geq a^{b_1} > 1$.

We are now in a position to study the function $f(x) = a^x$. As before we will only need to consider the case $a > 1$ and reader can state and prove results for the case $0 < a < 1$ himself. First we see that the law of exponents is valid i.e.

$$a^x a^y = a^{x+y} \tag{4}$$

for all x, y. Clearly let x_n, y_n be sequences of rationals tending to x, y so that $(x_n + y_n) \to (x + y)$. Then we have $a^{x_n} a^{y_n} = a^{x_n + y_n}$ and taking limits when $n \to \infty$ we are done. In similar fashion we can prove usual laws of exponents.

We next show that $f(x)$ is strictly increasing for all x. Let $x > y$ so that $x - y > 0$ and then $a^{x-y} > 1$. Multiplying by $a^y > 0$ on both sides we get $a^{x-y} a^y > a^y$ i.e. $a^x > a^y$. From this point it is easy to establish most of the common inequalities regarding exponents and we will not go into details. In particular the inequalities related to α, β, r, s in the last post all hold true for all positive real numbers r, s. However there is a caveat. The inequalities may become

weak if we try to construct sequences of positive rationals tending to r, s and then letting $n \to \infty$. But we don't really need the strict versions of those inequalities.

It will be important now to establish the simple but fundamental limit

$$\lim_{x \to 0} a^x = 1 \tag{5}$$

Let $\epsilon > 0$ be arbitrary. Then there is a positive integer m such that

$$|a^{-1/n} - 1| < \epsilon,\ |a^{1/n} - 1| < \epsilon$$

for $n \geq m$. Clearly if we chose $x \in (-1/m, 1/m)$ then

$$a^{-1/m} < a^x < a^{1/m}$$

and hence $|a^x - 1| < \epsilon$. Thus we can choose $\delta = 1/m$ and then we can see that $|a^x - 1| < \epsilon$ whenever $0 < |x| < \delta$. Thus $a^x \to 1$ as $x \to 0$.

As in the last post we had shown that $n(x^{1/n} - 1)$ is decreasing as n increases, we can show that the function $g(x) = \dfrac{a^x - 1}{x}$ is increasing for all $x > 0$. Also the function $g(x)$ satisfies $(a - 1) \geq g(x) \geq a^{x-1}(a - 1)$ for all $a > 1, x > 0$. It follows that as $x \to 0^+$ the limit

$$\lim_{x \to 0^+} \frac{a^x - 1}{x} = \phi(a)$$

exists and $\phi(a) \geq (a - 1)/a$. If $x \to 0^-$ then we put $x = -y$ so that $y \to 0^+$ and then we have

$$\lim_{x \to 0^-} \frac{a^x - 1}{x} = \lim_{y \to 0^+} \frac{a^y - 1}{ya^y} = \phi(a)/1 = \phi(a)$$

It follows that the limit

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \phi(a) \tag{6}$$

defines a function $\phi(a)$ which is defined for all $a > 0$.

We can also see that $\phi(1) = 0$ and $(a - 1)/a \leq \phi(a) \leq (a - 1)$ for $a > 1$. Thus $\phi(a) > 0$ for $a > 1$ and it can be easily seen that $\phi(1/a) = -\phi(a)$ so that $\phi(a) < 0$ for $0 < a < 1$. It can be established that $\phi(ab) = \phi(a) + \phi(b)$ for all positive a, b and $\phi(a/b) = \phi(a) - \phi(b)$. Thus if $a > b$ so that $a/b > 1$ and we have $\phi(a/b) > 0$ so that $\phi(a) > \phi(b)$.

It follows that the function $\phi(a)$ is defined for all $a > 0$ and it is strictly increasing function of a. It satisfies the properties

$$\phi(1) = 0, \phi(1/a) = -\phi(a), \phi(ab) = \phi(a) + \phi(b), \phi(a/b) = \phi(a) - \phi(b)$$

It is now time to identify this function $\phi(a)$ with $\log a$ and thus we define

$$\log x = \lim_{b \to 0} \frac{x^b - 1}{b} \tag{7}$$

for all $x > 0$. Just to repeat the properties of log function established in terms of ϕ we have

$$\log 1 = 0, \log(1/x) = -\log x, \tag{8a}$$
$$\log(xy) = \log x + \log y, \tag{8b}$$
$$\log(x/y) = \log x - \log y \tag{8c}$$

for all positive x, y. Also $\log x$ is a strictly increasing function of x and

$$\frac{x-1}{x} \leq \log x \leq (x-1) \tag{9}$$

for $x > 1$. From this last equation we can easily show that

$$\lim_{x \to 1} \frac{\log x}{x-1} = 1 \text{ or equivalently } \lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \tag{10}$$

and this limit shows that the derivative of $\log x$ is $1/x$ (see previous post).

We can also find derivative of $f(x) = a^x$. We have

$$(a^x)' = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \log a \tag{11}$$

We next show that

$$\log a^b = b \log a \tag{12}$$

for all $a > 0$ and all b. Clearly from the property $\log(ab) = \log a + \log b$ it is easy to show that (12) holds if b is rational. If b is irrational then we can take a sequence b_n of rationals tending to b and then we have $\log a^{b_n} = b_n \log a$. Taking limits when $n \to \infty$ and noting that both $\log x$ and a^x are continuous we get $\log a^b = b \log a$.

Now $\log 2 > 0$ and hence given any number $M > 0$ we can find an integer n such that $n \log 2 > M$. Let $N = 2^n$. If $x > N$ then

$$\log x > \log N = \log 2^n = n \log 2 > M$$

It follows that $\log x \to \infty$ as $x \to \infty$. Noting that $\log(1/x) = -\log x$ we get $\log x \to -\infty$ as $x \to 0^+$. It follows that $\log x$ is a strictly increasing function from $\mathbb{R}^+$ to $\mathbb{R}$ and hence there is a unique number $e > 1$ such that $\log e = 1$.

Then for any real x we have $\log e^x = x \log e = x$ so that if $y = e^x$ then $x = \log y$. It turns out that the specific power e^x is the inverse of the logarithm function. Also we note that $(e^x)' = e^x \log e = e^x$ so that e^x is its own derivative. We immediately get the formula

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \tag{13}$$

We will now show that

$$e^x = \lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = \lim_{t\to-\infty}\left(1+\frac{x}{t}\right)^t \tag{14}$$

Note that we can more easily combine these two limits into one and show instead that

$$\lim_{h\to 0}(1+xh)^{1/h} = e^x$$

Clearly we can see that

$$\begin{aligned}\lim_{h\to 0}\log(1+xh)^{1/h} &= \lim_{h\to 0}\frac{\log(1+xh)}{h}\\ &= \lim_{h\to 0} x\cdot\frac{\log(1+xh)}{xh} = x\end{aligned}$$

and by continuity of $\log x$ and e^x we get $\lim_{h\to 0}(1+xh)^{1/h} = e^x$. If we replace t in (14) by positive integer n then we immediately get the usual formulas

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-n},\ e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{15}$$

Note that the infinite series for e^x can either be proved using limits above or using the Taylor's series approach based on derivatives. We have now been able to derive all the common properties of $a^x, e^x, \log x$ and have established the derivative formulas. The interested reader can carry on from here and establish any property of these functions without any hassle.

Note: This post is inspired by this and this answer on MSE.

By Paramanand Singh
Sunday, May 11, 2014

Labels: Mathematical Analysis

Paramanand's Math Notes
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