Values of Rogers-Ramanujan Continued Fraction: Part 2

Continuing our journey from the last post we will deduce further properties of the Rogers-Ramanujan Continued Fraction $R(q)$ which will help us to find out further values of $R(q)$. In this connection we first establish an identity concerning powers of $R(q)$.

Identity Concerning $R^5(q)$

Using the identity (3) from the last post, Ramanujan established another fundamental property of $R(q)$ namely:

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} \tag{1}$$

Ramanujan used a very simple technique to establish the above mentioned identity. First he transforms the identity

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}$$

by replacing q with q^5 and obtains

$$\frac{1}{R(q^5)} - 1 - R(q^5) = \frac{f(-q)}{qf(-q^{25})} \tag{2}$$

Now it should be noted that

$$R(q^5) = \frac{q}{1+}\frac{q^5}{1+}\frac{q^{10}}{1+}\frac{q^{15}}{1+}\cdots$$

so that we can write $R(q^5) = qA(q^5)$ where $A(q^5)$ is dependent directly on q^5 and not on q.

Next step used by Ramanujan is highly economical and non-obvious. He takes a primitive 5th root of unity, say ζ, and replaces q in equation (2) by $\zeta^i q$ where $i = 0, 1, 2, 3, 4$ and multiplies the resulting equations. Let's first consider one such equation:

$$\frac{1}{\zeta^i qA(q^5)} - 1 - \zeta^i qA(q^5) = \frac{f(-\zeta^i q)}{\zeta^i qf(-q^{25})} \tag{3}$$

Let α, β be roots of $t^2 - t - 1 = 0$ so that

$$\alpha + \beta = 1, \alpha\beta = -1, \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

If we set $z_i = 1/\zeta^i qA(q^5)$ then we can see that the LHS of equation (3) can be written as $z_i - 1 - z_i^{-1} = (z_i - \alpha)(1 - \beta z_i^{-1})$. Let us write $z = z_0 = 1/R(q^5)$ for simplicity and then we note that $z_i = z/\zeta^i = \zeta^{5-i}z = \zeta^j z$ where j varies from 4 to 1 as i varies from 1 to 4. So putting $i = 0, 1, 2, 3, 4$ in equation (3) and multiplying the resulting equations we get the LHS as

$$\begin{aligned}&(z-\alpha)(1-\beta z^{-1})(\zeta z-\alpha)(1-\beta\zeta^{-1}z^{-1})\\&\quad\times(\zeta^2 z-\alpha)(1-\beta\zeta^{-2}z^{-1})(\zeta^3 z-\alpha)(1-\beta\zeta^{-3}z^{-1})\\&\quad\times(\zeta^4 z-\alpha)(1-\beta\zeta^{-4}z^{-1})\\&\qquad=(z^5-\alpha^5)(1-\beta^5 z^{-5})=z^5-(\alpha^5+\beta^5)+(\alpha\beta)^5 z^{-5}\end{aligned}$$

It can be verified by direct calculation or using simple elementary algebra that $\alpha^5+\beta^5=11$ so that the desired LHS is z^5-11-z^{-5} or

$$\frac{1}{R^5(q^5)}-11-R^5(q^5)$$

On the other hand the desired RHS is

$$\begin{aligned}\prod_{i=0}^{4}\frac{f(-\zeta^i q)}{\zeta^i q f(-q^{25})}&=\frac{f(-q)f(-\zeta q)f(-\zeta^2 q)f(-\zeta^3 q)f(-\zeta^4 q)}{q^5 f^5(-q^{25})}\\&=\frac{\displaystyle\prod_{n>0,n\not\equiv 0\pmod 5}(1-q^{5n})\cdot\prod_{n=1}^{\infty}(1-q^{5n})^5}{q^5 f^5(-q^{25})}\\&=\frac{f^6(-q^5)}{q^5 f^6(-q^{25})}\end{aligned}$$

In the above derivation we need to understand that each $f(-\zeta^i q)$ is an infinite product of the form $\prod\limits_{n=1}^{\infty}\{1-(\zeta^i q)^n\}$ and we have to multiply 5 such infinite products. We take first term of each such product and multiply to get $(1-q^5)$ thereby getting rid of the ζ's. Taking second term of each product and multiplying we get $(1-q^{10})$ and so on till we get $(1-q^{20})$ by multiplying 4th term of each product. Note that the 5th, 10th, 15th, $(5n)$th term in each product is already without any ζ and hence these terms are clubbed separately to form $\prod\limits_{n=1}^{\infty}(1-q^{5n})^5$. In this way we go from the first line in above derivation to the second line.

Next note that to go from the second line to the third line we need to see that the powers of q in the first product are all multiples of 5 except the multiples of 25. These we add ourselves both in numerator and denominator thereby completing the derivation. It is better if the reader writes out the products in expanded form and does the manipulations as explained here to convince of the result obtained finally.

We thus have established the following identity

$$\frac{1}{R^5(q^5)}-11-R^5(q^5)=\frac{f^6(-q^5)}{q^5 f^6(-q^{25})}$$

Replacing q^5 by q we obtain identity (1). If we replace $R(q)$ by $q^{1/5}H(q)/G(q)$ in equation (1) we immediately obtain the identity mentioned at the end of this post namely:

$$H(q)\{G(q)\}^{11}-q^2 G(q)\{H(q)\}^{11}=1+11q\{G(q)H(q)\}^6$$

Evaluating $R(e^{-2\pi/\sqrt{5}})$

If we define $\eta(q) = q^{1/12}f(-q^2)$ then we know that for any $n > 0$ we have

$$\frac{\eta(e^{-\pi\sqrt{n}})}{\eta(e^{-\pi/\sqrt{n}})} = n^{-1/4}$$

Putting $n = 5$ we get

$$\begin{aligned}\frac{\eta(e^{-\pi\sqrt{5}})}{\eta(e^{-\pi/\sqrt{5}})} &= 5^{-1/4}\\ \Rightarrow \frac{e^{-\pi\sqrt{5}/12}f(-e^{-2\pi\sqrt{5}})}{e^{-\pi/12\sqrt{5}}f(-e^{-2\pi/\sqrt{5}})} &= 5^{-1/4}\\ \Rightarrow \frac{e^{-2\pi/\sqrt{5}}f^6(-e^{-2\pi\sqrt{5}})}{f^6(-e^{-2\pi/\sqrt{5}})} &= 5^{-3/2}\end{aligned}$$

If we put $q = e^{-2\pi/\sqrt{5}}$ in (1) and set $x = R(q), y = x^5$ we get

$$\begin{aligned}&\frac{1}{y} - 11 - y = 5^{3/2} = 5\sqrt{5}\\ &\Rightarrow y^2 + (5\sqrt{5} + 11)y - 1 = 0\\ &\Rightarrow y = \frac{-5\sqrt{5} - 11 \pm \sqrt{250 + 110\sqrt{5}}}{2}\end{aligned}$$

Since $y > 0$ we must take $+$ sign and then we get

$$R(e^{-2\pi/\sqrt{5}}) = x = y^{1/5} = \left(\sqrt{\frac{125 + 55\sqrt{5}}{2}} - \frac{11 + 5\sqrt{5}}{2}\right)^{1/5}$$

Note that it is difficult to use equation (1) to find the value of $R(e^{-2\pi\sqrt{5}})$. However the equation (1) can also be used to find the value of $R(-e^{-\pi/\sqrt{5}})$. If we set $S(q) = -R(-q)$ then the equation (1) is transformed into

$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{qf^6(q^5)} \tag{4}$$

If we define $\eta(q) = q^{1/24}f(-q)$ then we know that

$$\frac{\eta^{24}(-e^{-\pi\sqrt{n}})}{\eta^{24}(-e^{-\pi/\sqrt{n}})} = n^{-6}$$

Putting $n = 5$ we get

$$\begin{aligned}\frac{\eta^{24}(-e^{-\pi\sqrt{5}})}{\eta^{24}(-e^{-\pi/\sqrt{5}})} &= 5^{-6}\\ \Rightarrow \frac{-e^{-\pi\sqrt{5}}f^{24}(e^{-\pi\sqrt{5}})}{-e^{-\pi/\sqrt{5}}f^{24}(e^{-\pi/\sqrt{5}})} &= 5^{-6}\\ \Rightarrow \frac{e^{-4\pi/\sqrt{5}}f^{24}(e^{-\pi\sqrt{5}})}{f^{24}(e^{-\pi/\sqrt{5}})} &= 5^{-6}\\ \Rightarrow \frac{e^{-\pi/\sqrt{5}}f^6(e^{-\pi\sqrt{5}})}{f^6(e^{-\pi/\sqrt{5}})} &= 5^{-3/2}\end{aligned}$$

Putting $q = e^{-\pi/\sqrt{5}}$ in equation (4) and setting $x = S(q), y = x^5$ we get

$$\begin{aligned}&\frac{1}{y} + 11 - y = 5\sqrt{5}\\ &\Rightarrow y^2 + (5\sqrt{5} - 11)y - 1 = 0\\ &\Rightarrow y = \frac{11 - 5\sqrt{5} \pm \sqrt{250 - 110\sqrt{5}}}{2}\end{aligned}$$

Since $y > 0$ we need to take $+$ sign above and then we get

$$x = S(e^{-\pi/\sqrt{5}}) = y^{1/5} = \left(\sqrt{\frac{125 - 55\sqrt{5}}{2}} - \frac{5\sqrt{5} - 11}{2}\right)^{1/5}$$

Relation between $R(e^{-2\alpha})$ and $R(e^{-2\beta})$ when $\alpha\beta = \pi^2$

In order to evaluate $R(e^{-2\pi\sqrt{5}})$ one can resort to the transformation formula for $R(q)$ which connects $R(e^{-2\alpha})$ and $R(e^{-2\beta})$ under the condition $\alpha\beta = \pi^2$. Ramanujan was keenly interested in such transformation formulas for various theta functions and also provided such formulas for $R(q)$ and $R(-q)$.

Ramanujan offers the following two formulas in his characteristic style:
If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

1) $$\left\{\frac{\sqrt{5}+1}{2} + \frac{e^{-2\alpha/5}}{1+}\,\frac{e^{-2\alpha}}{1+}\,\frac{e^{-4\alpha}}{1+}\,\frac{e^{-6\alpha}}{1+}\cdots\right\}$$
$$\times\left\{\frac{\sqrt{5}+1}{2} + \frac{e^{-2\beta/5}}{1+}\,\frac{e^{-2\beta}}{1+}\,\frac{e^{-4\beta}}{1+}\,\frac{e^{-6\beta}}{1+}\cdots\right\} = \frac{5+\sqrt{5}}{2}$$

2) $$\left\{\frac{\sqrt{5}-1}{2} + \frac{e^{-\alpha/5}}{1-}\,\frac{e^{-\alpha}}{1+}\,\frac{e^{-2\alpha}}{1-}\,\frac{e^{-3\alpha}}{1+}\cdots\right\}$$
$$\times\left\{\frac{\sqrt{5}-1}{2} + \frac{e^{-\beta/5}}{1-}\,\frac{e^{-\beta}}{1+}\,\frac{e^{-2\beta}}{1-}\,\frac{e^{-3\beta}}{1+}\cdots\right\} = \frac{5-\sqrt{5}}{2}$$

Note that we can write the above transformation formulas in the more compact notation as

$$\left(\frac{\sqrt{5}+1}{2} + R(e^{-2\alpha})\right)\left(\frac{\sqrt{5}+1}{2} + R(e^{-2\beta})\right) = \frac{5+\sqrt{5}}{2} \tag{5}$$

and

$$\left(\frac{\sqrt{5}-1}{2} + S(e^{-\alpha})\right)\left(\frac{\sqrt{5}-1}{2} + S(e^{-\beta})\right) = \frac{5-\sqrt{5}}{2} \tag{6}$$

where $S(q) = -R(-q)$.

Proofs of both the formulas are similar and hence we will establish only the first formula here. Let us define eta function by $\eta(q) = q^{1/12}f(-q^2)$ and set $A = R(e^{-2\alpha})$ and $B = R(e^{-2\beta})$, then from the last post we have

$$\frac{1}{A} - 1 - A = \frac{f(-e^{-2\alpha/5})}{e^{-2\alpha/5}f(-e^{-10\alpha})} = \frac{e^{\alpha/60}\eta(e^{-\alpha/5})}{e^{-2\alpha/5}e^{5\alpha/12}\eta(e^{-5\alpha})} = \frac{\eta(e^{-\alpha/5})}{\eta(e^{-5\alpha})} \tag{7}$$

and

$$\frac{1}{B} - 1 - B = \frac{f(-e^{-2\beta/5})}{e^{-2\beta/5}f(-e^{-10\beta})} = \frac{\eta(e^{-\beta/5})}{\eta(e^{-5\beta})} \tag{8}$$

Next we have the following transformation formula for eta function

$$\alpha^{1/4}\eta(e^{-\alpha}) = \beta^{1/4}\eta(e^{-\beta})$$

(this is just the formula $\eta(e^{-\pi\sqrt{n}})/\eta(e^{-\pi/\sqrt{n}}) = n^{-1/4}$ in another form as can be seen easily by setting $\alpha = \pi\sqrt{n}, \beta = \pi/\sqrt{n}$ so that $n = \alpha/\beta$). Replacing α by $\alpha/5$ and β by 5β (this does not change the product $\alpha\beta = \pi^2$) we get

$$\alpha^{1/4}\eta(e^{-\alpha/5}) = \sqrt{5}\beta^{1/4}\eta(e^{-5\beta}) \tag{9}$$

Interchanging α and β we get

$$\beta^{1/4}\eta(e^{-\beta/5}) = \sqrt{5}\alpha^{1/4}\eta(e^{-5\alpha}) \tag{10}$$

Multiplying the above equations (9) and (10) we get

$$\eta(e^{-\alpha/5})\eta(e^{-\beta/5}) = 5\eta(e^{-5\alpha})\eta(e^{-5\beta})$$

and hence on multiplying (7) and (8) we get

$$\begin{aligned}
&\left(\frac{1}{A} - 1 - A\right)\left(\frac{1}{B} - 1 - B\right) = 5\\
&\Rightarrow (A^2 + A - 1)(B^2 + B - 1) = 5AB\\
&\Rightarrow (AB)^2 + A^2B - A^2 + AB^2 + AB - A - B^2 - B + 1 = 5AB\\
&\Rightarrow (AB)^2 + AB(A + B) - (A + B) + 1 = A^2 + B^2 + 4AB\\
&\Rightarrow (AB)^2 + AB(A + B) - (A + B) + 1 = (A + B)^2 + 2AB\\
&\Rightarrow (AB)^2 + AB(A + B) - (A + B) + 1 - 2AB + \frac{(A + B)^2}{4} = \frac{5}{4}(A + B)^2\\
&\Rightarrow \left\{AB + \frac{A + B}{2} - 1\right\}^2 = \frac{5}{4}(A + B)^2\\
&\Rightarrow AB + \frac{A + B}{2} - 1 = \pm\frac{\sqrt{5}}{2}(A + B)
\end{aligned}$$

Now as $\alpha \to \infty$, $\beta \to 0$ so that $A \to 0$ and $B \to (\sqrt{5} - 1)/2$ so that the LHS of above equation tends to a negative value. It therefore follows that we must take the negative sign on the right and then

$$\begin{aligned}
&AB + \frac{\sqrt{5} + 1}{2}(A + B) = 1\\
&\Rightarrow 4AB + 2(\sqrt{5} + 1)(A + B) + (\sqrt{5} + 1)^2 = 4 + (\sqrt{5} + 1)^2\\
&\Rightarrow (2A + \sqrt{5} + 1)(2B + \sqrt{5} + 1) = 10 + 2\sqrt{5}\\
&\Rightarrow \left(\frac{\sqrt{5} + 1}{2} + A\right)\left(\frac{\sqrt{5} + 1}{2} + B\right) = \frac{5 + \sqrt{5}}{2}
\end{aligned}$$

so that the equation (5) is established.

There are analogous transformation formulas involving $R^5(q), S^5(q)$ which we state and establish below:

If $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2/5$ then we have

$$\left[\left[\frac{\sqrt{5}+1}{2}\right]^5 + R^5(e^{-2\alpha})\right]\left[\left[\frac{\sqrt{5}+1}{2}\right]^5 + R^5(e^{-2\beta})\right] = 5\sqrt{5}\left[\frac{\sqrt{5}+1}{2}\right]^5 \tag{11}$$

and

$$\left[\left[\frac{\sqrt{5}-1}{2}\right]^5 + S^5(e^{-\alpha})\right]\left[\left[\frac{\sqrt{5}-1}{2}\right]^5 + S^5(e^{-\beta})\right] = 5\sqrt{5}\left[\frac{\sqrt{5}-1}{2}\right]^5 \tag{12}$$

We establish (11) here on the lines similar to the proof of equation (5). Let us set $A = R^5(e^{-2\alpha}), B = R^5(e^{-2\beta})$ and $\eta(q) = q^{1/12}f(-q^2)$. Then we have

$$\frac{1}{A} - 11 - A = \frac{f^6(-e^{-2\alpha})}{e^{-2\alpha}f^6(-e^{-10\alpha})} = \frac{e^{\alpha/2}\eta^6(e^{-\alpha})}{e^{-2\alpha}e^{5\alpha/2}\eta^6(e^{-5\alpha})} = \frac{\eta^6(e^{-\alpha})}{\eta^6(e^{-5\alpha})} \tag{13}$$

$$\frac{1}{B} - 11 - B = \frac{f^6(-e^{-2\beta})}{e^{-2\beta}f^6(-e^{-10\beta})} = \frac{\eta^6(e^{-\beta})}{\eta^6(e^{-5\beta})} \tag{14}$$

Again we know that if $a, b > 0$ and $ab = \pi^2$ then

$$a^{1/4}\eta(e^{-a}) = b^{1/4}\eta(e^{-b})$$

Putting $a = \alpha, b = 5\beta$ we get

$$\alpha^{1/4}\eta(e^{-\alpha}) = 5^{1/4}\beta^{1/4}\eta(e^{-5\beta})$$

Interchanging roles of α and β we get

$$\beta^{1/4}\eta(e^{-\beta}) = 5^{1/4}\alpha^{1/4}\eta(e^{-5\alpha})$$

Multiplying last two equations we arrive at

$$\eta(e^{-\alpha})\eta(e^{-\beta}) = \sqrt{5}\eta(e^{-5\alpha})\eta(e^{-5\beta})$$

It now follows upon multiplying equations (13) and (14) that

$$\begin{aligned}
&\left(\frac{1}{A} - 11 - A\right)\left(\frac{1}{B} - 11 - B\right) = 125\\
&\Rightarrow (A^2 + 11A - 1)(B^2 + 11B - 1) = 125AB\\
&\Rightarrow (AB)^2 + 11AB(A+B) - (A^2 + B^2) + 121AB - 11(A+B) + 1 = 125AB\\
&\Rightarrow (AB)^2 + 11AB(A+B) - 11(A+B) + 1 = A^2 + B^2 + 4AB\\
&\Rightarrow (AB)^2 + 11AB(A+B) - 11(A+B) - 2AB + 1 = (A+B)^2\\
&\Rightarrow (AB)^2 + 11AB(A+B) + \left(\frac{11(A+B)}{2}\right)^2 - 11(A+B) - 2AB + 1\\
&\qquad = \frac{125}{4}\cdot(A+B)^2\\
&\Rightarrow \left(AB + \frac{11(A+B)}{2} - 1\right)^2 = \frac{125}{4}\cdot(A+B)^2\\
&\Rightarrow AB + \frac{11(A+B)}{2} - 1 = \pm\frac{5\sqrt{5}}{2}\cdot(A+B)
\end{aligned}$$

If we let $\alpha \to \infty$ then $\beta \to 0+$ so that $A \to 0$ and $B \to (5\sqrt{5} - 11)/2$ and thus the LHS of above equation tends to a negative value. We must therefore select the $-$ sign above and then we get

$$\begin{aligned}
&AB + \left(\frac{5\sqrt{5}+11}{2}\right)(A+B) = 1\\
&\Rightarrow AB + \left(\frac{\sqrt{5}+1}{2}\right)^{5}(A+B) + \left(\frac{\sqrt{5}+1}{2}\right)^{10} = 1 + \frac{121+125+110\sqrt{5}}{4}\\
&\Rightarrow \left\{\left(\frac{\sqrt{5}+1}{2}\right)^{5} + A\right\}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^{5} + B\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^{5}
\end{aligned}$$

and this establishes equation (11).

Evaluation of $R(e^{-2\pi\sqrt{5}})$

Equation (5) can now be used to calculate the value of $R(e^{-2\pi\sqrt{5}})$ since we already know the value of $R(e^{-2\pi/\sqrt{5}})$.

From equation (5) we have

$$\begin{aligned}
R(e^{-2\pi\sqrt{5}}) &= \frac{\dfrac{5+\sqrt{5}}{2}}{\dfrac{\sqrt{5}+1}{2} + R(e^{-2\pi/\sqrt{5}})} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{\dfrac{\sqrt{5}-1}{2}\left(\dfrac{\sqrt{5}+1}{2} + R(e^{-2\pi/\sqrt{5}})\right)} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \dfrac{\sqrt{5}-1}{2}\cdot R(e^{-2\pi/\sqrt{5}})} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \dfrac{\sqrt{5}-1}{2}\left(\sqrt{\dfrac{125+55\sqrt{5}}{2}} - \dfrac{11+5\sqrt{5}}{2}\right)^{1/5}} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \left\{\left(\dfrac{5\sqrt{5}-11}{2}\right)\sqrt{\dfrac{125+55\sqrt{5}}{2}} - 1\right\}^{1/5}} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \left\{\sqrt{5\sqrt{5}\cdot\dfrac{5\sqrt{5}+11}{2}\left(\dfrac{5\sqrt{5}-11}{2}\right)^2} - 1\right\}^{1/5}} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \left\{\sqrt{5\sqrt{5}\left(\dfrac{5\sqrt{5}-11}{2}\right)} - 1\right\}^{1/5}} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \left\{5^{3/4}\left(\dfrac{\sqrt{5}-1}{2}\right)^{5/2} - 1\right\}^{1/5}} - \frac{\sqrt{5}+1}{2}\\
&= \frac{\sqrt{5}}{1 + \sqrt[5]{5^{3/4}\left(\dfrac{\sqrt{5}-1}{2}\right)^{5/2} - 1}} - \frac{\sqrt{5}+1}{2}
\end{aligned}$$

(note that in the above derivation we have used the fact that

$$\left(\frac{\sqrt{5}-1}{2}\right)^5 = \frac{5\sqrt{5}-11}{2}$$

which can be verified easily by using simple algebra).

This is the form in which Ramanujan gave the value of $R(e^{-2\pi\sqrt{5}})$. Similarly we can show that

$$S(e^{-\pi\sqrt{5}}) = \frac{\sqrt{5}}{1 + \sqrt[5]{5^{3/4}\left(\dfrac{\sqrt{5}+1}{2}\right)^{5/2} - 1}} - \frac{\sqrt{5}-1}{2}$$

Alternative Proof for Ramanujan's Congruence $p(25n+24) \equiv 0 \pmod{25}$

The technique used to establish the identity (1) concerning $R^5(q)$ above was utilized in a very smart manner by Ramanujan to find the generating function of $p(5n+4)$ and thereby establish his congruence $p(25n+24)\equiv 0 \pmod{25}$. Readers should contrast the presentation given here with another proof of the same congruence provided in an earlier post.

Let us write $\xi=q^{-1/5}R(q)$ so that

$$\xi=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\cdots}$$

and then the identity concerning $R(q)$ can be written as

$$\frac{1}{\xi^{-1}-q^{1/5}-\xi q^{2/5}}=\frac{f(-q^5)}{f(-q^{1/5})}=\frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^{1/5})(1-q^{2/5})(1-q^{3/5})\cdots}\tag{15}$$

and the identity concerning $R^5(q)$ can be written as

$$\frac{1}{\xi^{-5}-11q-q^2\xi^5}=\frac{f^6(-q^5)}{f^6(-q)}=\frac{\{(1-q^5)(1-q^{10})(1-q^{15})\cdots\}^6}{\{(1-q)(1-q^2)(1-q^3)\cdots\}^6}\tag{16}$$

If we put $t=q^{1/5}$ so that $t^5=q$ then we have

$$\begin{aligned}\xi^{-5}-11q-q^2\xi^5&=\xi^{-5}-q(\alpha^5+\beta^5)+q^2\alpha^5\beta^5\xi^5\\&=(\xi^{-5}-q\alpha^5)(1-q\beta^5\xi^5)\\&=(\xi^{-5}-(\alpha t)^5)(1-(\beta t\xi)^5)\\&=(\xi^{-1}-\alpha t)(1-\beta t\xi)\\&\quad\times(\xi^{-1}-\zeta\alpha t)(1-\zeta^4\beta t\xi)\\&\quad\times(\xi^{-1}-\zeta^2\alpha t)(1-\zeta^3\beta t\xi)\\&\quad\times(\xi^{-1}-\zeta^3\alpha t)(1-\zeta^2\beta t\xi)\\&\quad\times(\xi^{-1}-\zeta^4\alpha t)(1-\zeta\beta t\xi)\\&=(\xi^{-1}-t-t^2\xi)\\&\quad\times\{\xi^{-1}-t(\alpha\zeta+\beta/\zeta)-t^2\xi\}\\&\quad\times\{\xi^{-1}-t(\alpha\zeta^2+\beta/\zeta^2)-t^2\xi\}\\&\quad\times\{\xi^{-1}-t(\alpha\zeta^3+\beta/\zeta^3)-t^2\xi\}\\&\quad\times\{\xi^{-1}-t(\alpha\zeta^4+\beta/\zeta^4)-t^2\xi\}\\&=(\xi^{-1}-t-t^2\xi)\\&\quad\times\{\xi^{-1}-t(\alpha\zeta+\beta/\zeta)-t^2\xi\}\\&\quad\times\{\xi^{-1}-t(\alpha/\zeta+\beta\zeta)-t^2\xi\}\\&\quad\times\{\xi^{-1}-t(\alpha\zeta^2+\beta/\zeta^2)-t^2\xi\}\\&\quad\times\{\xi^{-1}-t(\alpha/\zeta^2+\beta\zeta^2)-t^2\xi\}\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\xi^{-5}-11q-q^2\xi^5}{\xi^{-1}-t-t^2\xi}&=\{\xi^{-2}-t\xi^{-1}(\zeta+1/\zeta)+t^2(1-(\zeta+1/\zeta))\\&\quad+t^3\xi(\zeta+1/\zeta)+t^4\xi^2\}\\&\quad\times\{\xi^{-2}-t\xi^{-1}(\zeta^2+1/\zeta^2)+t^2(1-(\zeta^2+1/\zeta^2))\\&\quad+t^3\xi(\zeta^2+1/\zeta^2)+t^4\xi^2\}\end{aligned}$$

It is not necessary to multiply the full product on the right side (although Ramanujan did calculate the whole product in tedious fashion). What we really need here is the term containing t^4 and if we calculate this term it will surprisingly come out to be $5t^4$. Next dividing equation (15) by equation (16) we get

$$\frac{f^{6}(-q)}{f(-q^{1/5})f^{5}(-q^{5})}=\frac{\xi^{-5}-11q-q^{2}\xi^{5}}{\xi^{-1}-t-t^{2}\xi}$$

Since $1/f(-q^{1/5})=\sum_{n=0}^{\infty}p(n)q^{n/5}$ comparing the coefficients of $t^{4}=q^{4/5}$ on both sides of the equation we get

$$\frac{f^{6}(-q)}{f^{5}(-q^{5})}\sum_{n=0}^{\infty}p(5n+4)q^{n}=5$$

and thus follows the "beautiful identity" of Ramanujan

$$\sum_{n=0}^{\infty}p(5n+4)q^{n}=5\frac{f^{5}(-q^{5})}{f^{6}(-q)}=5\frac{\{(1-q^{5})(1-q^{10})(1-q^{15})\cdots\}^{5}}{\{(1-q)(1-q^{2})(1-q^{3})\cdots\}^{6}}\tag{17}$$

The congruence $p(5n+4)\equiv 0\pmod 5$ is immediately obvious from the above identity. It follows that the coefficient of q^{5n} in

$$\frac{q}{(1-q)(1-q^{2})(1-q^{3})\cdots}=\sum_{n=0}^{\infty}p(n)q^{n+1}$$

is a multiple of 5.

Also we know that $(1-q)^{5}=1-q^{5}+5J$ where J represents a power series with integer coefficients. Thus we get $(1-q)^{5}/(1-q^{5})=1+5J$ and taking reciprocals we get $(1-q^{5})/(1-q)^{5}=1+5J$. It follows that

$$\frac{(1-q^{5})(1-q^{10})\cdots}{\{(1-q)(1-q^{2})\cdots\}^{5}}=1+5J$$

Now we can see that

$$\frac{\sum_{n=0}^{\infty}p(5n+4)q^{n+1}}{5\{(1-q^{5})(1-q^{10})\cdots\}^{4}}=\frac{q}{(1-q)(1-q^{2})\cdots}\cdot\frac{(1-q^{5})(1-q^{10})\cdots}{\{(1-q)(1-q^{2})\cdots\}^{5}}$$

Clearly the coefficient of q^{5n} on the right is a multiple of 5 and hence we get the congruence $p(25n+24)\equiv 0\pmod{25}$.

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Labels: Continued Fractions , Mathematical Analysis , Theta Functions