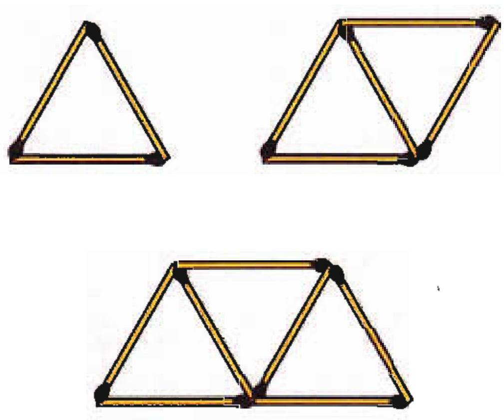
1 Arithmetic Sequences

Figure it out

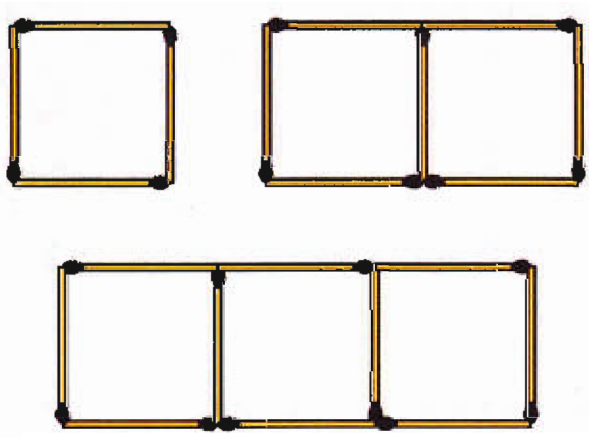
Look at these figures:



How many matchsticks are used in each?

How many matchsticks do we need to make the next figure in this pattern? (See the section **Matchstick math,** of the lesson **Letter Math** in the Class 6 textbook)

Now look at these:



How many sticks are used in each?

How many sticks do we need to make the next figure?

A game

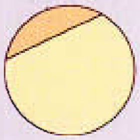
This is a game for two. One of the players begins by saying aloud a number less than or equal to ten. The second player adds to this, a number less than or equal to ten. Then the first player makes it still larger by adding a number less than or equal to ten. The first to reach hundred wins.

For example, if the first player says 6, the second can make it 16, at the most. If he actually makes it 16, the first player can take it upto 26.

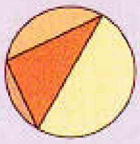
There is a scheme by which the first player can win. What are the numbers he should say to make sure that he wins? (Think backwards from 100)

Circle division

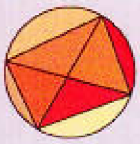
If we choose any two points on a circle and join them with a line, it divides the circle into two parts:



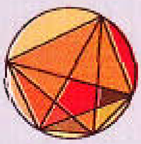
If we choose three points instead and join them with lines, they divide the circle into four parts:



What if choose four points and join every pair?

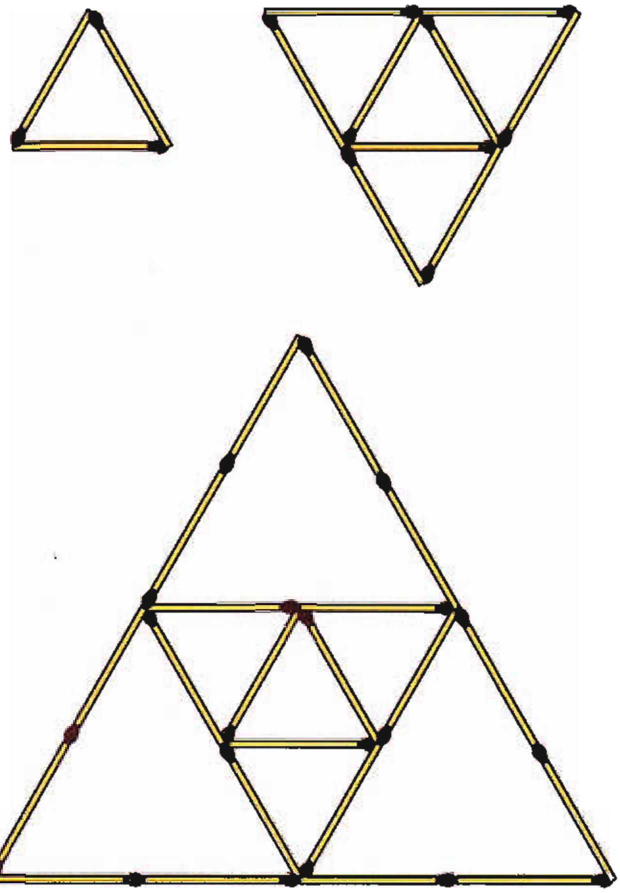


How about five points?



How many parts do you expect to get by joining six points? Check your guess by drawing a picture.

Can you find out the number of matchsticks in each figure of the pattern below and also how many we need to make the next one?



Numbers in order

Let's write down in order, the number of matchsticks used in each pattern above.

In the first pattern of triangles, we have

$$3, 5, 7, 9, \ldots$$

In the second pattern of squares,

$$4, 7, 10, 13, \ldots$$

What about the last pattern of triangles?

$$3, 9, 21, 45, \ldots$$

A collection of numbers as in these examples, written in order as the first, second, third and so on according to a definite rule, is called a *number sequence*.

Let's look at some more instances of such sequences:

- Suppose we deposit 1000 rupees in a bank and that we get simple interest at 6% annually. What would be the amount at the beginning of each year?

  At the beginning of the first year, it is 1000 rupees. At the beginning of the second year, it becomes 1060 rupees. At the beginning of the third year, it becomes 1120 rupees. And this goes on, right?

  That is, we get the sequence

  $$1000, 1060, 1120, 1180, \ldots$$

  What if the interest is compounded annually?

  We get the sequence

  $$1000, 1060, 1124, 1191, \ldots$$

  instead.

- We know that the speed of an object falling towards the earth increases every second by 9.8m/s. In other words, for an object dropped from a height, the speed is 9.8 m/s after the first second, 19.6 m/s after the second second, 29.4 m/s after the third second and so on. Thus the sequence of numbers we get here is

  $$9.8, 19.6, 29.4, \ldots$$

  What about the total distance this object travels after each second?

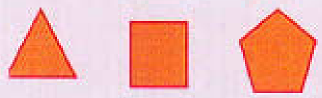
  It changes according to the equation, $s = 4.9t^2$. So, the total distance in metres that this object travels in one second, two seconds, three seconds and so on gives the sequence

  $$4.9, 19.6, 44.1, \ldots$$

Different kinds of sequences

The word sequence in ordinary language means things occurring one after another, in a definite order. In mathematics, we use this word to denote mathematical objects placed as the first, second, third and so on. The objects thus ordered need not always be numbers.

For example, we can have a sequence of polygons as given below:



Or a sequence of polynomials as

$$1 + x, 1 + x^2. 1 + x^3, \ldots$$

The arrangements of words of a language in alphabetical order is also a sequence.

Number sequences

Number sequences arise in any number of ways. For example, the digits in the decimal representation of π in order, gives the sequence

$$3, 1, 4, 1, 5, 9, \ldots$$

There is no simple way to find the number occuring at a specific position in this sequence.

In some sequences, the numbers repeat. For example, the digits in the decimal representation of $\frac{10}{11}$, written in order are

$$0, 9, 0, 9, \ldots$$

If we write out in order the last digits of the powers 2, 2^2, 2^3, ...we get the sequence

$$2, 4, 8, 6, 2, 4, 8, 6, \ldots$$

- A tank contains 1000 litres of water and it flows out at the rate of 5 litres per minute. So after one minute, the tank would contain 995 litres, after two minutes, 990 litres and so on. Here, we get the sequence

$$1000, 995, 990, 985, \ldots$$

Now write down the number sequences got in the instances described below:

- The perimeters of squares with each side of length 1 centimetre, 2 centimetres, 3 centimetres and so on. Also, the areas of these squares.
- The sum of the interior angles of polygons of sides 3, 4, 5 and so on. And the sum of their exterior angles.
- Multiples of 3

  Numbers which leave a remainder 1 on dividing by 3

  Numbers which leave a remainder 2 on dividing by 3
- Natural numbers which end in 1 or 6, written in order. Can you describe this sequence in any other way?

Add and go forth

We have seen so many sequences now. Let's have a look at them all together:

- 3, 5, 7, 9, ...
- 4, 7, 10, 13, ...
- 3, 9, 21, 45, ...
- 1000, 1060, 1120, 1180, ...
- 1000, 1060, 1124, 1191, ...
- 9.8, 19.6, 29.4, 39.2, ...
- 4.9, 19.6, 44.1, 78.4, ...
- 1000, 995, 990, 985, ...
- 4, 8, 12, 16, ...
- 1, 4, 9, 16, ...

- 180, 360, 540, 720, ...

- 360, 360, 360, 360, ...

- 3, 6, 9, 12, ...

- 1, 4, 7, 10, ...

- 2, 5, 8, 11, ...

- 1, 6, 11, 16, ...

The sequence 3, 5, 7, 9, . . . is got from the first matchstick problem. Here we need 3 sticks to make the first triangle and to make each new triangle, we need 2 more sticks.Thus we add 2 to 3, again add 2 to this sum and so on, to get the numbers in the sequence 3, 5, 7, 9, . . .

What about the second sequence? When we make squares with matchsticks, we need 4 sticks for the first square and for each new square, we need 3 more sticks. Thus we start with 4 and add 3 repeatedly to get the sequence 4, 7, 10, 13, . . .

Now look at the next sequence in our list. We need 3 sticks for the first triangle. To make a triangle on each of its sides, we need $3 \times 2 = 6$ more sticks; $3 + 6 = 9$ sticks in all. Next to make a triangle on each side of this bigger triangle, we need $3 \times 4 = 12$ more sticks; which means a total of $9 + 12 = 21$. Thus in this sequence, we start with 3 and continue by first adding 6, then 12, then 24 and so on.

A sequence which starts with any number and proceeds by the addition of one number again and again, is called an arithmetic sequence or an arithmetic progression.

So, the first two sequences in our list are arithmetic sequences; the third is not.

Now look at the sequence 1000, 995, 990, . . . Here, the numbers progressively decrease by 5.

Subtracting 5 can also be described as adding −5, right? So, we can say that the numbers in this sequence are got by adding −5 again and again. Thus it is also an arithmetic sequence.

What about the sequence 360, 360, 360, . . . , got as the sum of the exterior angles of polygons of increasing number of sides? Here we can say that each number is got by adding 0 repeatedly to 360. So, it is also an arithmetic sequence.

Natural number sequences

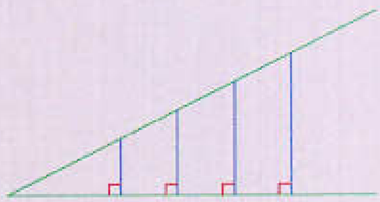
Some mathematicians have got together to collect sequences of natural numbers arising in various contexts and put them in the Net. The result is The Online Encyclopedia of Integer Sequences available at http://oeisf.org. It contains about 175,000 sequences.

At the web-page www.research.att.com/~njas/sequences/index.html, we can enter a few natural numbers and get several sequences containing these numbers in the same order, and short descriptions of how these sequences arise.

For example, entering 1, 2, 3, 4, 5, 7, we get 455 different sequences containing these numbers in this order. Some of them are:

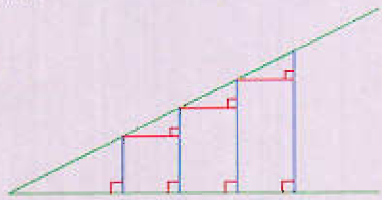
- 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, . . .
  Powers of primes in ascending order
- 1, 2, 3, 4, 5, 7, 8, 9, 10, 12,14, . . .
  Natural numbers which give a prime on multiplying by 6 and subtracting 1
- 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, . . .
  Numbers which do not have two consecutive natural numbers (other than 1) as factors

Parallel progression



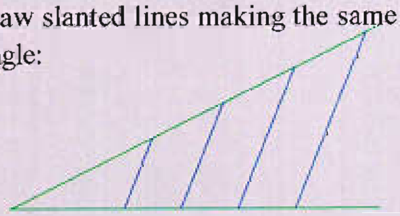
The vertical lines in the picture are equally spaced. Can you prove that their heights are in arithmetic sequence?

Draw perpendiculars as in the picture below:



The small right angled triangles got thus are all congruent. (Why?) So, their vertical sides are equal. That is, in the first picture, the heights of adjacent vertical lines increase by the same amount. This means the heights of these lines are in arithmetic sequence.

Suppose instead of vertical lines, we draw slanted lines making the same angle:



Are their lengths also in arithmetic sequence?

Now check which of the sequences in our list above are arithmetic sequences. Then have a look at these problems:

- The multiples of 2, written in order give 2, 4, 6, 8, . . . Is this an arithmetic sequence? What about the powers 2, 4, 8, 16, . . . of 2?
- Dividing the natural numbers by 2, we get $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Is this an arithmetic sequence?
- In the arithmetic sequence got by adding $\frac{1}{4}$ to itself repeatedly, does the number 10 occur anywhere? What about 11?
- The reciprocals of the natural numbers in order give $1, \frac{1}{2}, \frac{1}{3}, \ldots$ Is it an arithmetic sequence?
- Write down the sequence of differences (subtracting the larger from the smaller) of consecutive perfect squares. Is it an arithmetic sequence?

Addition and subtraction

Look at the two sequences got from the problems on interest. In both sequences, the increase in numbers is due to the interest added. In the case of simple interest, the increase every year is the interest for 1000 rupees (that is, 60 rupees). So, even without any computation, we can see that the total amounts form an arithmetic sequence.

What about compound interest? The interest added changes every year; so the total amounts are not in arithmetic progression.

What about the speed problem? What number should we add to 9.8 to get 19.6?

$$19.6 - 9.8 = 9.8$$

And what number do we need to add to 19.6 to make it 29.4?

$$29.4 - 19.6 = 9.8$$

That is, the sequence progresses by the addition of 9.8 itself at every stage. So speeds are in arithmetic sequence.

What about distances?

What number added to 4.9 gives 19.6?

$$19.6 - 4.9 = 14.7$$

And what number added to 19.6 gives 44.1?

$$44.1 - 19.6 = 24.5$$

So, the sequence of distances is got by starting with 4.9, adding 14.7 first, then adding 24.5, ...

So, it is not an arithmetic sequence. (Why?)

Do you notice something else in these examples?

In an arithmetic sequence, if we subtract any number from the number just after it, we get the same number

This number is called the *common difference* of the arithmetic sequence. In other words, common difference is the number repeatedly added to get the numbers in an arithmetic sequence.

As an example, look at the three sequences we got, based on division by 3:

3, 6, 9, 12, ...

1, 4, 7, 10, ...

2, 5, 8, 11, ...

What is the common difference of each sequence?

Let's look at some problems:

- The first number of an arithmetic sequence is 10 and the third number is 24. What is the second number?

  According to the facts given, by adding a number to 10 and then adding the same number again, we get 24 (How is that?)

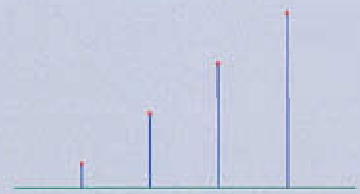
  So, double this number is $24 - 10 = 14$. This means the number added is 7.

  This gives the second number as $10 + 7 = 17$

  There's is another way to do this. We get the same number if we subtract the first number from the second, or if we subtract the second number from the third. (Look up the meaning of common difference.)

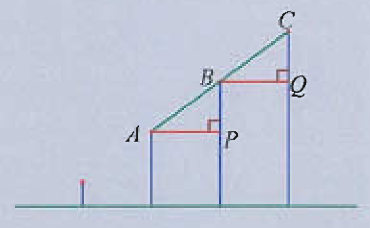
Geometry of arithmetic sequence

Take an arithmetic sequence of positive numbers. Draw a line and draw equally spaced perpendiculars to it, with heights equal to the numbers in this arithmetic sequence.



Join the tops of these perpendiculars. Aren't they all on the same line? Why is this so?

Take any three adjacent vertical lines and join their tops. Also draw perpendiculars as below:



The triangles ABP and BCQ are congruent. (Why?) And this means their angles are equal.

So, if we take $\angle ABP = x°$, then

$$\angle ABC = x + 90 + (90 - x) = 180°$$

This means A, B, C are all on the same straight line.

Back and forth

We have seen that the sum of three consecutive natural numbers is thrice the middle number. (The section, **Clarity in expression** of the lesson, **Polynomials** in the Class 9 textbook.)

This can be seen in another manner. The first number is 1 less than the middle number; and the third number is 1 more. So, when we add all three, the added 1 and the subtracted 1 cancel out, leaving only the middle number repeated thrice.

Translating this into algebra, if we take the middle number as x, then the left number is $x - 1$ and the right number is $x + 1$. And the sum is

$$(x-1) + x + (x+1) = 3x$$

Instead of consecutive natural numbers, suppose we take any three consecutive numbers from an arithmetic sequence. The decrease and increase from the middle number would be the common difference, instead of 1. And we get the same result.

How about translating this reasoning into algebra? Let's take the common difference as d. If we take the middle number as x, then the numbers are $x - d, x, x + d$ and so the sum is

$$(x-d) + x + (x+d) = 3x$$

Suppose instead of three, we take five numbers? Seven?

So, if we take the second number as x, then we get

$$x - 10 = 24 - x$$

From this, we get

$$2x = 24 + 10 = 34$$

and this gives

$$x = 17$$

- Prove that in any arithmetic sequence, if we take three consecutive numbers, then the middle number is half the sum of the first and the last.

  Let's take a, b, c as three consecutive numbers in an arithmetic sequence.

  Then, as in the second method of the previous problem, we get

  $$b - a = c - b$$

  From this, we get

  $$2b = a + c$$

  and then

  $$b = \frac{1}{2}(a + c)$$

Now try the problems below on your own:

- In each of the arithmetic sequences below, some numbers are missing. Their positions are marked with a ◯. Find these numbers.
  - 24, 42, ◯, ◯, ...
  - ◯, 24, 42, ◯, ...
  - ◯, ◯, 24, 42, ...
  - 24, ◯, 42, ◯, ...
  - ◯, 24, ◯, 42, ...
  - 24, ◯, ◯, 42, ...
- The second and fourth numbers in an arithmetic sequence are 8 and 2. Find the first and the third numbers.

- The second number of an arithmetic sequence is 5. Find the sum of its first and third numbers.
- Find an algebraic expression to compute the third number of an arithmetic sequence, using the first and the second numbers.

Position and term

In every sequence, the numbers are ordered as the first, second and so on. They are generally called *terms* of the sequence.

For example, in our first sequence got from triangles, the first term is 3, the second term is 5, the third term is 7 and so on.

What is its tenth term?

In other words, how many matchsticks do we need to make 10 triangles in this pattern? For this, we will have to join 9 more triangles with the first. For the first triangle we need 3 matchsticks and for every new triangle thereafter, we need 2 matchsticks. So,

$$\text{number of matchsticks for 10 triangles} = 3 + (9 \times 2) = 21$$

In other words, the 10^{th} term of the arithmetic sequence 3, 5, 7, . . . is 21.

Like this, can you compute the 15^{th} term of our second sequence 4, 7, 10, . . . got from squares?

Let's look at a few more problems:

- The first term of an arithmetic sequence is 2 and the common difference is 5. What is its 13^{th} term?

  Common difference is the number we add to each term of a sequence to go to the next term. Here it is 5; and the first term is 2. So, the terms of the sequence proceed as 2, 7, 12, ...

  How many steps are needed to go from the 1^{st} term to the 13^{th} term?

  That is, how many times do we need to add 5 to 2?

  So,

$$13^{\text{th}} \text{ term} = 2 + (12 \times 5) = 62$$

Sum and mean

We saw that in an arithmetic sequence, the sum of three consecutive numbers is three times the middle number, the sum of five consecutive terms is five times the middle number and so on.

What if we take an even number of terms? From one viewpoint, there is no middle term; from another viewpoint, there is a pair of middle terms. (In 1, 2, 3, 4, 5, 6 can't we say that the pair 3, 4 is right in the middle?)

So, in an arithmetic sequence of common dfference d, if we take four consecutive terms and name the middle pair x, y, then these four terms are $x - d, x, y, y + d$. And their sum is $2(x + y)$, which we can write $4 \times \frac{1}{2}(x + y)$. This we can say in ordinary language as four times the mean of the middle pair. (Don't you remember the lesson **Statistics,** in the Class 9 textbook?)

Is this true for six consecutive terms? What about eight?

Proportional differences

In an arithmetic sequence, the difference of two consecutive terms is the common difference. What if we skip a term? The difference is twice the common difference, right?What if we skip two terms?

In an arithmetic sequence, the difference between any two terms is got by multiplying the difference of their positions by the common difference.

This can be stated in another manner: in any arithmetic sequence, the term difference is proportional to the position difference; and the constant of proportionality is the common difference.

- The 12^{th} term of an arithmetic sequence is 25 and the common difference is 3. What is the 17^{th} term?

To get the 17^{th} term from the 12^{th} term, how many times should we add the common difference?

$$17^{th} \text{ term} = 25 + (5 \times 3) = 40$$

- In an arithmetic sequence, the 5^{th} term is 32 and the 11^{th} term is 74. What are the numbers in this sequence?

To get the 11^{th} term from the 5^{th} term, we must add the common difference 6 times.

From the given facts, the number added is $74 - 32 = 42$

So, common difference is $42 \div 6 = 7$

We get the 5^{th} term by adding the common difference 4 times to the 1^{st} term. So, in this sequence, the 5^{th} term 32 is got by adding $4 \times 7 = 28$ to the 1^{st} term. So,

$$\text{first term} = 32 - 28 = 4$$

Since the first term is 4 and the common difference is 7, the terms are

$$4, 11, 18, \ldots$$

We can do this problem using algebra also. Taking the first term as x and the common difference as y, the given facts translate to the two equations

$$x + 4y = 32$$

$$x + 10y = 74$$

(How?) Solving them, we get $x = 4, y = 7$. (Recall the lesson **Pairs of Equations,** in the Class 9 textbook.)

- In the arithmetic sequence $3, 7, 11, \ldots$, is 101 a term? What about 103?

The terms start with 3 and proceed by repeatedly adding 4; that is, by adding to 3, the multiples $4, 8, 12, \ldots$ of 4.

In other words, from any term of this sequence if we subtract 3, then we get a multiple of 4. And all such numbers are in this sequence.

Now let's look at 101 and 103

$$101 - 3 = 98$$

since 98 is not a multiple of 4, the number 101 is not a term of this sequence.

$$103 - 3 = 100$$

Since 100 is a multiple of 4, the number 103 is a term of this sequence.

Now do the problems given below on your own:

- The first term of an arithmetic sequence is 7 and the common difference is -2. What is its 12^{th} term?
- The 3^{rd} term of an arithmetic sequence is 10 and its 8^{th} term is 25. What is its 4^{th} term? And the 13^{th} term?
- The 5^{th} term of an arithmetic sequence is 11 and its 12^{th} term is 32. What are its first three terms?
- The 5^{th} term of an arithmetic sequence is 9 and its 9^{th} term is 5. What is its common difference? What is the 14^{th} term?
- The common difference of an arithmetic sequence is -1 and its 4^{th} term is 7. What is its 7^{th} term? And the 11^{th} term?
- How many three digit numbers leave a remainder 3 on division by 4?
- Write down the sequence of natural numbers which leave a remainder 3 on division by 6. What is the 10^{th} term of this sequence? How many terms of this sequence are between 100 and 400?

Sequences in Algebra

The terms of a sequence are formed by some rule. For example, in our first triangle problem, we start with 3 and repeatedly add 2 to get the sequence

$$3, 5, 7, \ldots$$

Sequence and remainders

The even numbers 2, 4, 6, . . . form an arithmetic sequence. So do the odd numbers 1, 3, 5, ... Both these have common difference 2.

Even numbers mean numbers divisible by 2; that is, numbers which leave a remainder 0 on division by 2. And odd numbers are those which leave a remainder 1 on division by 2.

Similarly, we have seen that there are three arithmetic sequences based on division by 3; those which leave a remainder 0, 1 or 2. What is the common difference of all these?

Based on the remainders got on division by 4, how many arithmetic sequences of natural numbers do we get? What are they? What is the common difference of all these?

Now let's think in reverse. In any arithmetic sequence of natural numbers, the difference of any two terms is a multiple of the common difference. This means they leave the same remainder on division by the common differnece. (Why?)

Thus, any arithmetic sequence of natural numbers is of the form described first; it consists of numbers leaving the same remainder on division by a specific number. And this divisor is the common difference.

In general, to get the term at any specified position of this sequence, we subtract 1 from the position number, multiply this by 2 and add to 3. For example,

$$15^{\text{th}} \text{ term} = \big((15-1)\times 2\big) + 3 = 31$$

How about writing the general rule in algebra?

For every natural number n,

$$n^{\text{th}} \text{ term} = \big((n-1)\times 2\big) + 3 = 2n + 1$$

So, in the algebraic expression $2n+1$, if we take $n = 1, 2, 3, \dots$, we get the terms $3, 5, 7, \dots$ of this sequence in the correct order. (See the section, **Algebraic expressions** of the lesson, **Polynomials** in the Class 9 textbook.)

In algebraic discussions, we denote the terms of a sequence by symbols such as $x_1, x_2, x_3, \dots$ or $y_1, y_2, y_3, \dots$ Thus we can write the sequence in the example above as

$$x_1 = 3$$
$$x_2 = 5$$
$$x_3 = 7$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$

We can shorten this further by writing

$$x_n = 2n + 1, \text{ for every natural number } n$$

(Note that this contains all the information about this sequence.)

Let's write the sequence of the square problem in this shortened form. In this, the first number is 4 and the number repeatedly added is 3. So, for evey natural number n,

$$n^{\text{th}} \text{ term} = \big((n-1)\times 3\big) + 4 = 3n + 1$$

Again, we can shorten this and compress the sequence into the algebraic expression

$$x_n = 3n + 1$$

What about the sequence in the second triangle problem?

Rules for sequences

What is the next term of the sequence $3, 5, 7, \dots$?

We haven't said that it should be an arithmetic sequence. So, the next number need not be 9. For example, if we had meant this to be the sequence of odd primes, then the next number would be 11.

What is the moral? From a few numbers written in order, we cannot predict the numbers to follow in the sequence. For this, the rule of forming the sequence or the context in which it arises should be specified.

Indeed, we have seen how the sequence 1, 2, 3, 4, 5, 7 can be continued in various ways, according to different rules, in the section **Natural number sequences.**

We can write the terms of this sequence as

$$x_1 = 3$$

$$x_2 = 9 = 3 \times (2^2 - 1)$$

$$x_3 = 21 = 3 \times (2^3 - 1)$$

$$x_4 = 45 = 3 \times (2^4 - 1)$$

and so on. In general, we can write the terms as

$$x_n = 3(2^n - 1)$$

(see the section **Growing triangle.**)

In the simple-interest problem, we got the arithmetic sequence 1000, 1060, 1120, ... The n^{th} term is

$$1000 + 60\,(n - 1) = 60n + 940$$

To get the sequence 1000, 1060, 1124, 1191, . . . of the compound interest problem, we have to first find the terms of the sequence given by

$$x_n = 1000\,(1.06)^{n-1}$$

and then round each number to the nearest natural number. (See the section **A computational trick,** of the lesson **Financial Math** in the Class 8 textbook.)

Now find the algebraic expression for all the sequences given at the beginning of the section, **Add and go forth**.

Algebra of arithmetic sequences

Look at the algebraic form of some of the arithmetic sequences we have seen:

3, 5, 7, 9, ...	$x_n = 2n + 1$
9.8, 19.6, 29.4, 39.2, ...	$x_n = 9.8n$
1000, 995, 990, 985, ...	$x_n = -5n + 1005$
4, 8, 12, 16, ...	$x_n = 4n$
360, 360, 360, 360, ...	$x_n = 360$
1, 4, 7, 10, ...	$x_n = 3n - 2$

In all these, the n^{th} term x_n is got by multiplying n by a specific number and then adding a specific number.

Growing triangle

We got the sequence 3, 9, 21, 45, . . . in our problem of making larger and larger triangles with matchsticks, How do we compute its terms?

In the original problem, we start with a triangle formed with one matchstick on each side; the construction progresses by constructing a larger triangle with two sticks on each side, then a still larger one with four sticks on each side and so on.

So, the total number of matchsticks needed at each stage is given by

$$3,\ 3 + (3 \times 2),\ 3 + (3 \times 2) + (3 \times 2^2), ...$$

That is,

$$3,\ 3(1 + 2),\ 3(1 + 2 + 2^2), ...$$

In this we can write

$$\begin{aligned} 1 + 2 &= 3 \\ &= 2^2 - 1 \\ 1 + 2 + 2^2 &= (2^2 - 1) + 2^2 \\ &= (2 \times 2^2) - 1 \\ &= 2^3 - 1 \\ 1 + 2 + 2^2 + 2^3 &= (2^3 - 1) + 2^3 \\ &= (2 \times 2^3) - 1 \\ &= 2^4 - 1 \end{aligned}$$

and so on. From this, we can compute the n^{th} term of our sequence as

$$3(1 + 2 + 2^2 + ... + 2^{n-1}) = 3(2^n - 1)$$

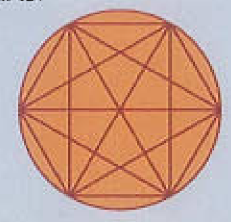
So, to make 25 triangles in this manner, we need $3(2^{25} - 1) =$ 100663293 matchsticks–more than ten crore!

Hasty conclusions

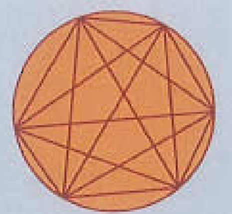
We discussed a problem on the number of parts of a circle got by joining points on it, in the section **Circle division**. For the number of points equal to 2, 3, 4, 5 we get the number of parts as 2, 4, 8, 16.

What about 6 points? We tend to guess the number of parts as 32. Do we get this many parts if we actually draw such a picture?

If the points are equally spaced, we get 30 parts:



Otherwise, 31 parts



Either way, the maximum number of parts is 31.

It can be proved that in general, if we take n points on the circle and join all pairs, the maximum number of parts got is

$$\frac{1}{24}n(n-1)(n-2)(n-3)+\frac{1}{2}n(n-1)+1$$

The interesting point is that in this expression and in 2^{n-1}, if we take $n=$ 1, 2, 3, 4, 5 we get the same numbers 1,2,4,8,16. From $n=6$ onwards, the numbers differ.

In other words, all these have the general algebraic form

$$x_n = an + b$$

where a and b are specified numbers.

Are all arithmetic sequences of this form? If we take the first term of an arithmetic sequence as f and the common difference as d, then its terms are

$$f,\ f+d,\ f+2d, \dots$$

In general, its n^{th} term would be

$$f + (n-1)\,d = dn + (f-d)$$

This means multiplying each n by the number d and adding the number $f-d$.

For example, in the arithmetic sequence with first term 2 and common difference 7, the n^{th} term is

$$2 + 7(n-1) = 7n-5$$

That is, $x_n = 7n-5$, On the other hand, it is not difficult to see that any sequence $x_n = an+b$ is an arithmetic sequence. Taking $n=1$, 2, 3, . . . in this, we get the terms of the sequence as

$$a+b,\ 2a+b,\ 3a+b, \dots$$

and we can easily see that it is indeed an arithmetic sequence with first term $a+b$ and common difference a.

Every arithmetic sequence is of the form $x_n = an+b$; conversely, every sequence of this form is an arithmetic sequence.

Now some problems based on these ideas:

- The first term and common difference of some arithmetic sequences are given below. Write each of them in the form $x_n = an+b$; also write down the first three terms of each:
    - first term -2, common difference 5
    - first term 2, common difference -5
    - first term 1, common difference $\frac{1}{2}$

- The algebraic form of some sequences are given below. Check whether each of them is an arithmetic sequence or not; also find out the first term and the common difference of the arithmetic sequences.
  - $x_n = 4 - 3n$
  - $x_n = n^2 + 2$
  - $x_n = \frac{n+1}{2}$
  - $x_n = \frac{n+2}{n}$
  - $x_n = (n + 1)^2 - (n - 1)^2$
- Multiply each odd number by 2 and add 1. If these numbers are written in order, does it form an arithmetic sequence? Write the algebraic form of this sequence. Now consider the odd numbers not in this sequence. Do they form an arthimetic sequence? What is its algebraic form?
- Consider the arithmetic sequence with first term $\frac{1}{2}$ and common difference $\frac{1}{4}$. Is 1 a term of this sequence? What about 2?

  Write the algebraic form of this sequence. Prove that all natural numbers occur in this sequence.
- Consider the arithmetic sequence with first term $\frac{1}{2}$ and common difference $\frac{1}{3}$. Is 1 a term of this sequence? What about 2?

  Write the algebraic form of this sequence. Prove that no natural number occurs in this sequence.
- In an arithmetic sequence, the ratio of the first term to the second is 2 : 3. What is the ratio of the third term to the fifth term?

Sums

We have seen a trick to compute the sum of consecutive natural numbers in the section **Triangular numbers,** of the lesson **Polynomials** in the Class 9 textbook. Let's take a closer look at it. For example, suppose we want to find the sum of the natural numbers from 1 to 10. (We can actually add the numbers to get the sum. But what about the sum from 1 to 100? The trick we are about to give can be used to find this also.)

Language of laws

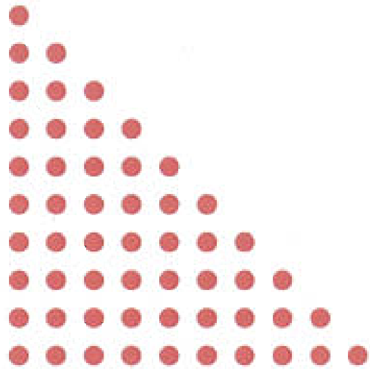
We have noted that to find the terms of a sequence, the law of formation should be specified. And we have seen some examples of how such rules can be written in algebra.

But not all sequences can be algebraically described. For example, no algebraic formula to find the n^{th} prime number has been discovered; that is, no formula to directly compute the number at a specified position in the sequence 2, 3, 5, 7, 11, 13, . . . of primes is known.

Again, there is no algebraic formula to compute the digit at a specified location in the sequence 3, 1, 4, 1, 5, 9, . . . got from the decimal representation of π.

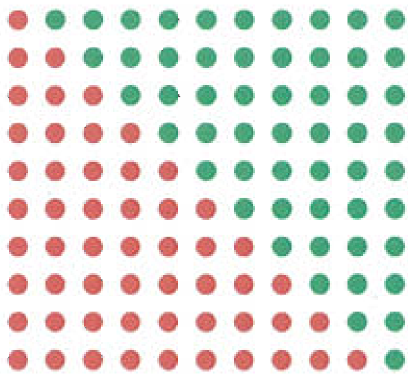
In such cases, we can only specify the rule of forming the terms in ordinary language.

This sum can be visualized as the number of dots in the picture below:



Let's make another copy of this triangle:

and join it upside down with the first



How many dots are there in all, red and green, in this rectangle?

There are 10 rows and each row has 11 dots; so $10 \times 11 = 110$.

This is twice the sum we want. So,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{1}{2} \times 10 \times 11 = 55$$

Metamorphosis

We saw that the algebraic form of an arithmetic sequence is

$$x_n = an + b$$

What does this mean?

If we multiply all natural numbers by a specific number and then add a specific number, we get an arithmetic sequence. For example, multiplying by $\frac{1}{2}$ and adding -1 gives the arithmetic sequence, $-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$ (Its algebraic form is $x_n = \frac{1}{2}n - 1$.)

Also, every arithmetic sequence arises this way. For example, the arithmetic sequence 7, 16, 25, . . . has the algebraic form $x_n = 9n - 2$ and this means multiplying all natural numbers by 9 and adding -2.

This process can be done using numbers instead of pictures: if we write

$$s = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

then

$$\begin{aligned} 2s &= (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) + \\ &\quad (10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) \\ &= 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 \\ &= 10 \times 11 \\ &= 110 \end{aligned}$$

and so

$$s = \frac{1}{2} \times 110 = 55$$

We can use the same argument, whether we add up to 10 or 100. In general

The sum of all natural numbers from one to a specific natural number is half the product of the number itself with the next natural number.

In algebraic language,

$$1 + 2 + 3 + \dots + n = \frac{1}{2} n(n + 1)$$

Using this, we can find the sum of other arithmetic sequences also. See these examples:

- How do we find the sum of the even numbers $2, 4, 6, \dots, 100$?

  Even numbers are got by multiplying natural numbers by 2. So,

  $$2 + 4 + 6 + \dots + 100 = 2(1 + 2 + 3 + \dots + 50)$$

  As we saw just now,

  $$1 + 2 + 3 + \dots + 50 = \frac{1}{2} \times 50 \times 51$$

  So, we have

  $$2 + 4 + 6 + \dots + 100 = 2 \times \frac{1}{2} \times 50 \times 51 = 2550$$

A math tale

We have mentioned the mathematician Gauss in the Class 9 textbook. It is said that he showed extraordinary mathematical ability right from an early age.

This happened when Gauss was just ten and in school. His teacher asked the children to add all numbers from 1 to 100, just to keep them quiet. Young Gauss did it in a flash, explaining his anwer thus: "1 and 100 make 101, so do 2 and 99; there are 50 such pairs and so the answer is $50 \times 101 = 5050$."

A sequence from the past

We have mentioned the ancient mathematical manuscript called the Ahmose Papyrus in the section **Ancient math,** of the lesson **Equations** in the Class 8 textbook. Problem 64 in it goes something like this:

10 hekats of barley is to be distributed among 10 people in a definite order. Each should get $\frac{1}{8}$ hekat more than the previous one. How much should be given to each?

Here *hekat* is a measure used in those days. The answer to this is given in the papyrus as follows:

1. Divide 10 by 10. We get 1
2. Subtract 1 from 10 and multiply it by half of $\frac{1}{8}$. We get $\frac{9}{16}$
3. Add this to the 1 got in the first step. The number $1\frac{9}{16}$ got now is the largest share.
4. Subtract $\frac{1}{8}$ from this repeatedly to get the other shares.

What is the logic behind this computation?

How do we do it using current techniques?

- We want to find the sum of a specified number of odd numbers 1, 3, 5, ···

How do we write this sequence of odd numbers in algebra?

The n^{th} term of this sequence is

$$1 + (n - 1) \times 2 = 2n - 1$$

and we can write it as

$$x_n = 2n - 1$$

So, the sum of the first n odd numbers

$$= x_1 + x_2 + x_3 + \dots + x_n$$

$$= (2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) \dots + (2n - 1)$$

$$= 2(1 + 2 + 3 + \dots + n) - \overbrace{(1 + 1 + 1 + \dots + 1)}^{n \text{ times}}$$

$$= \left(2 \times \frac{1}{2}n(n + 1)\right) - n$$

$$= n(n + 1) - n$$

$$= n^2$$

Thus the sum of a specified number of consecutive odd numbers starting from 1 is equal to the square of the number of odd numbers added. (This was noted in the lesson **Square Numbers** in the Class 7 textbook.)

We can find the sum of a specified number of consecutive terms of any arithmetic sequence in the same way as we found the sum of odd numbers.

We can write the general term of any arithmetic sequence as

$$x_n = an + b$$

So,

$$x_1 + x_2 + x_3 + \dots + x_n$$

$$= (a \times 1 + b) + (a \times 2 + b) + (a \times 3 + b) + \dots + (an + b)$$

$$= a(1 + 2 + 3 + \dots + n) + \overbrace{(b + b + b + \dots + b)}^{n \text{ times}}$$

$$= \left(a \times \frac{1}{2}n(n + 1)\right) + (n \times b)$$

$$= \frac{1}{2}an(n + 1) + bn$$

For convenience, we rewrite this in a slightly different way:

$$\begin{aligned}\frac{1}{2}an(n+1)+bn &= \frac{1}{2}n\big(a(n+1)+2b\big)\\ &= \frac{1}{2}n\big((an+b)+(a+b)\big)\\ &= \frac{1}{2}n(x_n+x_1)\end{aligned}$$

What is the meaning of this?

The sum of a specified number of consecutive terms of an arithmetic sequence is half the product of the number of terms with the sum of the first and the last terms.

For example, suppose we want to find the sum of the first 50 terms of the sequence 3, 5, 7, ... We first find the 50^{th} term as

$$3+(49\times 2)=101$$

By what we have seen above, we can compute the sum as

$$\frac{1}{2}\times 50\times(3+101)=2600$$

Now try these problems on your own:

- The first term of an arithmetic sequence is 5 and the common difference is 2. Find the sum of its first 25 terms.
- Find an algebraic expression to compute the sum of the first n terms of the arithmetic sequence with first term f and common difference d.
- If $5^2\times 5^4\times 5^6\times\ldots\times 5^{2n}=(0.04)^{-28}$ what is n?
- Find the sum of all three digit numbers which are multiples of 9.
- Prove that the sum of the first five terms of any arithmetic sequence is five times the middle number. What about the sum of the first seven terms? Can you formulate a general principle from these examples?

Another way

There is another technique for finding the sum of consecutive natural numbers.

We know that for any number x,

$$(x+1)^2-x^2=2x+1$$

Taking $x = 1, 2, 3, \ldots, n$ in this, we get

$$\begin{aligned}2^2-1^2&=(2\times 1)+1\\ 3^2-2^2&=(2\times 2)+1\\ 4^2-3^2&=(2\times 3)+1\\ &\cdots\cdots\\ (n+1)^2-n^2&=(2\times n)+1\end{aligned}$$

What if we add all these equations? We get

$$(n+1)^2-1=2(1+2+3+\ldots+n)+n$$

From this, we find

$$\begin{aligned}1+2+3+\ldots+n &= \frac{1}{2}((n+1)^2-1-n)\\ &= \frac{1}{2}(n^2+n)\\ &= \frac{1}{2}n(n+1)\end{aligned}$$

- The sum of the first n terms of an arithmetic sequence is $2n^2 + 3n$. Find the algebraic form of this sequence.

Do the problems below in your head:

- Consider the arithmetic sequences 3, 5, 7, . . . and 4, 6, 8, . . . How much more is the sum of the first 25 terms of the second than the sum of the first 25 terms of the first?
- How much more is the sum of the natural numbers from 21 to 40 than the sum of the natural numbers from 1 to 20?
- $51 + 52 + 53 + \dots + 70$
- $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots + \frac{25}{2}$
- $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + \dots + 12\frac{1}{2}$

Project

- Prove that if in an arithmetic sequence of natural numbers, one of the terms is a perfect square, then there are many terms which are perfect squares. Is there an arithmetic sequence of natural numbers in which no term is a perfect square?

Sum of squares

We can use an algebraic identity to compute the sum of the squares of natural numbers, just as we found the sum of natural numbers. We have seen the identitiy

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

(See the section **Patterns of power**, of the lesson **Polynomials** in the Class 9 textbook.)

From this we can see that for any number x,

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1$$

As before, if we take $x = 1, 2, 3, \dots, n$ in this and add, we get

$$(n+1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + \dots + n^2) + 3(1 + 2 + 3 + \dots + n) + n$$

That is,

$$n^3 + 3n^2 + 3n = 3(1^2 + 2^2 + 3^2 + \dots + n^2) + \frac{3}{2}n(n+1) + n$$

So, we get

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{3}\left(n^3 + 3n^2 + 3n - \frac{3}{2}n(n+1) - n\right)$$

Simplifying the right side of this equation, we get

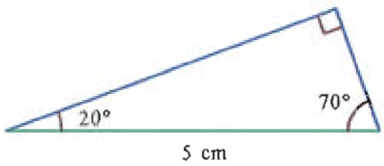
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

2 Circles

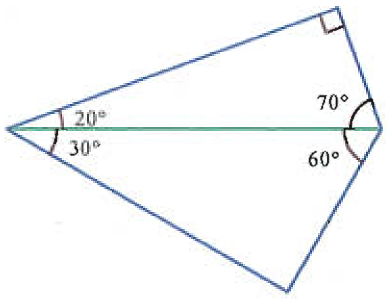
Angle drawings

We want to draw a right angled triangle of hypotenuse 5 centimetres; the other two sides of any length as we please. In what all ways can we do this?

We can start with a line 5 centimetres long. Draw any angle we please at one end. Subtract this angle from 90 and draw the angle thus got at the other end, to make a triangle. For example:



We can also draw something like this below the line:

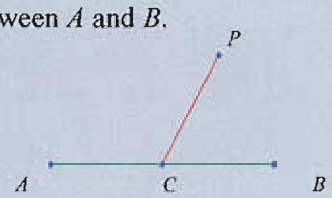


Another way is to use a set square from the geometry box. Place it with its right angle above (or below) the line and the perpendicular edges through the end-points of the line. Try!

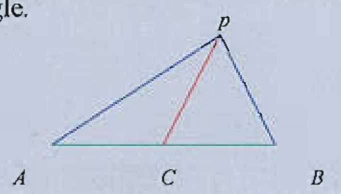
Draw many such triangles and look at their third vertices:

Right angles from circles

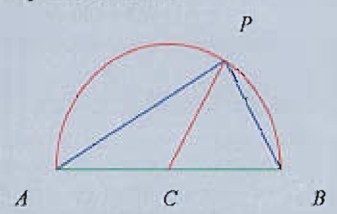
There is yet another way to draw a right-angled triangle with a specified hypotenuse AB. Mark the mid-point C of AB and mark a point P at a distance equal to half the distance between A and B.



We can show that APB is a right angle.

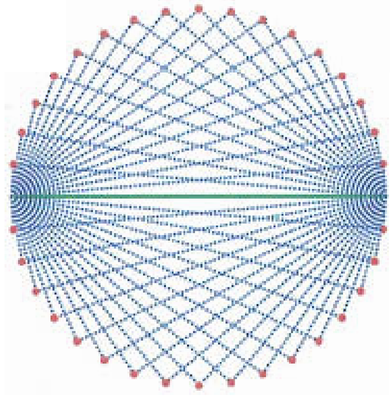


Since $CA = CB = CP$, the circle of radius this length, centred at C, passes through A, B and P

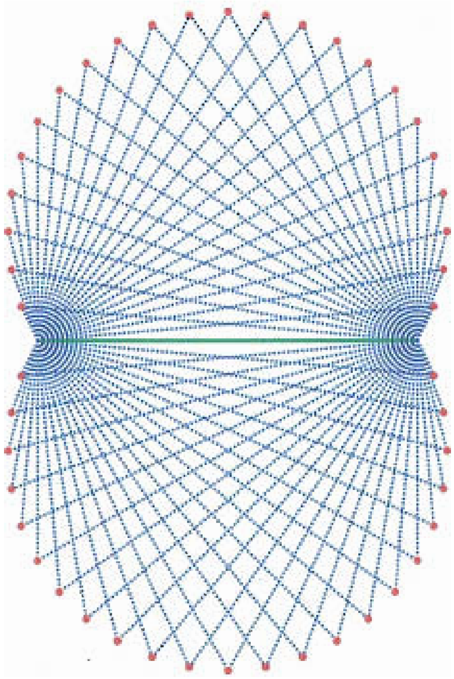


So, $\angle APB$ must be equal to 90°.

(Recall the section **Angle in a semicircle**, of the lesson **Congruent Triangles** in the Class 8 textbook.)

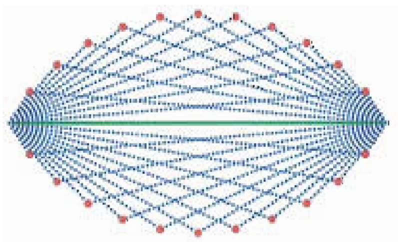


Now instead of a right angle, let's draw triangles with one angle 60° (We can use a corner of a set square.)



How about a 45° angle or a 30° angle? These can also be done with set squares.

Now cut out a triangle with a 120° angle from thick paper. Use it to draw pictures like the above with the top angles 120°:



Why do we get pictures like these? Let's investigate.

Circles from right angles

We saw that if we draw a circle with AB as diameter, then taking any point P on this circle, we could draw a right angled triangle APB.

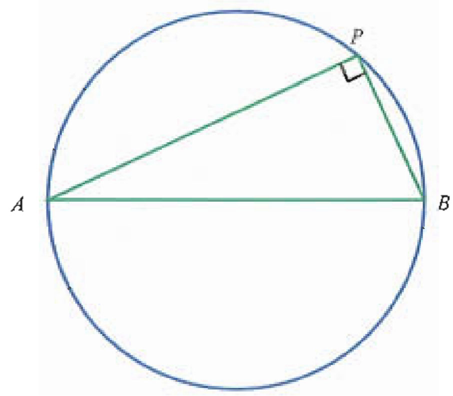
On the other hand, if we draw a right angled triangle with AB as hypotenuse, then the diameter of the circumcircle of ΔAPB would be AB. (There is such a problem in the section **Another division,** of the lesson **Geometric Proportions** in the Class 9 textbook)

So, if we draw all right angled triangles with AB as hypotenuse and take only their third vertices, we get all points on the circle with AB as diameter, except A and B.

Right angles and circles

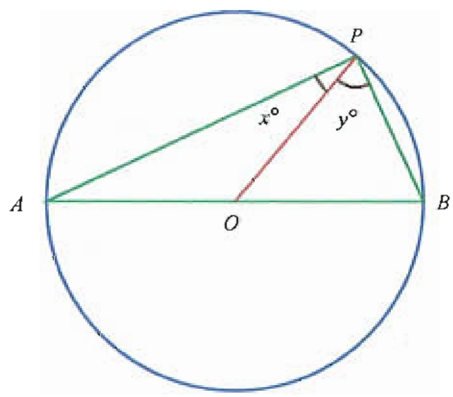
In the picture drawn with a right angle, we got a single circle and the line we started with turned out to be its diameter; that is, semicircles above and below this line.

Haven't we seen a picture like this before?

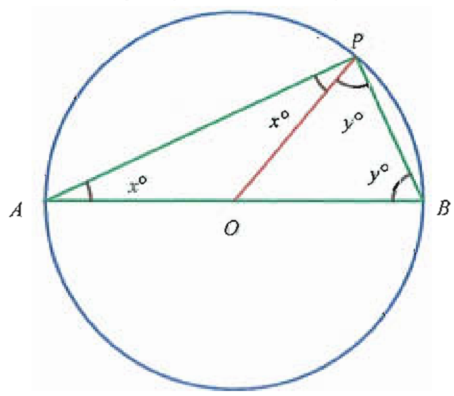


(Recall the section **Angle in a semicircle,** of the lesson **Congruent Triangles** in the Class 9 textbook.)

AB is a diameter of the circle. How did we prove that $\angle P$ is a right angle?



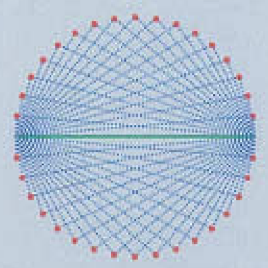
Here O is the centre of the circle. So, OAP and OBP are isosceles triangles. (Reason?) If we take $\angle APO = x°$ and $\angle BPO = y°$, then we get $\angle A = x°$ and $\angle B = y°$. (How is that?)



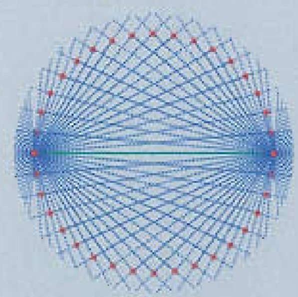
Locus

Locii of points can often be described in terms of lengths or angles. For example, we can think of the perpendicular bisector of the line joining two points as the locus of a point which moves, keeping the same distance from these points. We can also think of it as the locus of a point which moves such that the lines joining it with the end points make equal angles with the line.

What is the locus of the third vertex of a right angled triangle with a specified line as the hypotenuse? We saw that it is not the full circle with this line as diameter—the endpoints of the line are not on this locus.



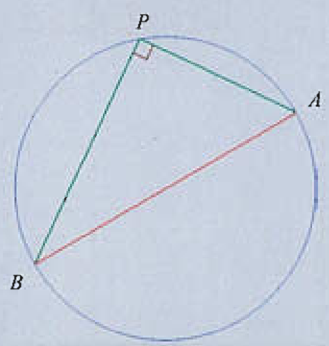
Instead, if we say the locus of the point of intersection of perpendicular lines through the end points of this line, then we get the entire circle.



Right angle and diameter

We saw that when two ends of a diameter of a circle are joined to another point on the circle, the angle formed at this point is a right angle.

On the other hand, suppose we draw a pair of perpendicular lines from a point on a circle and join the points of intersection of these lines with the circle. Is this line a diameter of the circle?



Here, the circle is the circumcircle of the right angled triangle APB. And we know that the hypotenuse of a right angled triangle is a diameter of its circumcircle. So, AB is indeed a diameter of the circle.

The section **Circle and square,** of the lesson **Between the lines** in the Class 7 textbook gives a trick to find the centre of a circle. Now do you see why this trick works?

Since the sum of the angles of ΔABP is 180°, we have

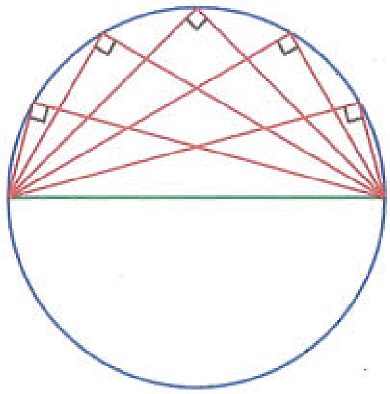
$$x + y + (x + y) = 180°$$

From this we get $2x + 2y = 180°$ and then

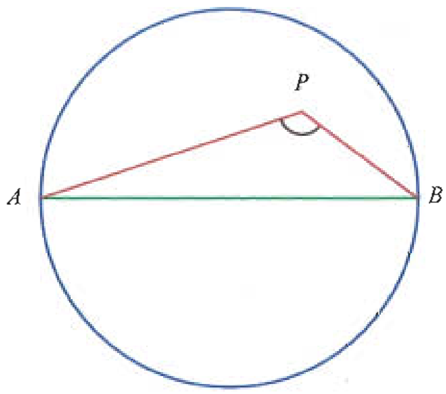
$$x + y = 90°$$

What do we see from this?

If the ends of a diameter of a circle are joined to any other point on the circle, then what we get is a right angle.

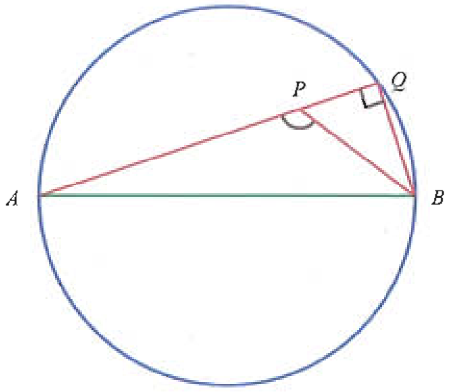


This leads to another thought. We get a right angle, when the ends of a diameter are joined to a point *on* the circle; what if we join them to a point *inside* the circle?



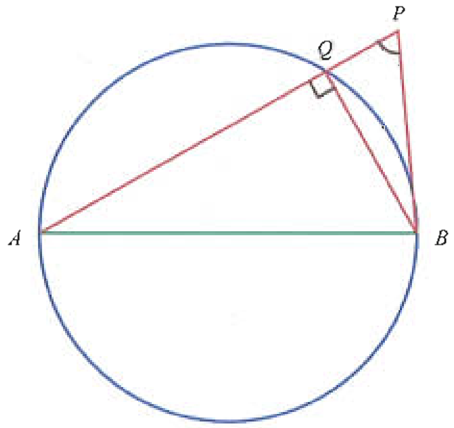
Do we get an angle larger than a right angle at every point within the circle?

In this picture, extend one line to meet the circle; join this point to the other end of the diameter.



Now $\angle APB$ is the exterior angle at the vertex P of ΔPQB. So, it is the sum of the (interior) angles of the triangle at Q and B. (This problem is in the section **Unchanging sums,** of the lesson **Polygons** in the Class 9 textbook.) Of these, the angle at Q is a right angle. This means $\angle APB$ is larger than a right angle, right?

Next, what about a point outside the circle?



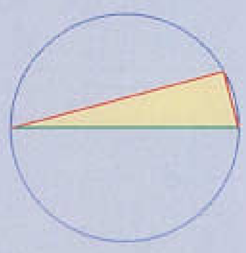
In this set up, $\angle APB$ is an internal angle of ΔPQB and $\angle AQB$ is an exterior angle. So, $\angle APB$ is smaller than a right angle.

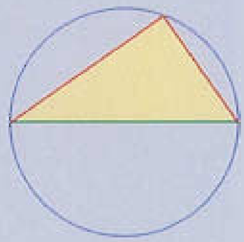
Now suppose that the ends of a diameter of a circle are joined to some point and we get a right angle at that point. This point cannot be *inside* the circle. (At points within the circle we get an angle larger than a right angle.) And it cannot be *outside* the circle. (For points outside the circle, we get an angle smaller than a right angle.)

So, the point must be *on* the circle. Don't you see now why we got a circle in our first picture drawn with a right angle? You can try some problems based on these ideas.

Square speciality

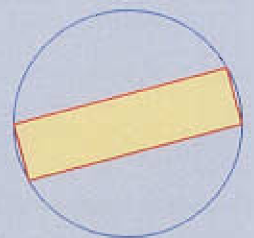
We can make a right angled triangle in a circle by joining various points of a circle to the ends of a diameter.

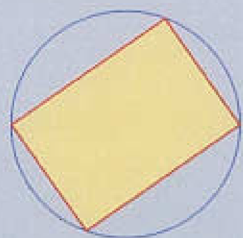




To get the maximum area, where should we choose the point?

This leads to another question. We can draw any number of rectangles with all four vertices on a circle:

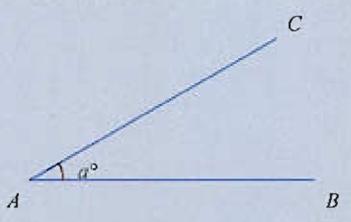




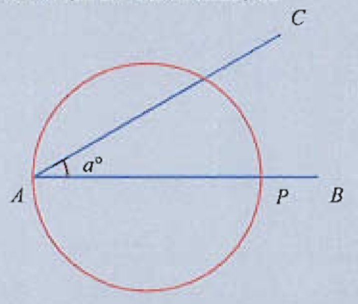
What is the speciality of the rectangle of maximum area?

Doubling an angle

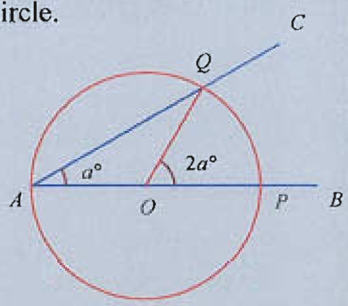
We know how we can halve an angle by drawing the bisector. How do we double an angle?



Mark a point P on AB and draw a circle with AP as diameter.



Join the point Q where this circle meets AC, and the centre O of the circle.



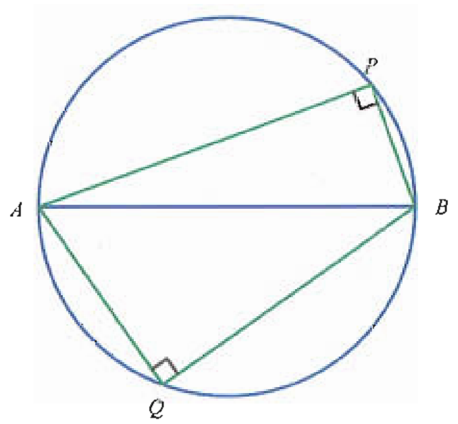
Since ΔOPQ is isosceles, $\angle OQA = a°$ and so the exterior angle POQ is equal to $2a°$.

- In ΔABC, we have $\angle A = 60°$ and $\angle B = 70°$. Is the vertex C inside or outside the circle with diameter AB?
- Prove that if a pair of opposite angles of a quadrilateral are right, then a circle can be drawn through all four of its vertices.
- In the quadrilateral $ABCD$, we have $AB = 3$cm, $BC = 4$ cm, $AC = 5$ cm, $\angle A = 120°$, $\angle C = 70°$. If we draw the circle with AC as diameter, which of the four vertices of $ABCD$ would be inside the circle? Which of them would be outside this circle? Is any vertex on the circle? What about the circle with BD as diameter?

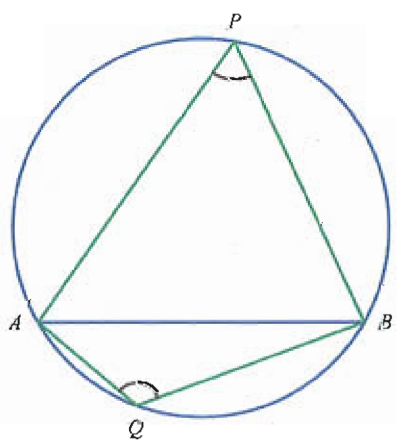
Angles, arcs and chords

We now know why we got a circle in the picture drawn with a right angle. What about the other pictures?

Again let's start with a circle. Any diameter of the circle divides it into two equal arcs; and we get a pair of right angles by joining points on each to the ends of the diameter.

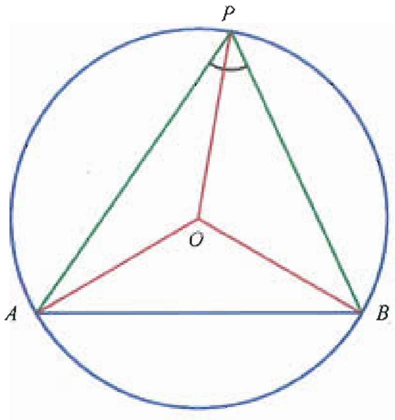


What about a chord which is not a diameter?

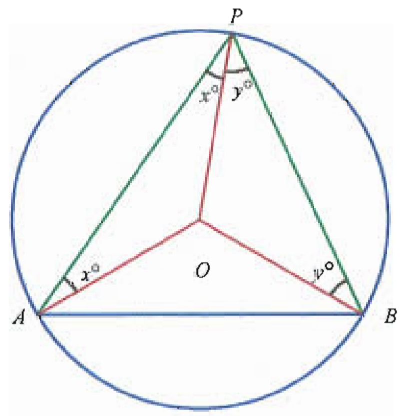


The arcs are not equal; and the angles are not right.

Let's inspect the arc and angle at the top and bottom separately. First the upper ones. As we did in the case of a diameter, we join P to the centre O of the circle. Since the centre is not on the chord here, we also join OA and OB.

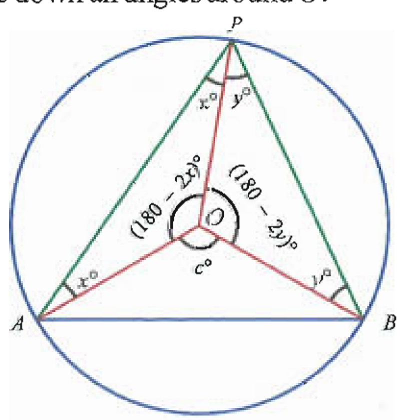


As in the case of diameter, here also, OAP and OBP are isosceles triangles.



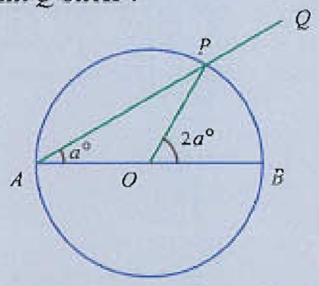
But here these two triangles do not form a single large triangle, as seen earlier. So, the old trick of summing up the angles of a triangle won't work.

Instead, let's write down all angles around O.



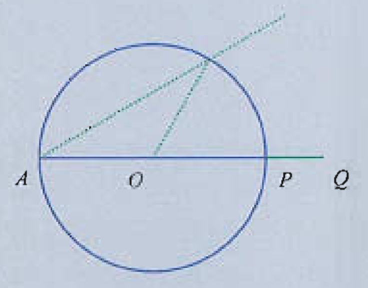
Angle of rotation

In the figure below, AB is a diameter of the circle and O is the centre. A point P on the circle is marked and a point Q on AP.

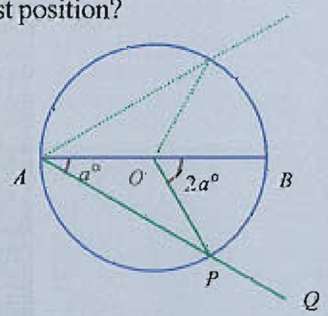


Let $\angle BAP = a°$, so that $\angle BOP = 2a°$.

Now suppose that P moves along the circle and reaches B.

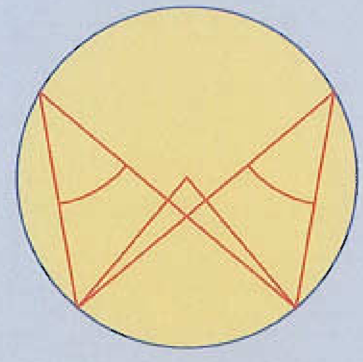


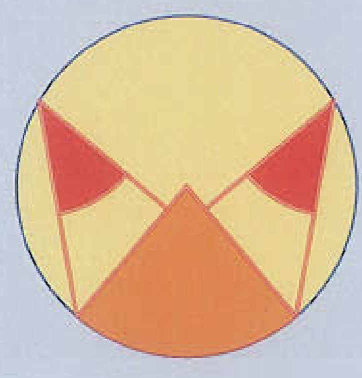
The line OP has rotated through $2a°$. And the line AP through $a°$. What happens as P moves again and reaches the point directly below the first position?



Cut and paste

Draw a figure and cut out pieces as shown below:





Now place them as shown below:

Thus we get

$$(180 - 2x) + (180 - 2y) + c = 360$$

(Recall the section **Around a point,** of the lesson **Polygons** in the Class 9 textbook.) That is,

$$360 - 2(x + y) + c = 360$$

This gives

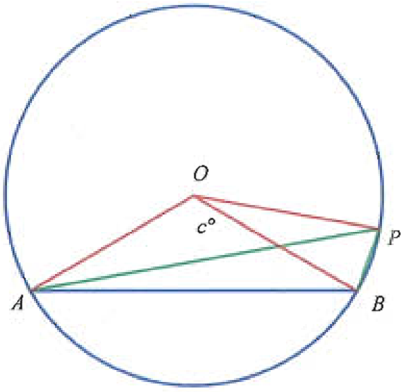
$$x + y = \frac{1}{2}c$$

That is,

$$\angle APB = \frac{1}{2}c°$$

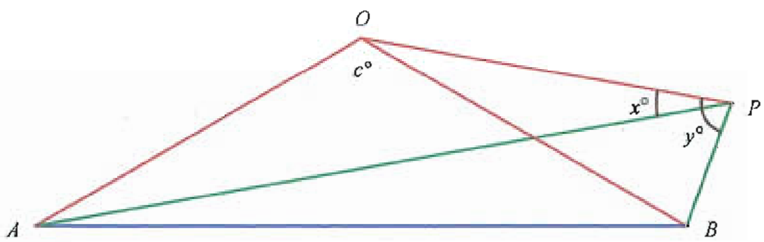
Would this be true, for any position of P on the upper arc?

What if it is like this?

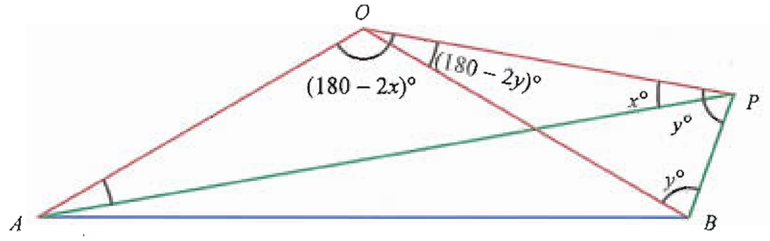


Let's try writing $\angle OPA = x°$ and $\angle OPB = y°$ as before.

To see things clearly, let's enlarge the triangles in the picture:



Now as before, we can find other angles, using the fact that ΔOAP and ΔOBP are isosceles.



From the figure we see that

$$\angle APB = (y - x)^\circ$$

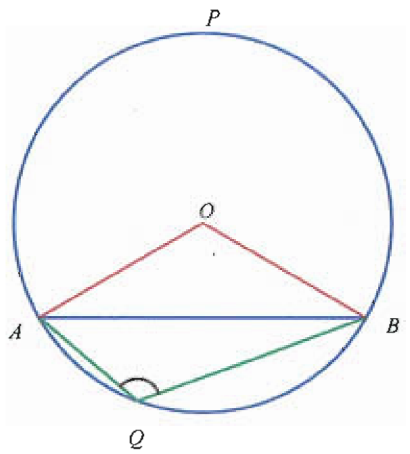
and that

$$\angle AOB = (180 - 2x) - (180 - 2y) = 2(y - x)^\circ$$

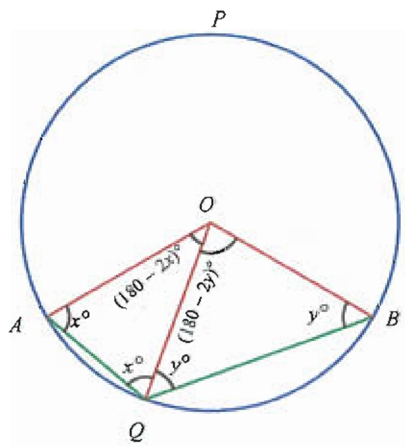
So, again we get

$$\angle APB = \frac{1}{2}c^\circ$$

What about angles below AB?

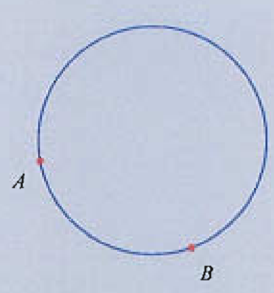


In this case also, by joining OQ, we get two isosceles triangles; and we can compute angles as we did earlier.



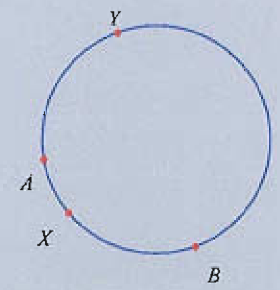
Arc pair

Two points on a circle actually divide it into two arcs:



In this picture, a small arc is got by travelling right from A to B along the circle and a large arc, by travelling left from A to B along the circle.

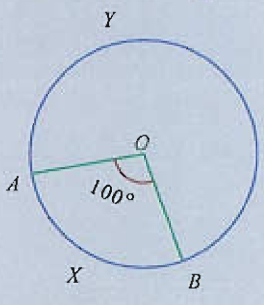
Choosing a point each on these arcs, we can name the arcs using the names of these points with the end points.



In this picture, the smaller arc is AXB and the larger arc is AYB.

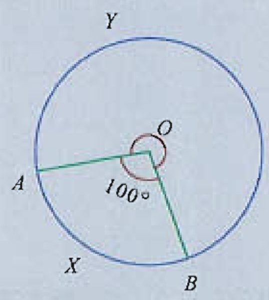
So, for any arc of a circle, there is a matching arc which completes it as a circle; and only one such arc. In other words, any arc can be completed as a circle in only one way.

Central angle



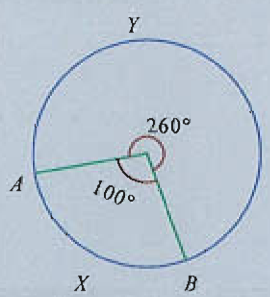
In the picture above, the central angle of the arc AXB is $100°$.

What about the central angle of the arc AYB?



By definition of the degree measure of an angle, the part $OAXB$ of the circle is made up of 100 out of 360 equal parts into which the circle is divided. So, how many parts make up the remaining part $OAYB$?

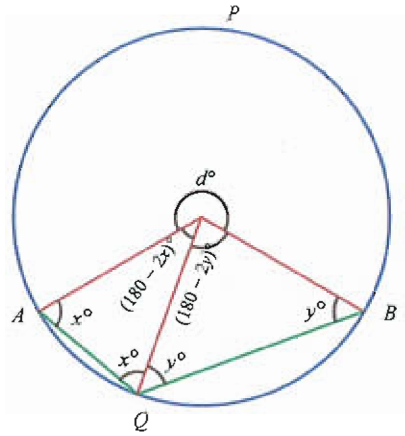
So, the central angle of the arc AYB is $260°$.



So, if we take the central angle of $\angle APB$ as d, then we can see that

$$(180 - 2x) + (180 - 2y) + d = 360$$

from the picture below:



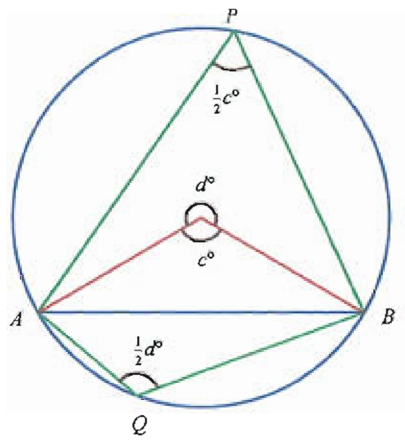
This equation gives

$$2(x + y) = d$$

That is,

$$\angle AQB = \frac{1}{2}d°$$

Let's sum up all we have seen so far. Look at this picture:

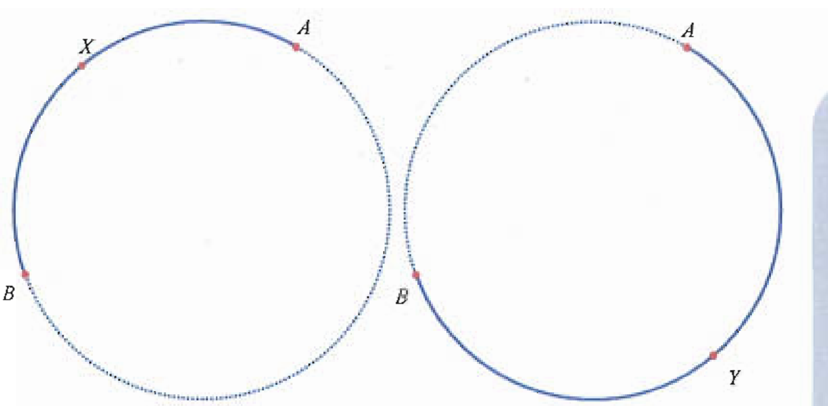


Wherever we take the point P, on the circle above AB, we get $\angle APB = \frac{1}{2}c°$.

Wherever we take the point Q, on the circle and below AB, we get $\angle AQB = \frac{1}{2}d°$.

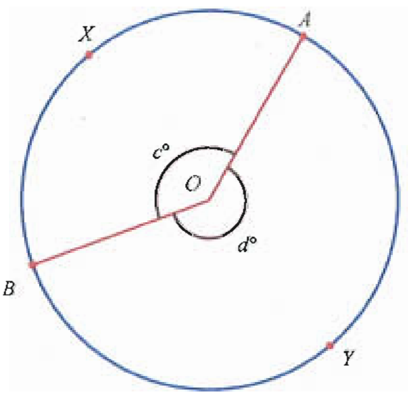
We can state this without referring to the chord AB.

Any two points on a circle divide it into a pair of arcs, right?

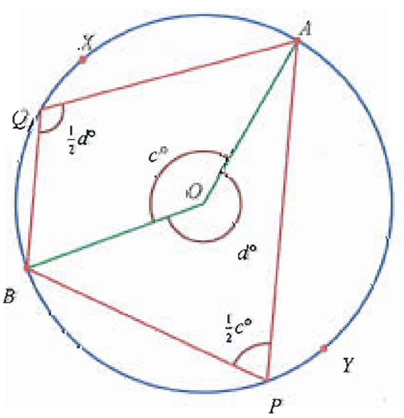


In this picture, the points A and B divide the circle into a pair of arcs AXB and AYB. The arc AYB is called the *alternate arc* (or the *complementary arc*) of AXB (and vice versa.)

Let's join A and B to the centre O of the circle.

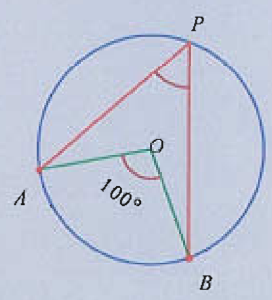


In this figure, $c°$ is the central angle of the arc AXB and $d°$ is the central angle of the arc AYB. Now let's take some point P on the arc AYB and some point Q on the arc AXB.



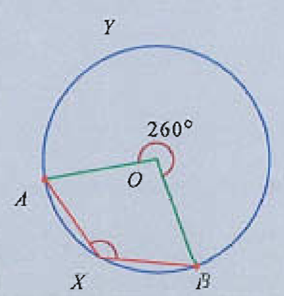
Then we can combine the two statements made earlier into one:

Changing angle



In the figure, $\angle APB = 50°$. Also, this angle would be the same 50°, wherever we take P on the larger of the two arcs made by A and B.

Now suppose this point travels left along the circle. The angle doesn't change till it reaches A. At A, there is no angle to speak of. As it moves further left and moves into the smaller arc, the angle changes. What does it become?



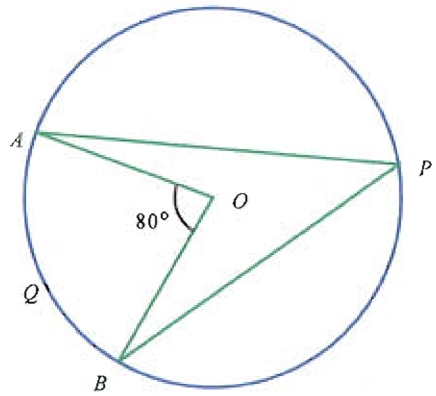
And it remains 130° till it reaches B.

Two points on a circle divide it into a pair of arcs. The angle got by joining these two points to a point on one of these arcs is equal to half the central angle of the alternate arc.

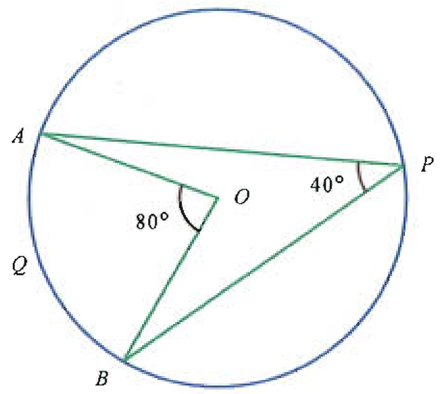
Here, instead of "central angle of an arc", we can say "angle made by an arc at the centre"; and instead of "angle got by joining ends of an arc to a point", we can say, "angle made by an arc at a point". Then the statement above can be rephrased like this:

The angle made by an arc at any point on the alternate arc is equal to half the angle made at the centre.

For example, look at this picture:



The central angle of the arc AQB is 80°. So, $\angle APB$ made at the point P on the alternate arc is half of this, which is 40°.

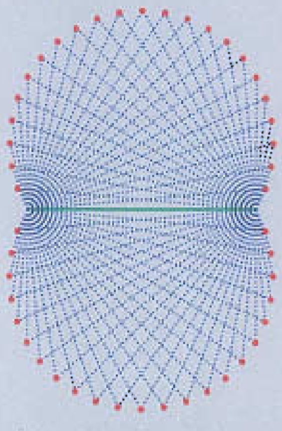


From the first picture, we can also see that the central angle of the arc APB is $360° - 80° = 280°$.

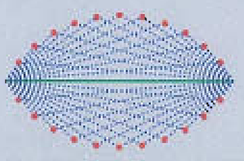
Circle trick

We started by drawing some pictures with the same angles above and below a line.

This was the picture got by taking $60°$ above and below:

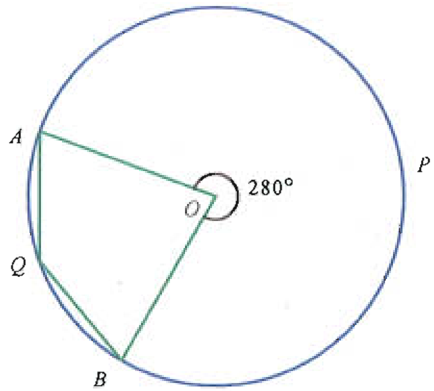


And taking $120°$ instead, we got this picture:



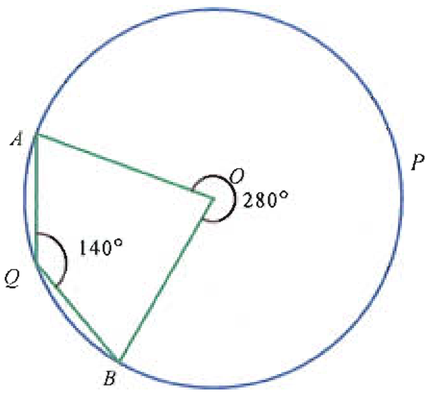
Now draw a picture with 60° above and 120° below. Don't you get a single, full circle? Why is this?

Suppose we take 30° above. What angle should we take below to get a full circle?



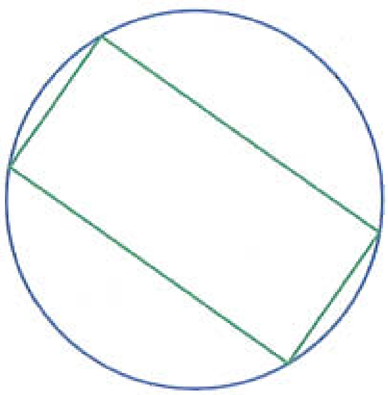
So, $\angle AQB$ made at the point Q on its alternate arc is

$\frac{1}{2} \times 280° = 140°$



Let's look at a few more examples:

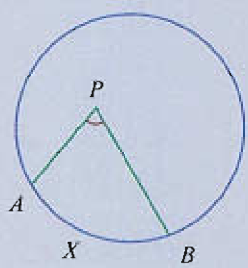
- In the figure below, all four vertices of the rectangle are on the circle. Prove that the diagonal of the rectangle is a diameter of the circle.



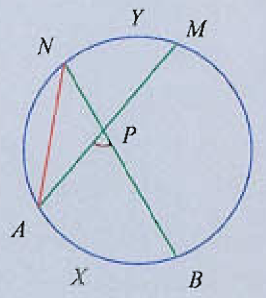
Inside the circle

What can we say about the angle got by joining the ends of an arc to a point inside the circle?

Look at this figure:



To find $\angle APB$, extend the lines AP and BP to meet the circle; join one of these points to one end of the arc:



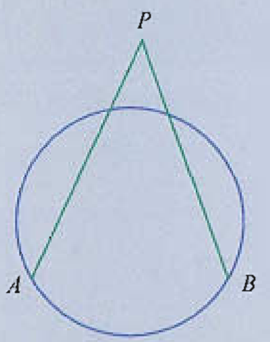
If we take the central angles of the arcs AXB and MYN as $x°$ and $y°$, then we get $\angle ANB = \frac{1}{2}x°$ and $\angle MAN = \frac{1}{2}y°$. These are angles of the triangle PAN; and $\angle APB$ is the exterior angle at the third vertex P. So,

$$\angle APB = \frac{1}{2}(x + y)°$$

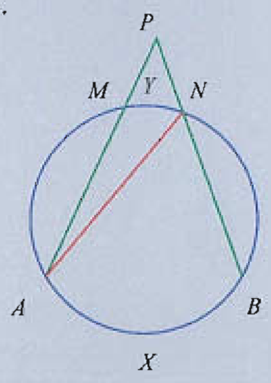
Thus $\angle APB$ is the mean of the central angles of the arcs AXB and MYN.

Outside a circle

What if we join the ends of an arc to a point outside the circle?



Join one end of the arc to the point where one of these lines cuts the circle.



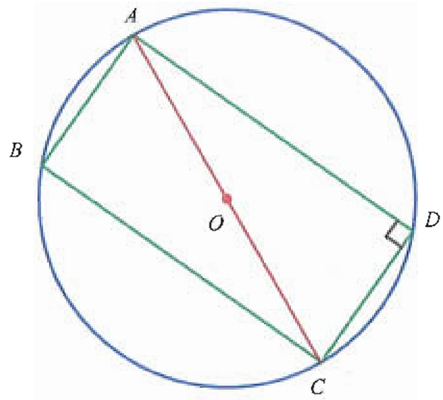
As before, let's take the central angles of the arcs AXB and MYN as $x°$ and $y°$. Again, we get $\angle ANB = \frac{1}{2}x°$ and $\angle MAN = \frac{1}{2}y°$. Here an external angle of ΔPAN is $\angle ANB$.

So,

$$\frac{1}{2}x = \angle APN + \frac{1}{2}y$$

from which we get

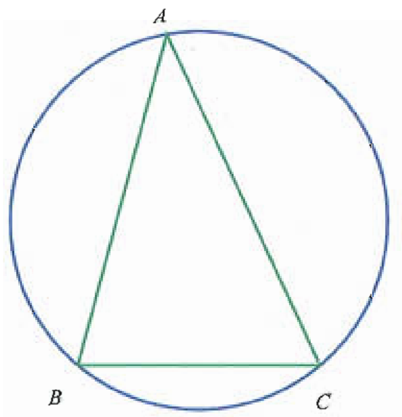
$$\angle APB = \frac{1}{2}(x - y)°$$



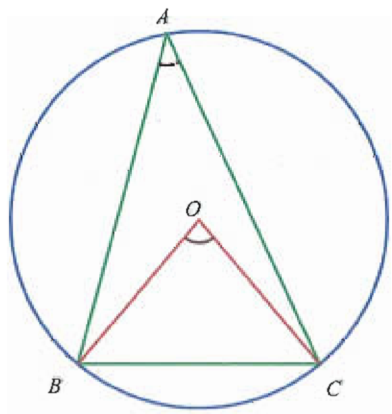
Since $ABCD$ is a rectangle, $\angle ADC = 90°$. So, the central angle of the alternate arc ABC of the arc ADC is $2 \times 90° = 180°$. That is, $\angle AOC = 180°$. This means that the points A, O, C lie on a straight line. In other words, AC is a diameter of the circle.

- How do we draw a triangle with angles 40°, 60°, 80° within a circle of radius 2.5 centimetres?

Let's first draw some triangle within a circle.



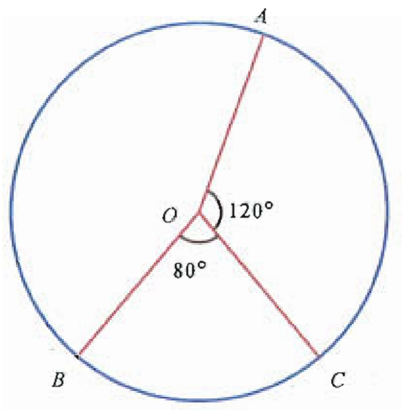
Join B and C to the centre O of the circle.



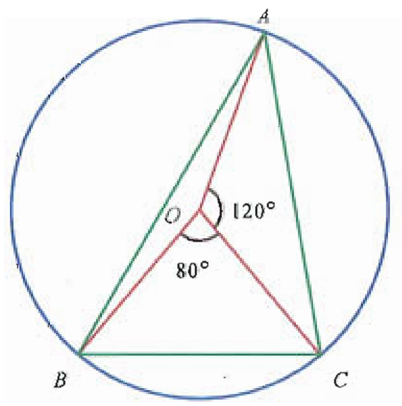
If $\angle BAC$ is to be 40°, how much should be $\angle BOC$?

Similarly, can't we find the angles got by joining other pairs of vertices to the centre of the circle?

So, for actually drawing the triangle we want, we draw a circle of radius 2.5 centimetres and mark A, B, C as shown below:

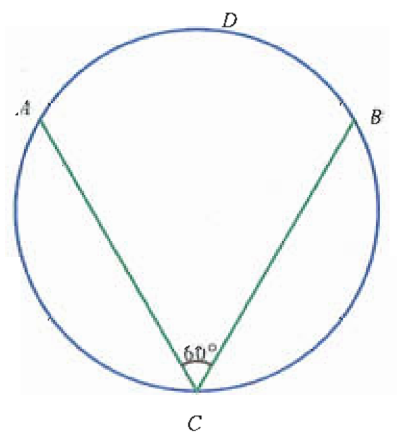


Now we need only join A, B, C to get the required triangle.



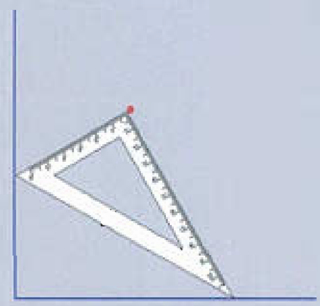
The problems below are for you:

- In this figure, what fraction of the circumference of the circle is the length of the arc ADB?

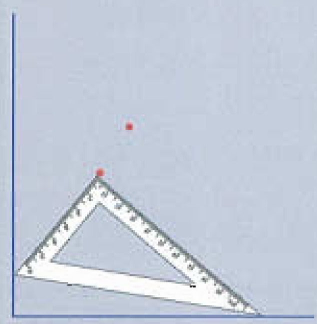


Sliding setsquare

Draw a pair of perpendicular lines on paper and place a set-square as shown below:



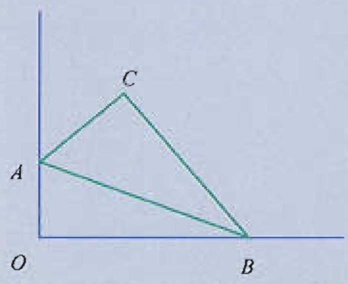
Mark the position of the top corner. Keeping the other two vertices on the lines, slide the set squre, and mark the various positions of the top vertex.



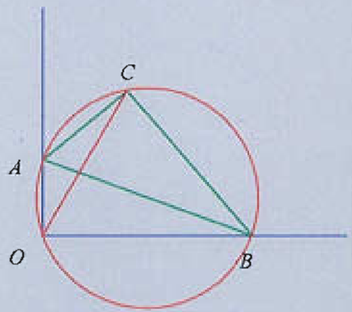
Note anything special about the points marked?

Circle, right angle and line

In our experiment with a sliding set square, all points marked are on a straight line, aren't they? Why does this happen?



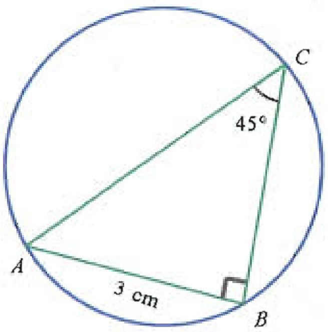
In the picture above, the set square is shown as ΔABC. Since both ACB and AOB are right angles, the circle on AB as diameter passes through both O and C.



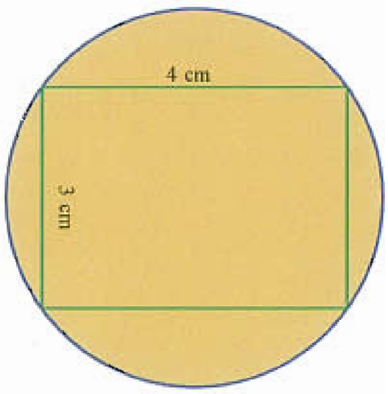
So, $\angle BAC = \angle BOC$. In this, since $\angle BAC$ is an angle of the set square we are using, it doesn't change. (In fact in the picture above, it is 60°.)

So, in sliding the set square, though the position of C changes, the line joining C and O keeps the same slant with OB. In other words, C can only move along the line making this angle with OB.

- What is the radius of the circle shown below?



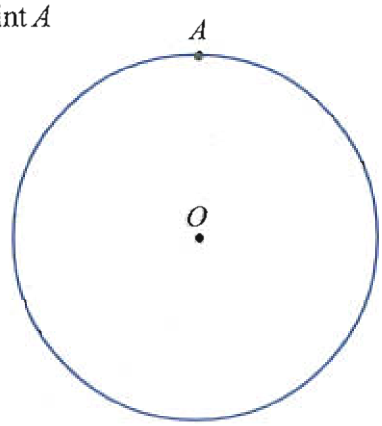
- What is the area of the circle shown below?



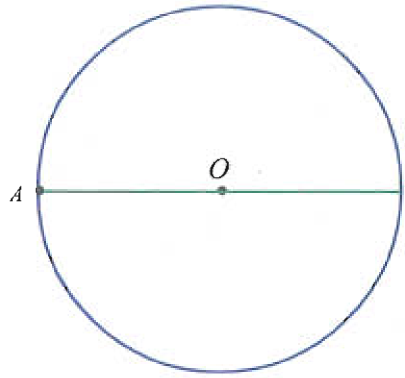
- How do we draw a triangle with two of the angles 40° and 120°, and circumradius 3 centimetres?

- How do we draw a $22\frac{1}{2}^\circ$ angle?

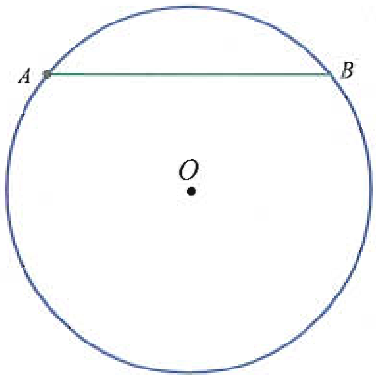
- In each of the pictures below, draw a $22\frac{1}{2}^\circ$ angle, according to the specifications:
    - At the point A



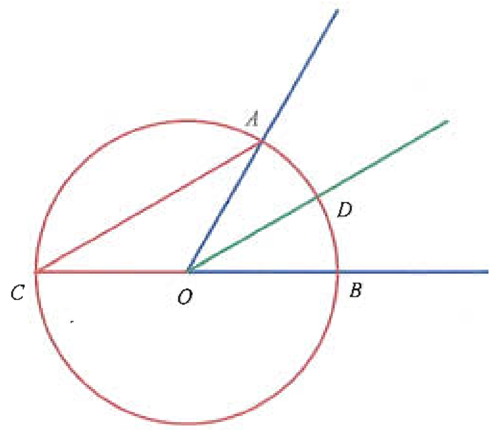
- At the point A with one side along OA



- At the point A, one side along AB



- In the picture below, O is the centre of the circle and the line OD is parallel to the line CA.

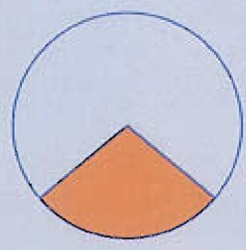


Prove that OD bisects $\angle AOB$.

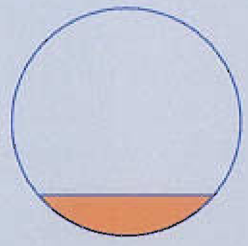
Can we use this to draw the bisector of a given angle? How?

Sector and segment

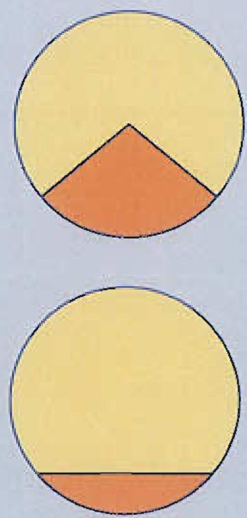
A sector of a circle is the part consisting of an arc of a circle and the two radii connecting its ends to the centre.



The part consisting of an arc and the chord joining its end points is a segment of a circle.



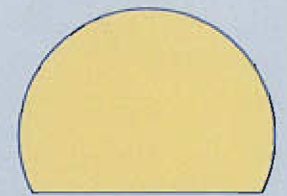
Since arcs of a circle come in pairs, so do sectors and segments



Size of a segment

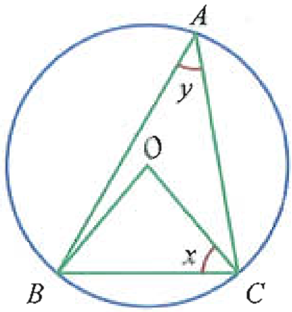
As the length of an arc increases, so does the size of the sector or segment made by it. And the length of an arc is measured using its central angle (Isn't it easier to measure the central angle than the length of the arc?)

The central angle of a sector is readily seen. What about the central angle of a segment?



First we have to determine the centre. And that we have already seen. (The section **Another view,** of the lesson **Circles** in the Class 9 textbook.)

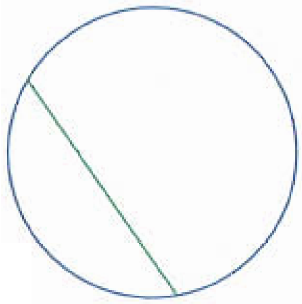
- In the figure below, O is the centre of the circle.

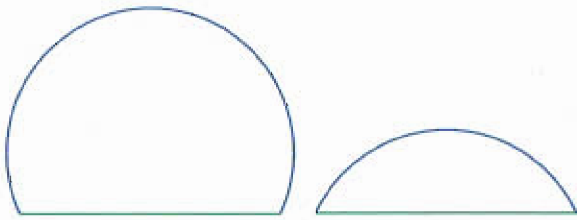


Prove that $x + y = 90°$

Segments of a circle

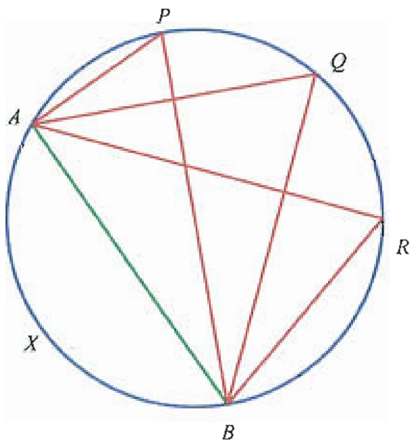
Every chord of a circle divides it into two parts:





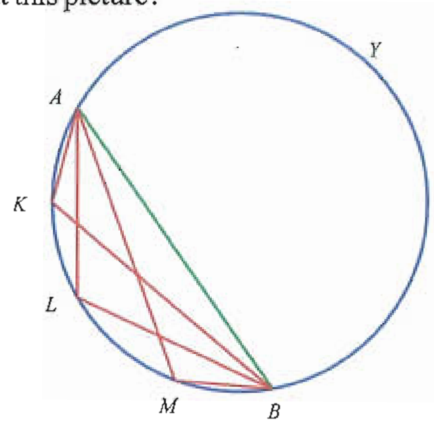
Such a part is called a *segment* of a circle.

See this picture:



Each of $\angle APB$, $\angle AQB$, $\angle ARB$ is equal to half the central angle of the arc AXB and so they are all equal.

What about this picture?



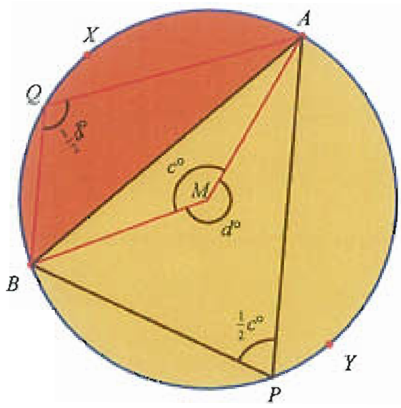
In this, each of $\angle AKB$, $\angle ALB$, $\angle AMB$ is equal to half the central angle of the arc AYB and so are all equal.

We can state this as follows:

Angles in the same segment of a circle are equal

We can note another fact also. Every segment has an alternate segment ; that is, a chord divides a circle into a pair of segments. Of these, angles in the same segment are equal. What about an angle in one segment and an angle in the alternate segment?

Angles in one segment are equal to half the central angle of an arc and the angles in the alternate segment are equal to half the central angle of the alternate arc, right? And what is the sum of the central angles of an arc and its alternate arc?



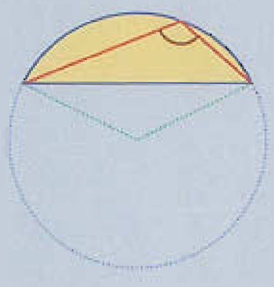
In the picture above, $c + d = 360°$ and so $\frac{1}{2}c + \frac{1}{2}d = 180°$.

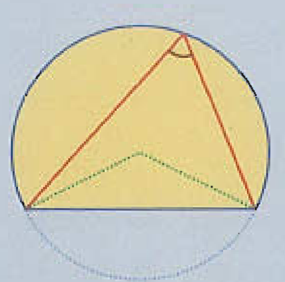
That is,

$$\angle APB + \angle AQB = 180°$$

Angle in a segment

Is there a direct method to compute the central angle of the arc of a segment, without finding the centre of the circle first? We know that angles in the same segment are equal.Using this angle, we can compute the central angle of the arc.





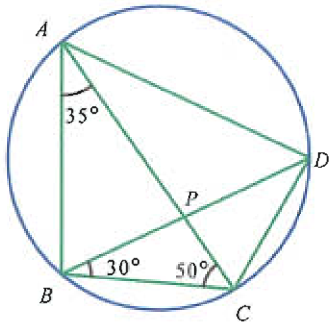
If the angle in a segment is $x°$, what is the central angle of its arc?

This can be stated as follows:

Angles in alternate segments are supplementary.

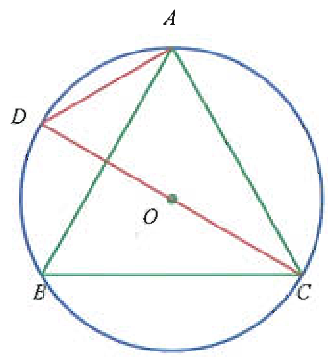
Now try your hand at these problems:

- In the figure below, A, B, C, D are points on the circle.



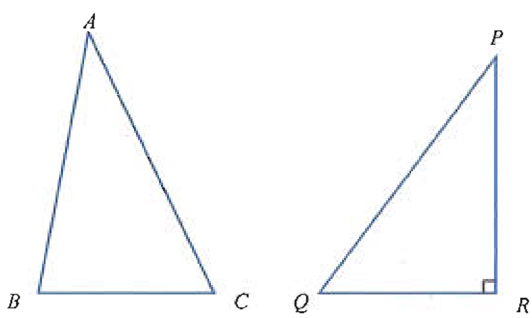
Compute the angles of the quadrilateral $ABCD$ and the angles between its diagonals.

- In the figure below, ΔABC is equilateral and O is its circumcentre.



Prove that the length of AD is equal to the radius of the circle.

- In the picture below, ΔPQR is right angled. Also, $\angle A = \angle P$ and $BC = QR$.



Prove that the diameter of the circumcircle of ΔABC is equal to the length of PQ.

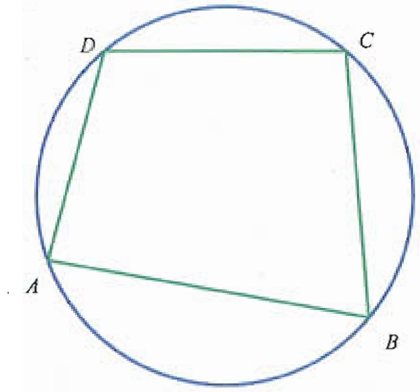
Circumcircle

We have seen that a circle can be drawn through any three points not on a line. (The section **Three points,** of the lesson **Circles** in the Class 9 textbook.) In other words, for any triangle, a circumcircle can be drawn.

What about quadrilaterals? Rectangles and certain types of trapeziums have circumcircles; but parallelograms which are not rectangles do not have circumcircles. Thus among quadrilaterals, there are two classes: those with circumcircles and those without.

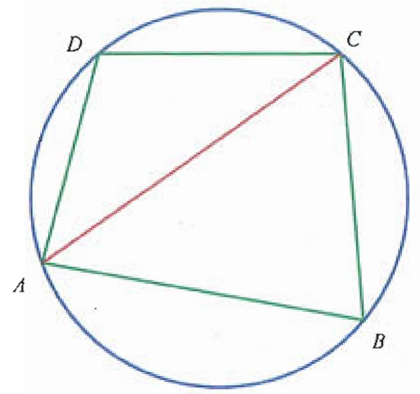
Circles and quadrilaterals

Look at this picture:



Is there any relation between the angles at A, B, C, D?

If you can't tell at once, try joining AC



Now the angles at B and D are the angles in the alternate segments which the chord AC cuts from the circle. And so they are supplementary.

Similarly, by drawing BD, we can see that the angles at A and C are also supplementary.

So, what can we say in general?

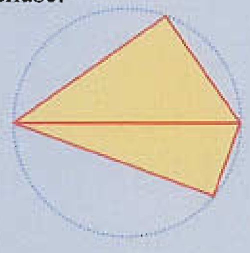
If all vertices of a quadrilateral are on a circle, then its opposite angles are supplementary.

Is the reverse true? That is, if the opposite angles of a quadrilateral are supplementary, can we draw a circle through all four of its vertices?

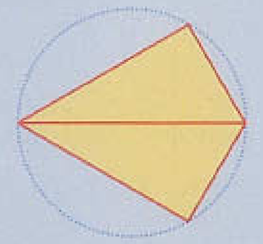
To answer this, let's first see how we would practically decide whether all four vertices of a quadrilateral can be put on a circle.

Making quadrilaterals

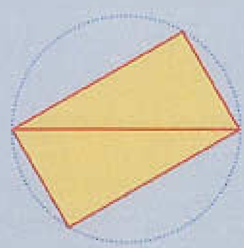
It is easy to construct special quadrilaterals which have circumcircles. One way is to join two right - angled triangles with equal hypotenuse:



What sort of a quadrilateral do we get if such triangles are actually congruent?



Suppose we flip the triangle below:

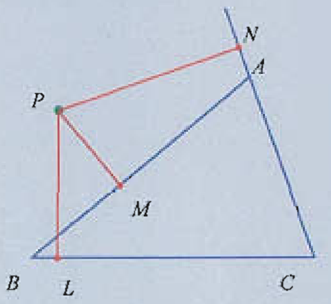


Can we make such quadrilaterals with circumcircles, using triangles other than right angled ones?

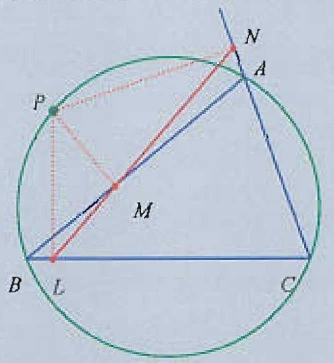
What should be the relation between the upper and lower triangles?

Circles and lines

We can check whether a specified point is on the circumcircle of a specified triangle, by measuring angles. There's another way. Draw perpendiculars from the point to the sides of the triangle:

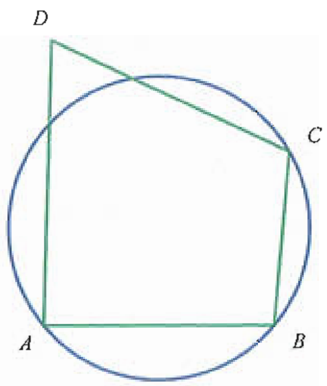


If the feet of these perpendiculars lie on the same line, then the points are on the circumcircle; otherwise not.

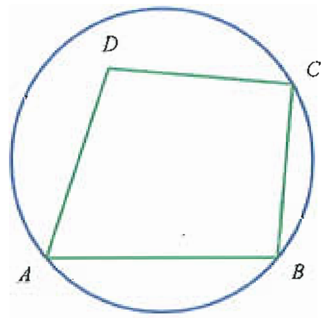


This is known as Simpson's Theorem

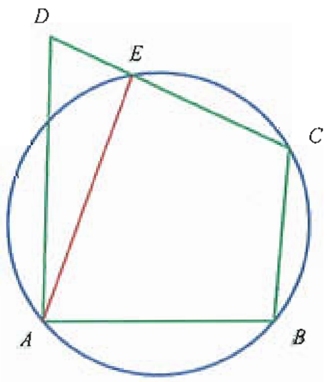
We can anyway draw a circle through three of the vertices of a quadrilateral. (Recall how we drew circles through three points not on a line, in Class 9.) Now the fourth vertex. If it is on this circle, then we are done. But this vertex may be outside the circle:



Or, inside the circle:



Let's have a closer look at the first figure. Joining A with the point where the circle cuts CD, we have another quadrilateral $ABCE$.



Since A, B, C, E are on the circle,

(1) $$\angle B + \angle AEC = 180°$$

Now as in the discussion about points inside and outside the circle in the section **Right angles and circles**, we can see that

$$\angle AEC = \angle EAD + \angle D$$

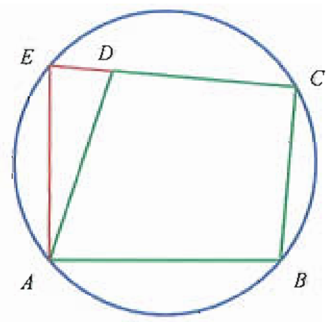
and so

(2) $$\angle D < \angle AEC$$

Thinking about the meanings of the algebraic statements marked (1) and (2) for a moment, we can easily see that

$$\angle B + \angle D < 180°$$

Next, in the second picture, we extend CD to meet the circle and join this point to A.



In this figure, we see that

(3) $$\angle B + \angle E = 180°$$

Also, from ΔAED, we find

$$\angle ADC = \angle E + \angle EAD$$

which gives

(4) $$\angle ADC > \angle E$$

From the statements marked (3) and (4), we get

$$\angle B + \angle ADC > 180°$$

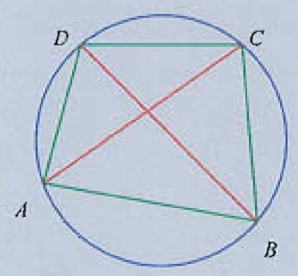
So, what have we seen?

Suppose a circle is drawn through three vertices of a quadrilateral. If the fourth vertex is outside this circle, then the sum of the angles at this vertex and the opposite vertex is less than 180°; if the fourth vertex is inside the circle, then this sum is greater than 180°.

Another theorem

Simpson's Theorem can also be seen as a theorem about quadrilaterals which have circumcircles. (How?) Another property of such quadrilaterals is that the sum of the products of opposite sides is equal to the product of the diagonals. That is, if the quadrilateral $ABCD$ has circumcircle, then

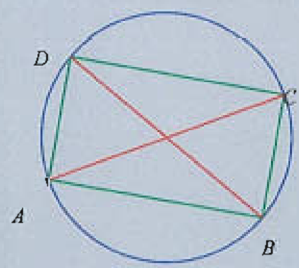
$$(AB \times CD) + (AD \times BC) = AC \times BD$$



On the other hand, if any quadrilateral satisfies this property, then it has a circumcircle. This is known as Ptolemy's Theorem.

A rectangle has a circumcircle. Also, its opposite sides are equal and so are the diagonals. So according to this theorem, in a rectangle $ABCD$

$$AB^2 + BC^2 = AC^2$$



But this is Pythagoras Theorem!

Area

The Indian mathematician Brahmagupta has given a method to compute the area of a cyclic quadrilateral. If we take the lengths of the sides of a cyclic quadrilateral as a, b, c, d and $s = \frac{1}{2}(a + b + c + d)$, then its area is equal to

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Haven't we seen a similar formula? If we take $d = 0$ in this, it becomes

$$\sqrt{(s-a)(s-b)(s-c)s}$$

Isn't this Heron's formula for the area of a triangle?

Brahmagupta has also given methods to compute the areas of non - cyclic quadrilaterals, in terms of their sides and angles. Using these, we get another interesting property of cyclic quadrilaterals:

Among the various quadrilaterals with the same lengths for sides, the cyclic quadrilateral has the maximum area.

(We have already seen that if the fourth vertex is within this circle, then this sum is actually *equal* to 180°.)

Now suppose that in a quadrilateral $ABCD$, we have $\angle B + \angle D = 180°$. Draw the circle through A, B, C.

Can D be outside the circle? If so, we must have $\angle B + \angle D < 180°$. So, D is not outside the circle.

Can it be inside the circle? If so, we must have $\angle B + \angle D > 180°$. So, it is not inside the circle either.

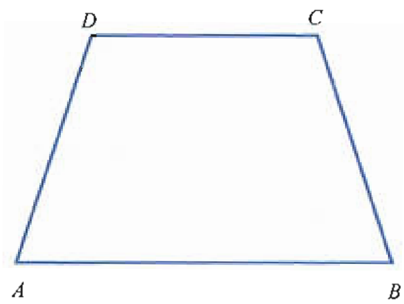
Since D is neither inside the circle nor outside, it must be on the circle.

That is,

If the opposite angles of a quadrilateral are supplementary, then we can draw a circle through all four of its vertices.

A quadrilateral for which a circle can be drawn through all the four vertices is called a *cyclic quadrilateral*. From our discussion above, a cyclic quadrilateral can also be described as a quadrilateral in which the opposite angles are supplementary.

All rectangles are cyclic quadrilaterals. Isosceles trapeziums are also cyclic quadrilaterals. Look at this picture:



$ABCD$ is an isosceles trapezium. So,

$$\angle A = \angle B$$

(Recall the section **Isosceles trapeziums**, of the lesson **Construction of Quadrilaterals** in the Class 9 textbook.)

Also, since AB and CD are parallel,

$$\angle A + \angle D = 180°$$

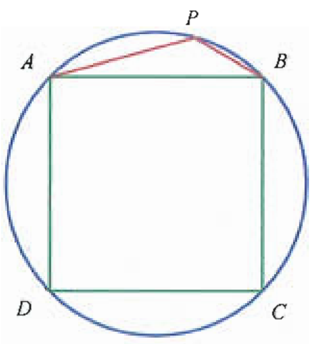
From these two equations, we get

$$\angle B + \angle D = 180°$$

This means $ABCD$ is a cyclic quadrilateral.

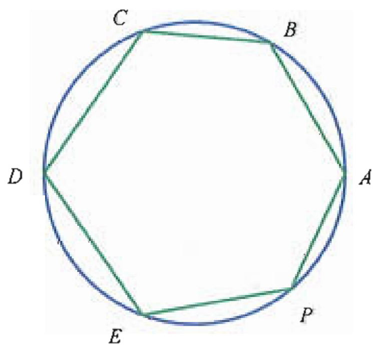
Now try these problems:

- Prove that in a cyclic quadrilateral, the exterior angle at any vertex is equal to the interior angle at the opposite vertex.
- Prove that a non-rectangular parallelogram is not a cyclic quadrilateral.
- Prove that non-isosceles trapeziums are not cyclic.
- In the figure below, $ABCD$ is a square.



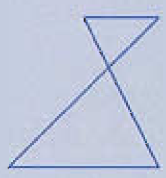
How much is $\angle APB$?

- Prove that in a cyclic hexagon $ABCDEF$ as shown below, $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$.

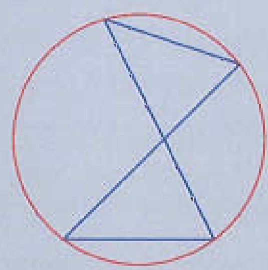


Similar triangles

We have seen several methods of drawing triangles similar to a given one. One of the methods is this:



After extending the sides upwards, instead of drawing a line parallel to the bottom side, draw a circle through the ends of the lines.

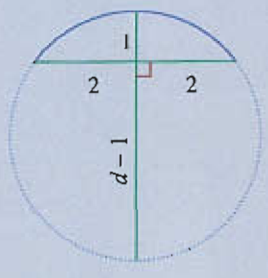


Is the upper triangle similar to the lower one?

Old problem, new method

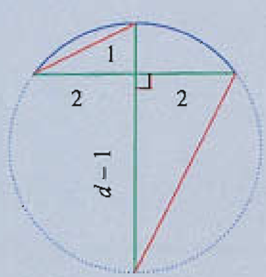
The distance between the ends of a piece of a bangle is 4 centimetres and its maximum height is 1 centimetre. Can we compute the radius of the bangle?

Remember doing this problem in Class 9? Now we can do it a little more quickly. We can picture the full bangle as below:



Here, d is the diameter of the circle.

We can draw two right angled triangles as below:



These are similar. (Why?) So,

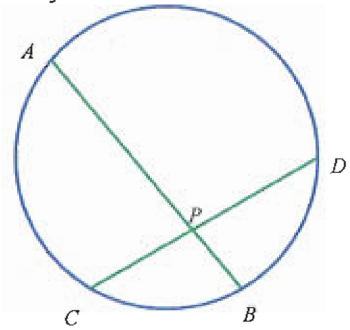
$$\frac{d-1}{2} = \frac{2}{1}$$

which gives $d - 1 = 4$ and we get $d = 5$.

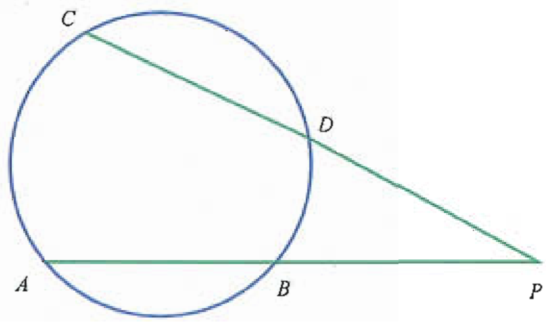
Cutting chords

Draw two non-parallel chords in a circle. They must intersect each other.

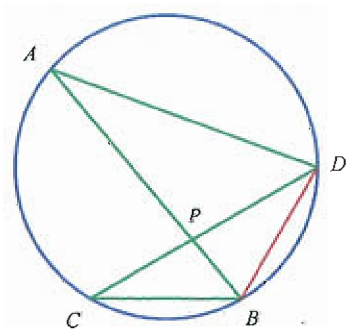
The intersection may be within the circle:

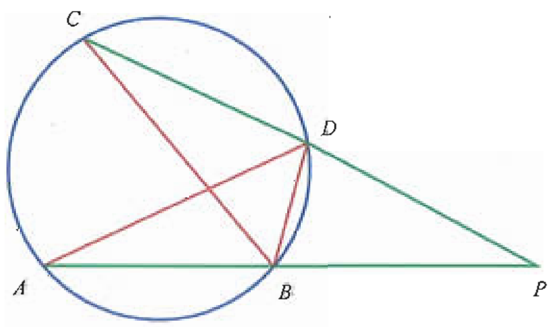


Or, outside the circle:



Either way, the triangles got by joining AD and BC can be shown to be similar.





In both figures, look at the angles at A and C of ΔAPD and ΔBPC. They are the angles in the same segment cut off by the chord BD of the circle and so are equal.

In the first figure, the angles $\angle APD$ and $\angle BPC$ at P are the opposite angles formed by the intersection of the lines AB and CD and so are equal; in the second figure, they are just different names for the same angle.

Thus in either figure, two pairs of angles of ΔAPD and ΔBPC are equal and so the third pair is also equal. That is, the triangles are similar. Since in similar triangles, pairs of sides opposite equal angles have the same ratio, we have in the first figure

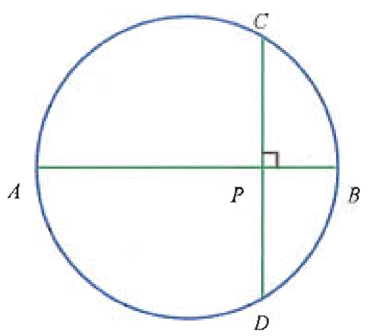
$$\frac{AP}{CP} = \frac{PD}{PB}$$

The second figure also gives the same. (Did you check?)

From this equation, we get

$$AP \times PB = CP \times PD$$

Let's look at a special case of this: AB is a diameter and CD is a chord perpendicular to it:



Since the perpendicular from the centre bisects a chord, we have $CP = PD$ here. So, the equation seen earlier becomes

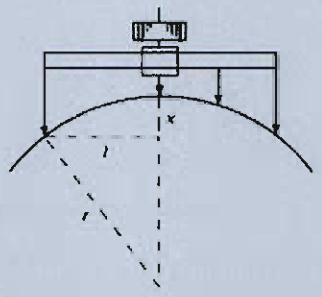
$$AP \times PB = CP^2$$

One application of this is to draw squares of any area. (One method of doing this is already seen in the section **Algebra and Pythagoras,** of the lesson **Irrational Numbers** in the Class 9 textbook.)

Math tool

Some lenses are made from spheres. And often we need to compute the radius of the sphere from which the lens was cut out. An instrument for this is the *spherometer.*

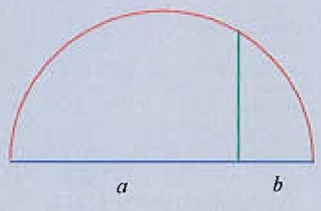
By placing its three legs on top of the piece of sphere, we can find the width of an arc. Using the screw in the middle, the maximum height can also be found.



With these, the radius can be computed as in our bangle problem.

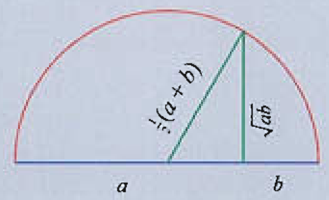
Geometry, algebra, numbers

Look at this picture:



What is the height of the vertical line? If we take it as x, we get $ab = x^2$ and so $x = \sqrt{ab}$

What is the radius of the semicircle? Since the diameter is $a + b$, the radius is $\frac{1}{2}(a+b)$.



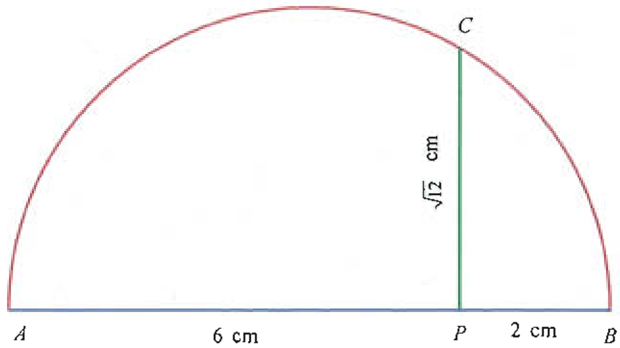
From the figure, we can see that the radius is larger than the perpendicular. Is there any instance in which they are equal? So, what do we see here?

For any two distinct numbers a and b, we have

$$\frac{1}{2}(a+b) > \sqrt{ab}$$

For example, let's see how we draw a square of area 12 square centimetres. What we need is a line with the square of its length 12 centimetres. In the equation above, the square of a length is expressed as the product of two other lengths. So, let's first write the square of the length we want, 12, as the product of two numbers. Since $12 = 6 \times 2$, if we draw $AP = 6$ and $PB = 2$ in the figure above, then by our equation, we will get $CP^2 = 12$.

So, we first draw AB of length 8 centimetres and mark P on it, 6 centimetres from A. Then we draw a semicircle with diameter AB. If we draw the perpendicular to AB through P and intersect the semicircle at C, we are almost done.



All it remains is to draw a square with CP as a side. (Remember the section **Square root** of the lesson **Real Numbers** in the Class 9 textbook?)

In how many different ways can we draw such square?

The problems below are for you:

- Draw a rectangle of sides 5 centimetres and 4 centimetres and draw a square of the same area.
- Draw a triangle of sides 4, 5, 6 centimetres and draw a square of the same area.
- Draw a quadrilateral with sides 2, 3, 4, 6 centimetres and one diagonal 5 centimetres. Draw a square of the same area.
- Given a quadrilateral, how do we draw a square of the same area, without making any measurements?

3 Second Degree Equations

New Equations

Let's start with a problem:

- When the length of each side of a square was increased by 5 centimetres, its perimeter became 36 centimetres. What was the length of a side of the original square?

We have seen very many problems like this. How do we solve it?

We can think like this: the length of a side of the new square is $36 \div 4 = 9$, and so the length of a side of the original square is $9 - 5 = 4$.

Or, we can forget about squares and perimeters and reformulate the problem solely in terms of numbers like this.

- The sum of a number and 5, multiplied by 4 gives 36. What is the number?

Thinking in reverse and inverting the operations, we can find the sum of the number and 5 as $36 \div 4 = 9$ and the number itself as $9 - 5 = 4$.

Or again, we may go all the way and formulate the problem in algebra, as:

- Find x such that $4(x + 5) = 36$

And we can solve it by first writing

$$x + 5 = \frac{36}{4} = 9$$

and then

$$x = 9 - 5 = 4.$$

Quantities and equations

We use numbers to denote various quantities and use algebra for expressing unchanging relations between varying quantities. For example, the relation between the length of a side and perimeter of a square can be written as

$$p = 4s$$

Here the unchanging relation is that, whatever be the length of a side, the perimeter is four times this length.

If an object is thrown upwards from the earth with a speed of u metres per second, then its speed after t seconds is given by

$$v = u - 9.8t$$

We can use such equations to find some quantities when other quantities are known. For example, if an object is thrown upwards at the speed of 20m/s, then to find when its speed would be 10m/s, we need only find the number t for which

$$20 - 9.8t = 10$$

Let's slightly change our original problem:

- When the length of each side of a square was increased by 5 centimetres, its area became 36 square centimetres. What was the length of a side of the original square?

What is the length of a side of the new square?

How did you get it as 6 centimetres?

So, the length of a side of the original square is $6 - 5 = 1$ centimetre.

How about changing this to a problem involving numbers alone?

- The square of the sum of a number and 5 is 36. What is the number?

To get back the number from its square, we need only take the square root. (In other words, the inverse operation of squaring is extracting the square root.)

Thus the sum of the number and 5 is $\sqrt{36} = 6$

And the number itself is $6 - 5 = 1$

Now how about writing the problem in algebra?

- Find x such that $(x + 5)^2 = 36$

And the answer?

$$x + 5 = \sqrt{36} = 6$$

$$x = 6 - 5 = 1$$

Let's look at another problem:

- The common difference of an arithmetic sequence is 5 and the square of the second term is 36. What is the first term?

Reasoning as in the last problem, we get the second term of the sequence as 6. So, the first term is 1.

In other words, the arithmetic sequence is 1, 6, 11, ...

Is there any other arithmetic sequence with the same property as given in this problem? How about the arithmetic sequence −11, −6, −1, ...?

Why did we miss this? When we reasoned the square of the second term is 36 and so the second term is 6, we forgot the fact that the square of −6 is also 36, right?

Equations and equations

We have seen problems involving just one unknown quantity and the resulting equations in Class 8. All of them, after simplifications, were reduced to equations like $2x = 3$ or $\frac{1}{2}x = -7$.

In general, these equations were all of the form $ax = b$.

But in many instances, we get equations involving the squares of quantities. For example, consider the problem of finding the lengths of the sides of a rectangle of area 323, with one side 2 centimetres longer than the other. To find the answer to this problem, we have to find x satisfying the equation

$$x(x + 2) = 323$$

which in turn becomes

$$x^2 + 2x = 323$$

So, what is the correct reasoning?

The second term is either 6 or –6. If the second term is 6, then the first term is $6 - 5 = 1$, and if the second term is –6, then the first term is $-6 - 5 = -11$.

What about algebra?

$$(x + 5)^2 = 36$$

$$x + 5 = 6 \text{ or } x + 5 = -6$$

$$x = 6 - 5 = 1 \text{ or } x = -6 - 5 = -11$$

To shorten this a little, we use a symbol. Instead of saying 6 or –6, we write ± 6 (to be read, "plus of minus 6"). Using this, we can write the steps above like this:

$$(x + 5)^2 = 36$$

$$x + 5 = \pm 6$$

$$x = -5 \pm 6$$

$$x = -5 + 6 = 1 \text{ or } x = -5 - 6 = -11$$

But, shouldn't we have thought along these lines in the rectangle problem also?

Do we have to? The length of a side of a rectangle cannot be negative, right?

In general, in doing such problems using algebra, it is a good practice to find both answers and then think back to the context of the original problem, to decide whether to take both solutions or only one.

One more problem:

- Of three consecutive integers, 1 added to the product of the first and the third gives 169. What are the numbers?

Can we do this using inverse operations? So, let's try algebra. Taking the numbers as x, $x + 1$ and $x + 2$, the given information translates to the equation

$$x(x + 2) + 1 = 169$$

This we can write as

$$x^2 + 2x + 1 = 169$$

What next?

Solutions—math and reality

We want to make a rectangle of perimeter 20 centimetres and height 11 centimetres less than the width.

Taking the width as x, the algebraic form of this problem is

$$x + (x - 11) = 10$$

which gives

$$x = 10.5$$

That is, the width should be 10.5 centimetres. What about the height then?

$$10 - 10.5 = -0.5$$

which is impossible.

Now look at this problem:

The distance between two numbers marked on the number line is 11; and their sum is 10. What are the numbers?

Taking the larger number as x, we get the same equation as in the first problem. The numbers turn out to be $10\frac{1}{2}$ and $-\frac{1}{2}$. And this is indeed a solution to this problem.

In general, the equations arising from different contexts may be the same; but the solutions may not suit all the contexts.

As in the previous problems, can we write the left hand side expression as a square?

Recall that

$$(x+1)^2 = x^2 + 2x + 1$$

So, what can we do with our equation? We can write it as

$$(x+1)^2 = 169$$

From this we can find

$$x + 1 = \pm\sqrt{169} = \pm 13$$

and so

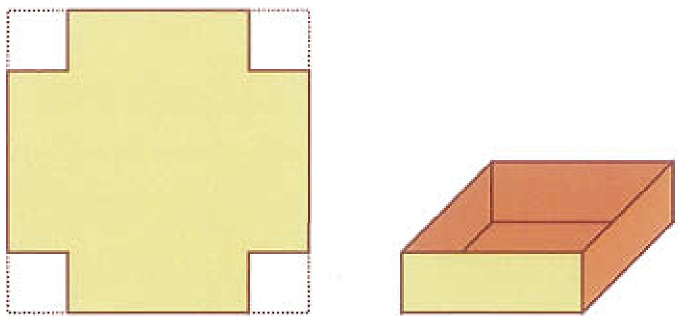
$$x = -1 \pm 13$$

$$x = 12 \quad \text{or } x = -14$$

Thus the numbers are 12, 13, 14 or −14, −13, −12.

Now try these problems on your own:

- 2000 rupees is invested in a scheme giving interest compounded annually. After two years the amount became 2205 rupees. What is the rate of interest?
- A pavement 2 metres wide runs around a square ground. The total area of the ground and the pavement is 1225 square metres. What is the area of the ground?
- A box is to be made by cutting off small squares from the corners of a square of thick paper and turning up the tabs.



The height of the box should be 5 centimetres and it should have a capacity of $\frac{1}{2}$ litre. What should be the length of a side of the square cut off? And the length of a side of the original square?

On squares

We have seen an algebraic identity about the square of a sum of two numbers, in Class 8.

For any two numbers x and y

$$(x+y)^2 = x^2 + 2xy + y^2$$

If we choose various numbers in the place of y in this, we get various identities, for all numbers x, such as

$$(x+1)^2 = x^2 + 2x + 1$$

$$(x+3)^2 = x^2 + 6x + 9$$

$$\left(x+\tfrac{2}{3}\right)^2 = x^2 + \tfrac{4}{3}x + \tfrac{4}{9}$$

$$(x-4)^2 = x^2 - 8x + 16$$

Look again at the expressions on the right side of these equations. All are second degree polynomials. Notice any relation between the coefficient of x and the last number?

Can you write $x^2 + 8x + 64$ as the square of a first degree polynomial? What about $x^2 + 8x + 16$?

- The common difference of an arithmetic sequence is 1 and the product of the first and the third terms is 143. What are the first three terms?

Completing the square

How did you do the last problem above?

If we take the second term of the sequence as x, then the first and the third terms would be $x - 1$ and $x + 1$; and from the given facts, we get

$$(x - 1)(x + 1) = 143$$

In this,

$$(x - 1)(x + 1) = x^2 - 1$$

So, the equation becomes

$$x^2 - 1 = 143$$

From this we get

$$x^2 = 144$$

$$x = \pm 12$$

and the first three terms of the sequence as 11, 12, 13 or -13, -12, -11.

Instead of this method, how about taking the first term of the sequence as x? Then we get the equation

$$x(x + 2) = 143$$

and we can write this as

$$x^2 + 2x = 143$$

What do we do next?

Can we write the polynomial on the left side of this equation as a square? Take a look at the earlier problems again. We know that

$$x^2 + 2x + 1 = (x + 1)^2$$

So, what if we add 1 to both sides of our equation? We get

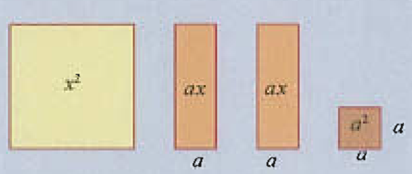
$$x^2 + 2x + 1 = 144$$

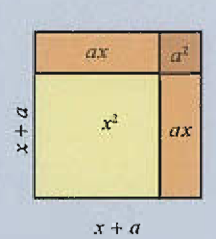
Geometry and algebra

For any two numbers x and a, we have

$$x^2 + 2ax + a^2 = (x + a)^2$$

For positive numbers x and a, this can be geometrically represented as below:





That is,

$$(x + 1)^2 = 144$$

The rest is easy, isn't it?

$$x + 1 = \pm\sqrt{144} = \pm 12$$

$$x = -1 \pm 12$$

$$x = 11 \text{ or } x = -13$$

And we get the terms of the sequence as 11, 12, 13 or −13, −12, −11.

Let's slightly modify this problem:

- The common difference of an arithmetic sequence is 6 and the product of the first and the second terms is 280. What are the first three terms of this sequence?

Taking the first term as x, we get the equation

$$x^2 + 6x = 280$$

In this, to write the left hand side of the equation in the form of a square, what number should we add? (Recall that for any two numbers x and a, we have $(x + a)^2 = x^2 + 2ax + a^2$).

We can write $x^2 + 6x$ as $x^2 + (2 \times 3)x$. So, to change $x^2 + 6x$ into $(x + 3)^2$, we need only add $3^2 = 9$. Thus we change our equation as

$$x^2 + 6x + 9 = 289$$

which means

$$(x + 3)^2 = 289$$

Now we can find

$$x + 3 = \pm 17$$

$$x = 14 \text{ or } -20$$

So, the first three terms of the sequence are 14, 20, 26 or −20, −14, −8

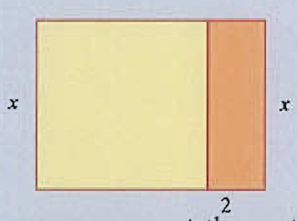
Let's look at another problem:

- A mathematician travelled 300 kilometres to attend a conference. During his talk he said, "Had the average speed of my trip been increased by ten kilometres per hour, I would have reached here one hour earlier". What was the average speed of his trip?

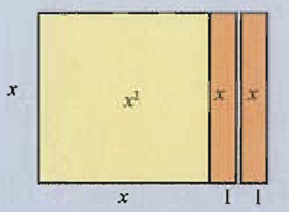
Geometric square completion

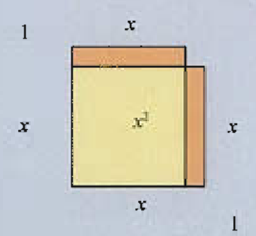
The algebraic fact that we can make $x^2 + 2x$ the square of an expression by adding 1 can be visualized geometrically also.

For this, we must first see $x^2 + 2x$ as a square of side x and a rectangle of side x and 2, joined together.

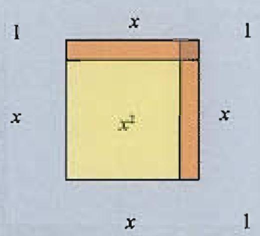


Now suppose we cut the rectangle into two equal pieces and shift one piece upwards as below:





To make this figure a square of side $x + 1$, all we need is a square of side 1 at the top right corner, right?



Taking the average speed as x kilometres per hour, what is the time taken for the trip? $\frac{300}{x}$.

If the speed is increased by 10 kilometres per hour, what would be the time taken for the trip? $\frac{300}{x+10}$.

So, what the mathematician said can be written as the equation.

$$\frac{300}{x} - \frac{300}{x+10} = 1$$

We can change this as below:

$$\frac{300(x+10) - 300x}{x(x+10)} = 1$$

$$\frac{300(x+10-x)}{x(x+10)} = 1$$

From this, we get

$$x(x + 10) = 3000$$

That is,

$$x^2 + 10x = 3000$$

Now to change the polynomial on the left to a square, what must we add?

So, we write this equation as

$$x^2 + 10x + 25 = 3025$$

That is,

$$(x + 5)^2 = 3025$$

From this, can't we calculate the average speed as 50 kilometres per hour?

Different way

There is another method to solve the equation, $x(x + 6) = 280$. We write $x + 6$ as $(x + 3) + 3$ and x as $(x + 3) - 3$. Using these, we find

$$\begin{aligned} x(x+6) &= \big((x+3)-3\big)\big((x+3)+3\big) \\ &= (x+3)^2 - 3^2 \end{aligned}$$

So, the original equation can be written

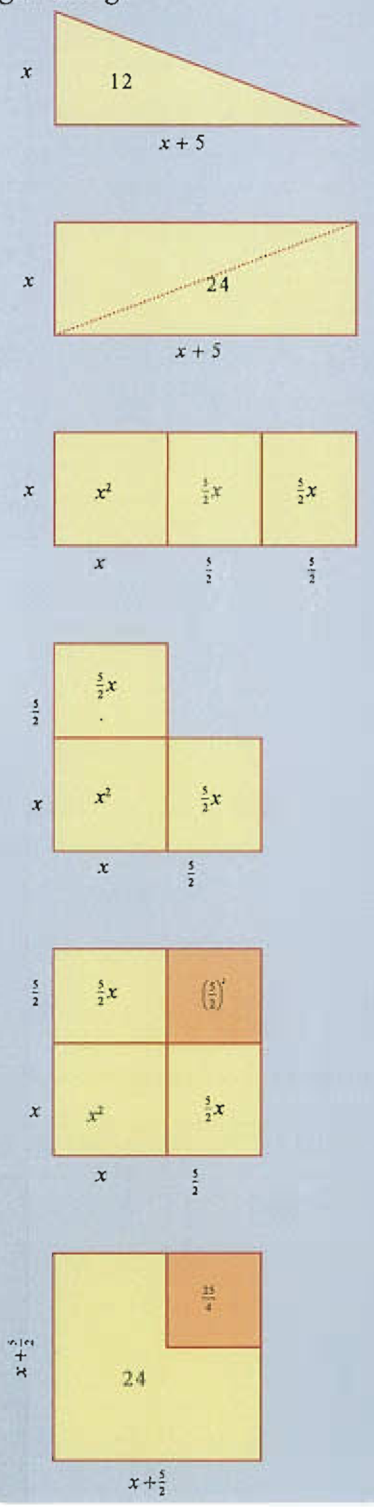
$$(x + 3)^2 - 9 = 280$$

From this we can find x as we did earlier.

See if you can solve $x^2 + 10x = 3000$ this way.

Geometry of solution

See the geometric explanation of the solution to the problem on a right angled triangle:



Now another problem:

- In a right angled triangle, one of the perpendicular sides is 5 centimetres longer than the other and its area is 12 square centimetres. What are the lengths of its sides?

If we take the shorter of the perpendicular sides to be x centimetres long, then the other is $x + 5$ centimetres long. What about the area?

Thus the given facts, translated to algebra becomes

$$\frac{1}{2}x(x + 5) = 12$$

and this we can rewrite as

$$x^2 + 5x = 24$$

What number can we add to $x^2 + 5x$ to get an expression of the form $x^2 + 2ax + a^2$?

Let's first write

$$x^2 + 5x = x^2 + \left(2 \times \frac{5}{2}\right)x$$

And note that

$$x^2 + 2 \times \left(\frac{5}{2}\right) x + \left(\frac{5}{2}\right)^2 = \left(x+\frac{5}{2}\right)^2$$

So, let's add $\left(\frac{5}{2}\right)^2$ to either side of our equation of the problem.

This gives

$$x^2 + 2 \times \left(\frac{5}{2}\right) x + \left(\frac{5}{2}\right)^2 = 24 + \frac{25}{4}$$

That is,

$$\left(x+\frac{5}{2}\right)^2 = \frac{121}{4}$$

From this we find

$$x = -\frac{5}{2} \pm \frac{11}{2}$$

and this gives

$$x = -\frac{5}{2} + \frac{11}{2} \text{ or } x = -\frac{5}{2} - \frac{11}{2}$$

Here, $-\frac{5}{2} - \frac{11}{2}$ is a negative number. Since a length cannot be negative, this doesn't work for our problem. So, we need to take only $x = -\frac{5}{2} + \frac{11}{2} = 3$

Thus the lengths of the perpendicular sides of our triangle are 3 and 8 centimetres. From these, can't you find the hypotenuse? Do it.

Just one more problem:

- A rectangle is to be made with perimeter 100 centimetres and area, 525 square centimetres. What should be the lengths of its sides?

The sum of the lengths of any two adjacent sides of this rectangle is 50 centimetres, isn't it? (How?) So, if we take the length of one side as x, then the length of the other is $50 - x$. The area is to be 525 square centimetres. So,

$$x(50 - x) = 525$$

and this can be written as

$$50x - x^2 = 525$$

It's more convenient to change this slightly and write

$$x^2 - 50x = -525$$

(Why?) Now what? If we recall

$$x^2 - 2ax + a^2 = (x - a)^2$$

Things become easy. What number do we add to $x^2 - 50x$?

Thus we can change our equation to

$$x^2 - 50x + 25^2 = -525 + 25^2$$

That is,

$$(x - 25)^2 = 100$$

From this, we get

$$x = 25 \pm 10$$

Increase and decrease

There's another way to find a rectangle with perimeter 100 centimetres and area 525 square centimetres.

A square of perimeter 100 centimetres has sides 25 centimetres long and thus area 625 square centimetres. The area of the rectangle we seek is (naturally) less than this. To get this rectangle, we shorten one side of this square; to keep the perimeter the same, we have to lengthen the other side by the same amount.

If the decrease (and increase) is taken as x centimetres, then the lengths of the sides of our rectangle are $25 - x$ and $25 + x$. So, the equation of our problem is

$$(25 - x)(25 + x) = 525$$

That is

$$625 - x^2 = 525$$

From this we get

$$x = \pm 10$$

So, the lengths of the sides of the rectangle are $25 - 10 = 15$ and $25 + 10 = 35$ centimetres.

That is,

$$x = 35 \text{ or } x = 15$$

If we take $x = 35$, then $50 - x = 15$ and if $x = 15$, then $50 - x = 35$. Either way, the lengths of the sides of the rectangle are 35 and 15 centimetres.

Take a look at the algebraic forms of all these problems:

- $x^2 + 2x = 143$
- $x^2 + 10x = 3000$
- $x^2 + 5x = 24$
- $x^2 - 50x = -525$

The general form of all these is

$$x^2 + ax = b$$

Next see how we changed these equations:

- $x^2 + 2x + 1^2 = 143 + 1^2$, that is, $(x + 1)^2 = 144$
- $x^2 + 10x + 5^2 = 3000 + 5^2$, that is, $(x + 5)^2 = 3025$
- $x^2 + 5x + \left(\frac{5}{2}\right)^2 = 24 + \left(\frac{5}{2}\right)^2$, that is, $\left(x + \frac{5}{2}\right)^2 = \frac{121}{4}$
- $x^2 - 50x + 25^2 = -525 + 25^2$, that is, $(x - 25)^2 = 100$

The general method is to make $x^2 + ax$ a square, by adding the square of half the coefficient of x:

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2$$

This method is called *completing the square*.

Look at this problem:

- How many terms of the arithmetic sequence 3, 7, 11, . . . must be added to get 300?

Denoting the terms of the sequence as $x_1, x_2, x_3, \ldots$, we have

$$x_n = 3 + 4(n - 1) = 4n - 1$$

So,

$$x_1 + x_2 + x_3 + \ldots + x_n = \frac{1}{2}n(x_1 + x_n)$$

Past and present

The method of completing squares to solve second degree equations is pretty old. We see the Babylonians using this technique to solve problems related to areas, as early as 1500 BC.

But the current practice of converting problems to algebraic equations was not used then. (This is at most five hundred years old.) Problems were formulated, and solutions described in ordinary language. In geometric problems, the methods used were also geometrical.

Thus from the historical point of view, what we now describe as geometric interpretations of algebraic methods were actually the motivation for the algebraic techniques developed later.

$$= \frac{1}{2}n(3 + (4n - 1))$$

$$= 2n^2 + n$$

What if we want this sum to be 300? We must have

$$2n^2 + n = 300$$

Is this of the form $x^2 + ax = b$, we have discussed?

How do we change the coefficient of n^2 to 1?

How about dividing both sides of our equation by 2?

$$n^2 + \frac{1}{2}n = 150$$

Next we complete the square:

$$n^2 + \frac{1}{2}n + \left(\frac{1}{4}\right)^2 = 150 + \frac{1}{16}$$

That is,

$$\left(n+\frac{1}{4}\right)^2 = \frac{2401}{16}$$

Now can't we find n?

$$n = -\frac{1}{4} \pm \frac{49}{4}$$

$$n = 12 \text{ or } n = -\frac{50}{4}$$

Since n is a natural number in our problem, we need only the solution $n = 12$.

So, adding 12 terms of this sequence would give 300.

Thus in some equations, we have the additional job of changing the coefficient of x^2 to 1, before completing the square.

Now try these problems:

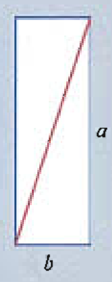
- The length of a rectangle is 10 centimetres more than its breadth; and its area is 144 square centimetres. What are the length and breadth?
- How many terms of the arithmetic sequence 5, 7, 9, . . . should be added to get 140?

Diagonal problem—algebra

We see that the method of completing squares was used not only for solving second degree equations, but for computing square roots also, in olden times.

For example, in an ancient Babylonian clay tablet, the method of computing the diagonal of a tall, slender rectangle is given like this:

divide the square of the width by the height and add half of this to the height



In the current algebraic language, this becomes

$$\sqrt{a^2 + b^2} \approx a + \frac{b^2}{2a}$$

Its rationale can also be explained with algebra, we know that

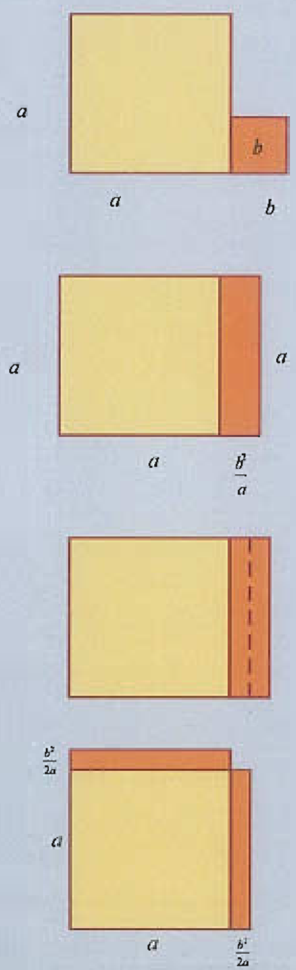
$$a^2 + b^2 + \left(\frac{b^2}{2a}\right)^2 = \left(a + \frac{b^2}{2a}\right)^2$$

If the number b is small compared to the number a, then $\left(\frac{b^2}{2a}\right)^2$ would be a very small number and so we can take

$$a^2 + b^2 \approx \left(a + \frac{b^2}{2a}\right)^2$$

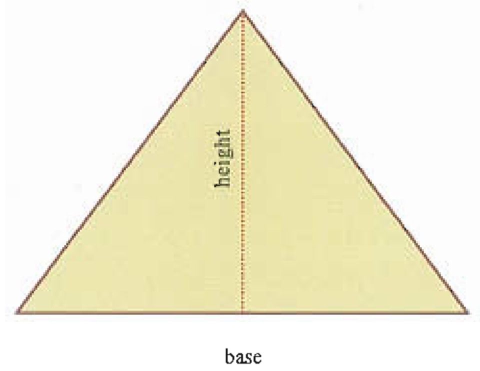
Diagonal problem—geometry

Let's have a look at the geometry of the Babylonian diagonal estimate:



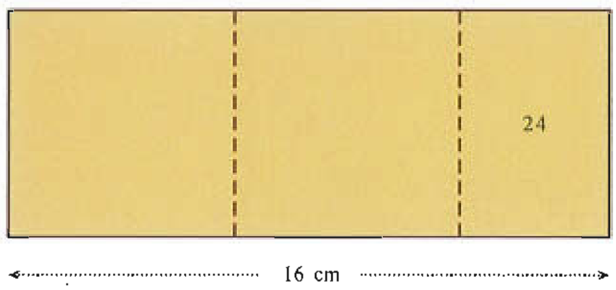
The last shape differs only slightly from a square of side $a+\frac{b^2}{2a}$, right?

- An isosceles triangle as shown below is to be made:



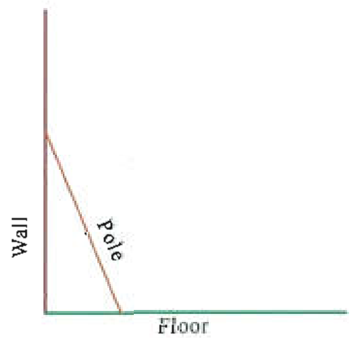
The height should be 2 metres less than the base and the area should be 12 square metres. What should be the lengths of the sides?

- From a rectangular sheet of paper, two squares are cut off as shown below:



The remaining rectangle should have area 24 square centimetres. What should be the lengths of the sides of the squares?

- A pole 2.6 metres high leans against a wall, with its foot 1 metre away from the wall.



When this distance was slightly increased, the top of the pole slid down the wall by the same distance. How far was the foot of the pole shifted?

- The perimeter of a rectangle is 28 metres and its diagonal is 10 metres. What are the lengths of its sides?

- Find a pair of numbers with sum 4 and product 2.

- The sum of a number and its reciprocal is $2\frac{1}{12}$. What are the numbers?

- Thirty candies were distributed among some children. Relishing the sweet, the whiz-kid among them said, "If we were one short, we would all get one more." How many kids were there?

- There are two taps opening into a tank. If both are opened, the tank would be full in 12 minutes. The time taken for it to fill with only the small tap open, is 10 minutes more than the time to fill with only the large tap open. What is the time taken to fill the tank with only the small tap open?

Equations and polynomials

The general algebraic form of all the problems done so far is

$$ax^2 + bx + c = 0$$

For example, our very first problem gave the equation

$$(x + 5)^2 = 36$$

and this we can write

$$x^2 + 10x - 11 = 0$$

Another equation we came across in one of our problems is

$$2x^2 + x = 300$$

and we can write this as

$$2x^2 + x - 300 = 0$$

Looking at it like this, another idea takes shape. We know that $ax^2 + bx + c$, with $a \neq 0$ is the general form of a second degree polynomial.

So all the problems we have done can be seen as investigations in

Square root

We can use the Babylonian technique (finding the diagonal of a rectangle), for computing approximations of square roots also. For any two numbers x and a, we have

$$a^2 + x + \left(\frac{x}{2a}\right)^2 = \left(a + \frac{x}{2a}\right)^2$$

If the number x is small compared to a, we can take

$$a^2 + x \approx \left(a + \frac{x}{2a}\right)^2$$

and so

$$\sqrt{a^2 + x} \approx a + \frac{x}{2a}$$

For example, since $2 = \frac{9}{4} - \frac{1}{4}$, we have

$$\begin{aligned}\sqrt{2} &= \sqrt{\left(\frac{3}{2}\right)^2 - \frac{1}{4}}\\ &\approx \frac{3}{2} - \frac{1}{2} \times \frac{1}{4} \times \frac{2}{3}\\ &= \frac{17}{12}\\ &\approx 1.4166\end{aligned}$$

We have seen in Class 9 that $\frac{17}{12}$ was used as an approximation for $\sqrt{2}$ in ancient Babylon and in our place also (the sections **An old method** and **Folk math** of the lesson **Irrational Numbers**).

finding the numbers for which a second degree polynomial gives zero.

Now let's see the algebraic form of our method of solution also. To solve the equation

$$ax^2 + bx + c = 0$$

we first rewrite the equation in the more familiar form

$$ax^2 + bx = -c$$

Then we divide both sides by a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Next to change the left side expression to a square, we add the square of half the coefficient $\frac{b}{a}$ of x:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

The left side of this can be written

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x+\frac{b}{2a}\right)^2$$

and the right side can be simplified to

$$-\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$$

Thus our equation becomes

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

From this we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

and then

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Polynomials

We can use the method of completing squares to find out certain features of a second degree polynomial.

For example, consider the polynomial

$$p(x) = x^2 + 6x + 11$$

By the process of completing squares, we can write

$$\begin{aligned} p(x) &= x^2+6x+11 \\ &= (x^2+6x+9)+2 \\ &= (x+3)^2+2 \end{aligned}$$

In this, whatever number we take as x, the number $(x + 3)^2$ would not be negative. So, the number got as $p(x)$ would never be less than 2.

In other words, if we take various numbers in the place of x and calculate the numbers got as $p(x)$, the least number got would be 2; and this is got when we take $x = -3$.

Thus we see that

For any three numbers a, b, c with $a \neq 0$, the solution to the equation

$$ax^2 + bx + c = 0 \quad is \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In view of our earlier discussion, this can also be stated as follows.

In the polynomial

$$p(x) = ax^2 + bx + c$$

if we take

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

then

$$p(x) = 0$$

In solving second degree equations, instead of dividing by the coefficient of x and then completing the square, we can directly use this formula.

For example, consider the equation

$$2x^2 + x = 300$$

Writing this as

$$2x^2 + x - 300 = 0$$

We can find x directly as

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 2 \times (-300)}}{2 \times 2}$$

$$= \frac{-1 \pm \sqrt{2401}}{4}$$

$$= \frac{-1 \pm 49}{4}$$

$$= 12 \text{ or } \frac{25}{2}$$

Here are some problems for you.

Find the answers to the following problems correct to two decimal places using a calculator.

Different methods

To solve a second degree equation, we can do the process of completing the square step by step; or we can use the formula which does all this work in one stroke and quickly gives the solution. The convenience in using either depends on the nature of the equation.

For example, to solve

$$x^2 + 12x + 7 = 0$$

completing the square to get

$$(x + 6)^2 = -7 + 36$$

and proceeding may be better than directly trying to compute

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 1 \times 7}}{2}$$

On the other hand, for an equation like

$$2x^2 + 5x - 3 = 0$$

directly computing

$$x = \frac{-5 \pm \sqrt{5^2 + 4 \times 2 \times 3}}{4}$$

may be better than completing the square to get

$$2\left(x + \frac{5}{4}\right)^2 = -3 + \frac{25}{8}$$

- A rectangle of perimeter 10.75 metres and area 5.8 square metres is to be made. What should be the lengths of its sides?
- The distance travelled by an object thrown upwards in t seconds is $30t - 4.9t^2$ metres. After how much time would it fall down? At what all times would it be 20 metres above the ground?
- In each of the second degree polynomials given below, find the numbers to be taken as x to get zero:
  - $x^2 - 5x + 6$
  - $x^2 - 2x - 1$
  - $x^2 + 5x + 6$
  - $2x^2 - 7x - 15$
  - $x^2 + x - 6$
  - $9x^2 + 12x + 4$
  - $x^2 - x - 6$

Discrimination

Armed with our new knowledge, let's tackle an old problem:

- We want to make a rectangle of perimeter 20 centimetres and area 26 square centimetres. What should be the lengths of the sides?

The equation of these requirements is

$$x(10 - x) = 26$$

and we can write this as

$$x^2 - 10x + 26 = 0$$

Trying out our new technique, we get

$$x = \frac{10 \pm \sqrt{100 - 104}}{2} = \frac{10 \pm \sqrt{-4}}{2}$$

What does $\sqrt{-4}$ mean? Negative numbers don't have square roots!

The meaning of this answer is that there are no numbers x for which the equation $x^2 - 10x + 26 = 0$ holds. (In other words, whatever number x we take, we won't get zero from the the polynomial $x^2 - 10x + 26$.)

In general, note that in the solution for the equation, $ax^2 + bx + c = 0$, we have a square root, $\sqrt{b^2 - 4ac}$. If $b^2 - 4ac$ is a positive number, then it has two square roots and each leads to a solution of the equation.

Squares again!

We have seen in Class 9 that among all rectangles of perimeter 20 centimetres, the square of side 5 centimetres has the maximum area (the section **Square speciality.** of the lesson **Polynomials**).

This we can see in a different way. If the length of one side of such a rectangle is taken as x, then its area can be found from the polynomial

$$p(x) = x(10 - x) = 10x - x^2 = -(x^2 - 10x)$$

Completing the square, we can write

$$p(x) = -\left((x-5)^2 - 25\right) = 25 - (x-5)^2$$

In this, whatever be the number we take as x, the number $(x - 5)^2$ would not be negative; so that the number $p(x)$ would not be greater than 25. And for $x = 5$, we do get $p(x) = 25$

On the other hand, if $b^2 - 4ac$ is negative, then the equation has no solution.

What if $b^2 - 4ac$ is zero? Then it has only one square root (zero itself) and so the equation has only one solution.

The number $b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$. So, what we found out is this:

If the discriminant of a second degree equation is positive, then it has two solutions; if it is negative, the equation has no solutions; and if it is zero, the equation has only one solution

Now look at this problem:

- An 8 centimetres long wire is to be bent into a rectangle. Can a rectangle with diagonal 2 centimetres be made from it? What about a rectangle with diagonal 4 centimetres?

Taking the length of one side of such a rectangle as x the other must be $4 - x$ and so the square of the diagonal must be

$$x^2 + (4 - x)^2 = 2x^2 - 8x + 16$$

So, the first question is whether this can be 4. That is,

$$2x^2 - 8x + 16 = 4$$

This we can write as

$$2x^2 - 8x + 12 = 0$$

The discriminant of this equation is

$$(-8)^2 - 4 \times 2 \times 12 = 64 - 96 < 0$$

So, it is not possible to make such a rectangle.

Next, let's check whether a rectangle of diagonal 4 centimetres is possible. The equation of this wish is

$$2x^2 - 8x = 0$$

and its discriminant is

$$(-8)^2 - 4 \times 2 \times 0 = 8^2 = 64$$

Word and meaning

The word *discriminant* in ordinay language means a feature which makes something different from others. And *discrimination* means understanding of the difference between one thing and another, and also good judgement and taste.

It is also used to mean unjust treatment of different groups of people.

Polynomial and discriminant

We can use the discriminant to understand some features of a second degree polynomial also. In the polynomial

$$p(x) = ax^2 + bx + c$$

we can complete the square and write

$$p(x) = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$
$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right)$$

In this, $\left(x + \frac{b}{2a}\right)^2$ is not negative, whatever be the number x. Also, if $b^2 - 4ac$ is negative, then $\frac{4ac - b^2}{4a^2}$ is positive. So, if a is positive, then $p(x)$ is positive for any number x and if a is negative, then so is $p(x)$, for any number x.

What do we see from this? If the discriminant $b^2 - 4ac$ is negative, then depending on whether a is positive or negative, the numbers got from the polynomial $p(x)$ would be all positive or all negative.

What if $b^2 - 4ac$ is positive? And what if it is zero?

So, the equation has two solutions and they are

$$x = \frac{8 \pm \sqrt{64}}{4} = 4 \text{ or } 0$$

Since x is the length of the side of a rectangle in our problem, $x \neq 0$. If we take $x = 4$, then the length of the other side would be $4 - x = 0$. So, either way, we cannot make such a rectangle.

What do we see here? Even if the mathematical equation arising from a physical problem has a solution, the physical problem itself may not have a solution.

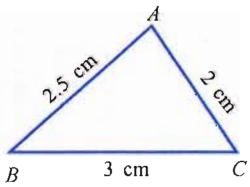
Now try to answer the questions below:

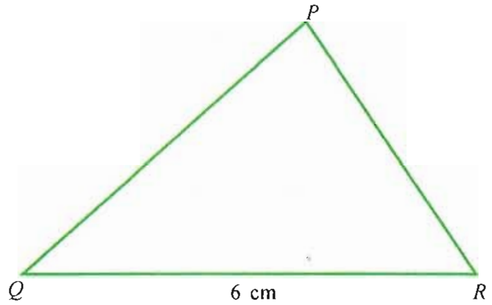
- Can the sum of the first few consecutive terms of the arithmetic sequence 5, 7, 9, . . . be 140? What about 240?
- From the polynomial $p(x) = x^2 + x + 1$, do we get $p(x) = 0$ Can the sum be 240? any x? What about $p(x) = 1$? And $p(x) = -1$?
- From the expression $x + \frac{1}{x}$, do we get 0, 1, or 2 for some number x?
- Prove that if a, b, c are positive numbers and if the equation $ax^2 + bx + c = 0$ has solutions, then they are negative numbers.

4 Trigonometry

Sides and angles

Look at these two triangles:





ΔPQR has the same angles as ΔABC. More precisely,

$$\angle P = \angle A \qquad \angle Q = \angle B \qquad \angle R = \angle C$$

And QR is twice as long as BC.

What can you say about the lengths of the other sides of ΔPQR?

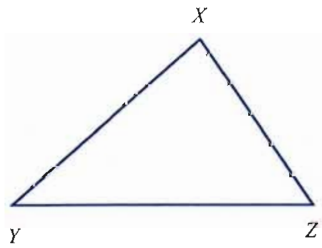
The side opposite $\angle P$ is twice as long as the side opposite $\angle A$; and the sides opposite the other equal pairs of angles must also have the same ratio, right?

In other words,

$$PQ = 2 \times AB = 5 \text{ cm}$$

$$RP = 2 \times CA = 4 \text{ cm}$$

Now look at this triangle:



In this also, we have

$$\angle X = \angle A \qquad \angle Y = \angle B \qquad \angle Z = \angle C$$

On the earth and up the sky

Trigonometry is the study of the relations between the sides and angles of triangles.

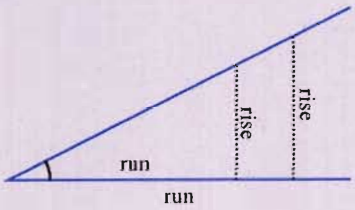
We have seen that angles are used to measure slant or spread or turn. (The section **Slant and spread and turn,** of the lesson **Circle measures** in the Class 9 textbook.) In history, we first come across measures of slant in various constructions on the earth; and measures of turn in the study of planets in the sky.

The first astronomical studies were also for earthly needs. The basic need of man is food; and production of food, that is agriculture, depends on the weather. A factor influencing weather is the revolution of the earth around the sun. To understand this well, we must be able to determine the positions of the other planets and the stars. This is why in all ancient agricultural communities, astronomy was a major topic of study. And mathematics, especially geometry, is very much necessary for this.

Measure of slant

We have already seen the Babylonian technique of measuring angles by dividing a circle into 360 equal parts and its connection with astronomy. (The section **History of angle measurement,** of the lesson **Slant and spread** in the Class 6 textbook.) This method was used in Babylon from the third century BC. And this is the degree measure that we now use.

But in the constructions on the earth, another method was used to measure slants. See this picture:



As shown in the picture, the "run" and "rise" change from one position to another, but if we divide the rise by the run at any position, we get the same number. (Why?) And this number differs from angle to angle, depending on its size. It is this number, which is used as a measure of slant.

In the Ahmose Papyrus from ancient Egypt (see the section **Ancient Math,** of the lesson **Equations** in the Class 8 textbook), we can see such computations. The slant between the triangular faces and the base of square pyramid is computed in this manner.

In a clay tablet from ancient Babylon, we see tables of numbers got by dividing the hypotenuse by another side, for various right angled triangles.

Without knowing the length of at least one side, what can we say?

At least this much:

$$\frac{XY}{2.5} = \frac{YZ}{3} = \frac{ZX}{2}$$

Getting rid of decimals (or using ΔPQR instead of ΔABC), we can write this as

$$\frac{XY}{5} = \frac{YZ}{6} = \frac{ZX}{4}$$

From this, we can write

$$\frac{XY}{YZ} = \frac{5}{6}; \quad \frac{YZ}{ZX} = \frac{6}{4}$$

or combine all these into the single equation

$$XY : YZ : ZX = 5 : 6 : 4$$

The lengths of the sides of the other two triangles also bear the same ratio, don't they?

What general principle do we see here? We can draw several triangles with the same three angles; and the actual lengths of the sides may vary from triangle to triangle. But the ratio of these lengths doesn't change.

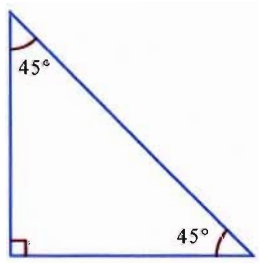
In short,

For triangles with the same set of angles, the ratio of the lengths of the sides is the same.

This leads to another thought: if the angles of a triangle are known, can we find the ratio of their sides?

Let's look at a couple of examples:

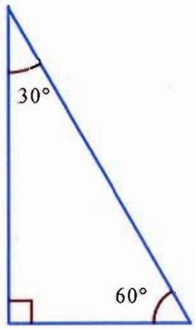
- See this triangle:



The perpendicular sides of this triangle are equal (Reason?) And if this length is taken as x, then the hypotenuse should be $\sqrt{2}\,x$. (Why?)

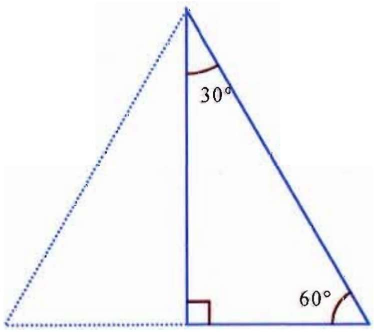
Thus the ratio of the sides of this triangle is $1 : 1 : \sqrt{2}$

- Now another right angled triangle:

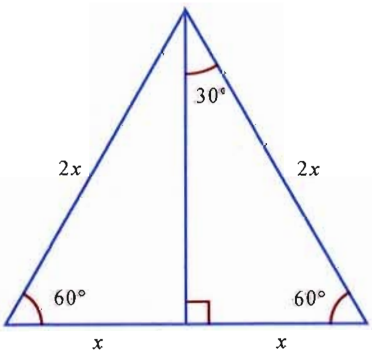


Let's take the length of its shortest side as x. How do we compute the lengths of the other two sides?

The first triangle we saw is half a square; and this one is half an equilateral triangle. (See the section, **Set a square and a triangle,** of the lesson **Between the lines**, in the Class 7 textbook.)



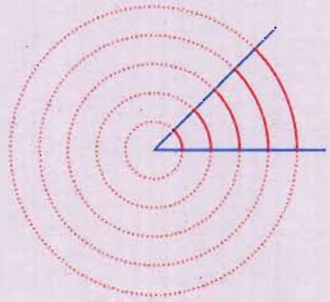
The sides of this equilateral triangles are as shown below:



Degree measure of angles

What does it mean, when we say that an angle is 45°?

We can draw several circles centred at the vertex of such an angle. And the lengths of the arcs of such circles within the sides of this angle are all different.



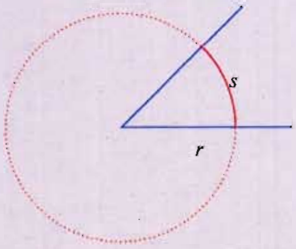
But each of these arcs is $\frac{1}{8}$ of the corresponding full circle. And 45 is the number got by multiplying this fraction $\frac{1}{8}$ by 360.

What if the measure of the angle is 60°? For any circle centred at the vertex of an angle of this size, the length of the arc within the sides of the angle would be $\frac{1}{6}$ of the entire circle. And 60 is the number got by multiplying this $\frac{1}{6}$ by 360.

Generally speaking, the degree measure of any angle is the number got by first drawing a circle centred at its vertex, dividing the length of the arc within its sides by the circumference, and then multiplying this number by 360.

Another measure of an angle

We have seen what the degree measure of an angle means.



We noted that for circles of different sizes, r and s in the picture above change, but $\frac{s}{2\pi r}$ does not change; and the degree measure of the angle is this fixed number multiplied by 360. In other words,

degree measure of the angle $= \frac{s}{2\pi r} \times 360$.

In this, we can change r and s, but not the numbers 2π and 360. So, isn't it enough if we take $\frac{s}{r}$ as a measure of the angle?

That's right. This gives another measure of the angle, called its radian measure. Thus,

$$\text{radian measure of the angle} = \frac{s}{r}$$

We use the symbol ° to denote the degree measure, right? The radian measure is written rad.

This idea was first proposed by the English mathematician, Roger Cotes in the eighteenth century. The name radian was first used by the English physicist, James Thomson in the nineteenth century.

So, we find the length of the hypotenuse of our original triangle as $2x$ and the length of one of the shorter sides to be x. What about the length of the third side?

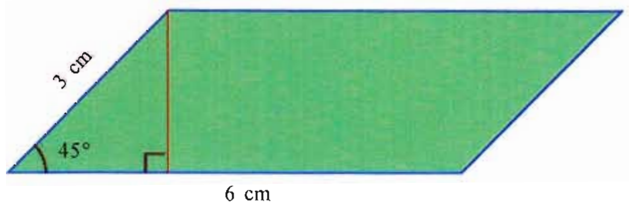
$$\sqrt{(2x)^2 - x^2} = \sqrt{3x^2} = \sqrt{3}x$$

Thus the ratio of the sides of the triangle is $1 : \sqrt{3} : 2$

Let's do some problems using these ideas:

- The adjacent sides of a parallelogram are 6 and 3 centimetres long and the angle between them is $45°$. What is its area?

  To compute the area of a parallelogram, we should know the distance between a pair of parallel sides. Let's draw a figure:

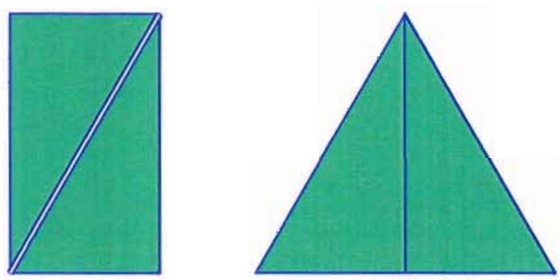


  The vertical side of the right angled triangle in this is $\frac{1}{\sqrt{2}}$ of the hypotenuse. (How?) So, the height of the parallelogram is $\frac{3}{\sqrt{2}}$ centimetres. Thus the area of the parallelogram is $\frac{18}{\sqrt{2}}$ square centimetres. If we are willing to compute a bit more, we can write

$$\frac{18}{\sqrt{2}} = 18 \times \frac{\sqrt{2}}{2} \approx 9 \times 1.414 = 12.726$$

  This gives the area of the parallelogram as 12.73 square centimetres, correct to two decimal places.

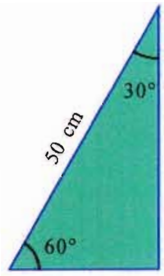
- A rectangular piece of wood is to be cut along the diagonal and the pieces re-arranged to form an equilateral triangle, as shown below and the sides of the triangles should be 50 centimetres long.



What should be the dimensions of the rectangle?

To make an equilateral triangle like this, the triangles got by cutting the rectangle should have angles 30°, 60°, 90°. And the side of the equilateral triangle would be hypotenuse of this right angled triangle.

So, what is our problem? We have to find the lengths of the other two sides of the triangle shown below:



The ratio of the lengths of the sides of this triangle, in the order of size, is $1 : \sqrt{3} : 2$. So, the length of the shortest side is

$$50 \times \frac{1}{2} = 25$$

and the length of the other side is

$$50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}$$

Thus the lengths of the sides of the rectangle should be 25 centimetres and $25\sqrt{3}$ centimetres. If needed, we can compute these correct to millimetres (Try!)

Some more problems are given below. Try your hand at them.

- The area of a parallelogram is 30 square centimetres. One of its sides is 6 centimetres and one of its angles is 60°. What is the length of its other side?
- The sides of an equilateral triangle are 4 centimetres long. What is the radius of its circumcircle?
- One angle of a right angled triangle is 30° and its hypotenuse is 4 centimetres. What is its area?

Degree and radian

Just as centimetre and inch are two of the several units used to measure length, degree and radian are two common units to measure an angle. In the International System of Units abbreviated as SI units, the Unit of angle measurement is taken as radian.

From the equations defining degree and radian, we see that

radian measure of angle

$$= \text{degree measure of angle} \times \frac{180}{\pi}$$

That is,

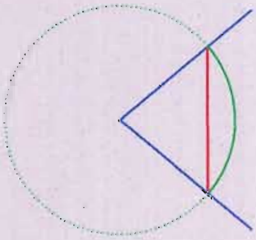
$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.2958°$$

More easy to remember is the conversion formula

$$\pi \text{ rad} = 180°$$

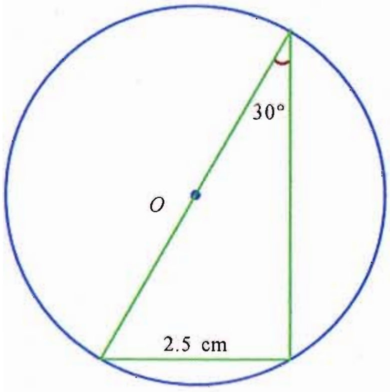
Straightening

In measuring an angle, we actually measure the length of an arc of a circle, whether we use degrees or radians. Instead of this, the Greek astronomer Hipparchus started using lengths of chords, in the second century BC.



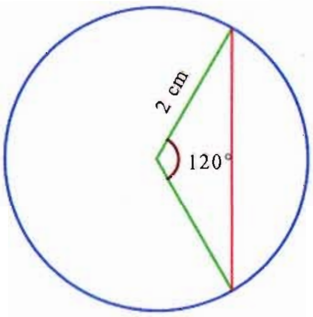
Later mathematicians often refer to a table of chords of various central angles computed by Hipparchus, but this table has not been found. However, such a table of chords done by the Egyptian astronomer Claudius Ptolemy in the second century AD, has been found. He has computed accurately the lengths of chords of central angles up to 180° in a circle of radius 60 units, at $\frac{1}{2}^{\circ}$ intervals.

- In the figure below, O is the centre of the circle.

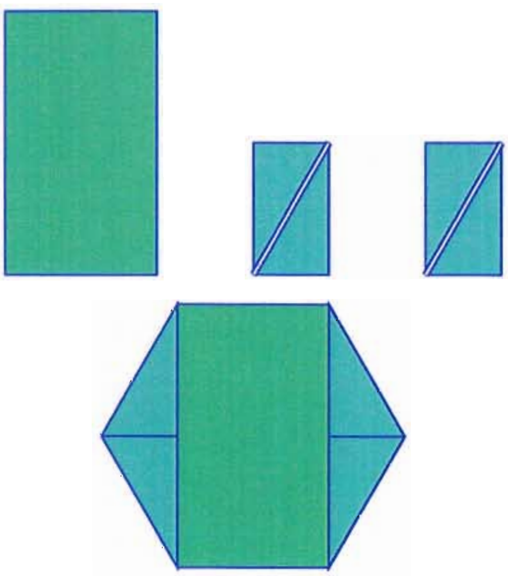


What is the diameter of the circle?

- What is the length of the chord in the figure below?

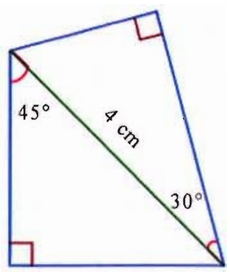


- Two identical rectangles are to be cut along the diagonals and the triangles got joined with another rectangle, to make a regular hexagon as shown below:

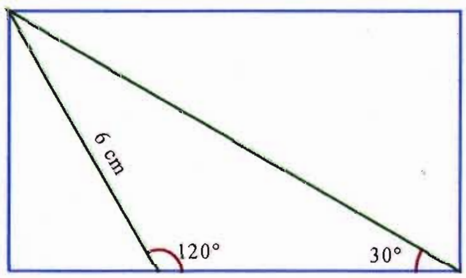


What should be dimensions of the rectangles?

• Compute the lengths of all sides of the quadrilateral below:



• Compute the area of the rectangle below.

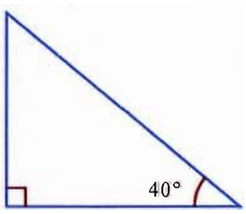


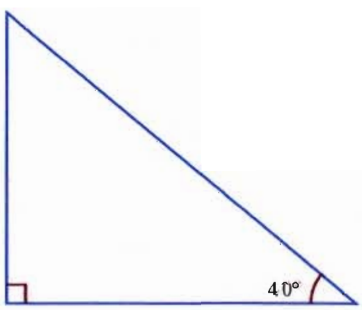
New measures of angles

Recall that our aim is to determine the ratio of the sides of a triangle in terms of its angles. And we have seen how this is done for certain right angled triangles. But this is not so easy for other triangles. Some tables which help us to do this have been computed by mathematicians, quite some time ago. Let's see what these tables are and how we can use them in our task.

First note that for any given angle smaller than a right angle, we can draw any number of right angled triangles having this as one of its angles; and the other angles of all these triangles are also equal.

For example, look at the pictures of some right angled triangles with one angle 40°.

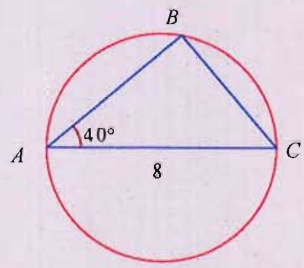




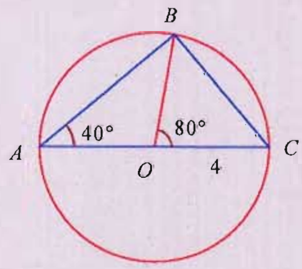
Old methods

Using Ptolemy's tables, we can compute the lengths of the perpendicular sides of a right angled triangle, given its hypotenuse and one angle.

For example, suppose we want to do this for a right angled triangle of hypotenuse 8 centimetres and one angle 40°. What Hipparchus and Ptolemy do is to imagine such a triangle drawn within a circle as shown below:



If we draw the radius to the right angled corner, we get a figure like this:

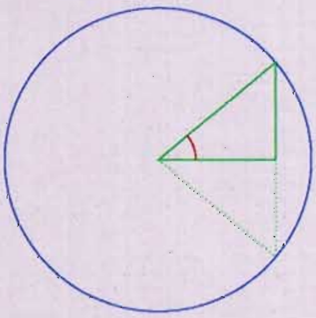


Now using the table of chords, we can find the length of the chord of central angle 80° in a circle of radius 1 unit. Multiplying this by 4 gives one side of our triangle. The other side can be found using Pythagoras Theorem.

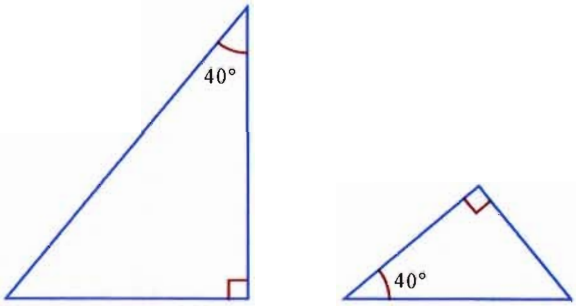
Half-chord

To compute the perpendicular sides of a triangle using Ptolemy's table, we have to double the angle and halve the hypotenuse.

This can be avoided by forming a table which associates with every angle, half the length of the chord of double the angle.



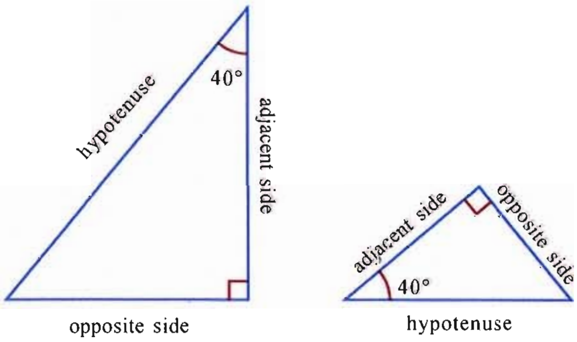
Such a table can be seen in the astronomy text called *Suryasiddhanta*, written in India during the fifth century AD. Operations using such tables can also be found in the book *Aryabhatiya* written by the famous Indian astronomer Aryabhata in the same period. He calls this measure of the angle *ardhajya.* (We have noted in the section **Chord and cord,** of the lesson **Circle** in the Class 9 textbook, that that chord of a circle is called *jya* in Sanskrit.)



Though these are of different sizes, all of them have the same three angles 40°, 50°, 90°. And so the lengths of their sides are in the same ratio.

In other words, the ratio of the lengths of two sides of any one triangle among these, is equal to the ratio of the lengths of the sides in the same position of any other triangle.

To shorten this some what, let's call the shorter of the two sides containing the angle 40°, its *adjacent side*. The longer side is of course, the hypotenuse. The side opposite this angle, naturally enough, we name its *opposite side.*



Then in each of these triangles, the number got by dividing the opposite side of 40° by the hypotenuse is the same; and this has been computed to be about 0.6428. Again, the number got by dividing the adjacent side of 40° by the hypotenuse is also the same for all triangles and this has been computed to be about 0.7660.

These numbers have special names. For example, the number got by dividing the opposite side of 40° by the hypotenuse, in any right angled triangle with this as an angle, is called the *sine* of 40°; and the number got by dividing the adjacent side of 40° by the hypotenuse is called the *cosine* of 40°.

They are shortened as sin 40° and cos 40°.

Thus as mentioned earlier,

$$\sin 40° \approx 0.6428$$

$$\cos 40° \approx 0.7660$$

There are tables which give the sin and cos for all angles less than 90°. A part of it looks like this (The full table is given at the end of the chapter.)

Angle	sin	cos
35°	0.5736	0.8192
36°	0.5878	0.8090
37°	0.6018	0.7986
38°	0.6157	0.7880
39°	0.6293	0.7771
40°	0.6428	0.7660

From this we see for example,

$$\sin 35° \approx 0.5736$$

$$\cos 35° \approx 0.8192$$

What do these mean? In any right angled triangle drawn with one angle 35°, the opposite side of this angle divided by the hypotenuse gives approximately 0.5736; and its adjacent side divided by the hypotenuse gives approximately 0.8192.

Using these names, we can describe the facts about the right angled triangles seen earlier as

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

Can you write sin 30° and cos 30° like this?

Now let's look at some instances of using these tables:

- The hypotenuse of a right angled triangle is 6 centimetres long and one of its angles is 40°. What are the lengths of its other two sides?

What's in a name?

The measure of an angle that is now called sin is the same as what Aryabhata called *ardhajya*. This is how the name evolved.

Arybhata himself, in his later works, drops the adjective *ardha* (meaning half) and writes only *jya* for the half-chord associated with an angle. For a period starting sometime around seventh century AD, the rulers of the Arab countries actively promoted the translation of ancient texts from Greece and India. In the translation of *Aryabhatiya*, the word *jya* was transliterated as *jiba*. Following the custom of not writing the vowels, this was written as *jb* in Arabic.

Later, these Arabic texts reached Europe and were translated into Latin, sometime around the thirteenth century. In supplying the missing vowels to *jb*, the Latin translators mistook it for the word *jaib* which means a bend or a fold. They used the word *sinus* which means the same thing in Latin. During the course of time this became simply sine.

The word cosine comes from what Arybhata calls the *kotijya*.

Kerala math

We have mentioned Madhavan, the fourteenth century Kerala mathematician. (the section **π in Keralam** of the lesson **Circle Measures** in the Class 9 textbook). He discovered a sequence to compute the length of a chord from the length of its arc. Translating what he has written in Sanskrit to modern mathematical notation, his finding is that for any number x, the sequence:

$$x,$$

$$x - \frac{x^3}{1 \times 2 \times 3},$$

$$x - \frac{x^3}{1 \times 2 \times 3} + \frac{x^5}{1 \times 2 \times 3 \times 4 \times 5}$$

....

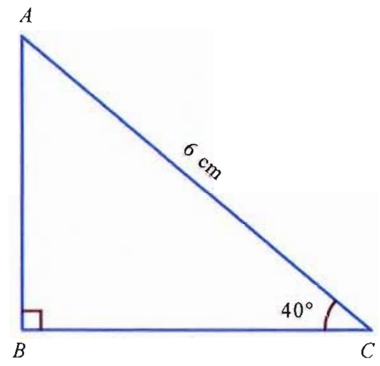
gets closer and closer to $\sin x$, where x is in radians

We can shorten this by writing

$$\sin x = x - \frac{x^3}{1 \times 2 \times 3} + \frac{x^5}{1 \times 2 \times 3 \times 4 \times 5} - \cdots$$

This was re-discovered by Newton in England and Leibniz in Germany, during the seventeenth century.

Let's draw a picture:



From this figure, we get

$$\frac{AB}{AC} = \sin 40° \text{ and } \frac{BC}{AC} = \cos 40°$$

and these give

$$AB = AC \times \sin 40°$$

$$BC = AC \times \cos 40°$$

Now using the given fact that $AC = 6$ centimetres and the values of $\sin 40°$ and $\cos 40°$ got from the tables, we can compute

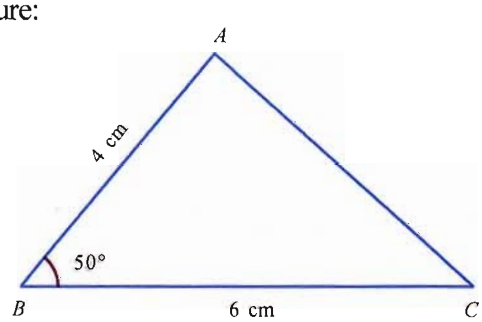
$$AB \approx 6 \times 0.6428 = 3.8568$$

$$BC \approx 6 \times 0.7660 = 4.596$$

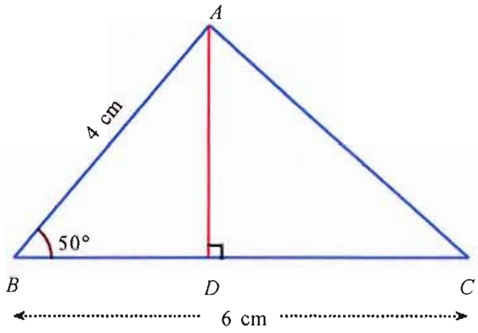
Thus the perpendicular sides of the triangle are approximately 3.9 and 4.6 centimetres long.

- The lengths of two sides of a triangle are 6 centimetres and 4 centimetres; and the angle between them is 50°. What is the area of this triangle?

See this figure:



To compute the area of this triangle, we need the height from a side. Let's draw the perpendicular from the top vertex.



The area of the triangle is

$$\frac{1}{2} \times BC \times AD = \frac{1}{2} \times 6 \times AD = 3 \times AD$$

How can we compute the length of AD? From the right angled triangle ABD in the figure, we get

$$\frac{AD}{AB} = \sin 50°$$

From this we get

$$AD = AB \times \sin 50° = 4 \sin 50°$$

And from the tables we find

$$\sin 50° \approx 0.7660$$

so that we get

$$AD \approx 4 \times 0.7660 = 3.064$$

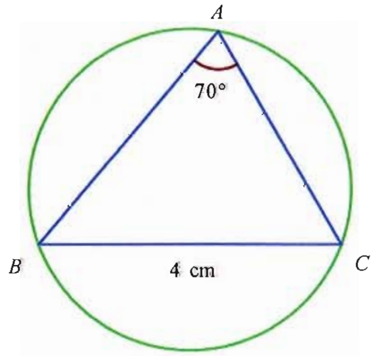
Now we can find the area as

$$3 \times AD \approx 3 \times 3.064 \approx 9.19$$

Thus the area of the triangle is about 9.19 square centimetres.

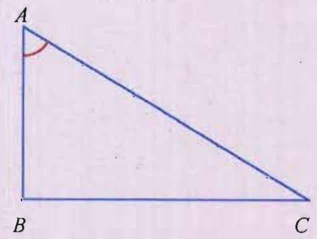
Now take the angle at B as $130°$ instead of $50°$, and compute the area.

- One angle of a triangle is $70°$ and the length of its opposite side is 4 centimetres. What is its circumradius?



Pythagorean relation

See this triangle:



The Pythagoras Theorem applied to it gives

$$AB^2 + BC^2 = AC^2$$

Dividing this equation by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = 1$$

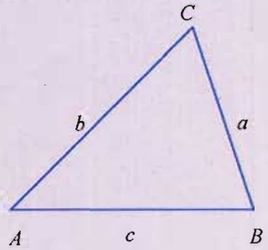
Viewing this relative to $\angle A$, it becomes

$$\cos^2 A + \sin^2 A = 1$$

And this is true for any angle. (The squares of $\cos A$ and $\sin A$ are written $\cos^2 A$ and $\sin^2 A$.)

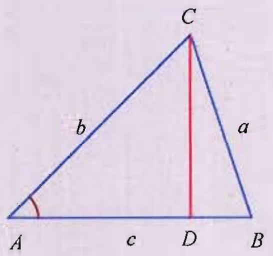
Area

Look at the traingle below.



How do we compute the area of this triangle?

For this, we first draw the perpendicular from C.



Then we have

$$\text{area} = \frac{1}{2} \times AB \times CD$$

Also, from the figure, we see that

$$CD = AC \sin A = b \sin A$$

So, we get

$$\text{area} = \frac{1}{2}\, bc \sin A$$

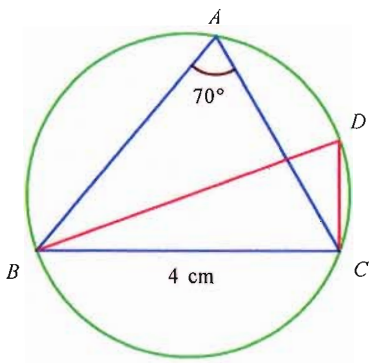
Drawing perpendiculars from other vertexes, we can also see that

$$\text{area} = \frac{1}{2}\, ca \sin B = \frac{1}{2}\, ab \sin C$$

All these expressions for area give the same number, right? Can you find any relation between the sides and angles of a triangle from these?

Have we done recently any such problem, where we had to compute a diameter? Just have another look at the problems done so far in this lesson.

Let's draw the diameter through B in the above figure and join its other end with C:



Then BCD is a right angled triangle (why?) and also, the angles at A and D are equal (reason?) So, $\angle BDC = 70°$. Now from the right angled triangle BDC, we get

$$\frac{BC}{BD} = \sin 70°$$

and from this

$$BD = \frac{BC}{\sin 70°} = \frac{4}{\sin 70°}$$

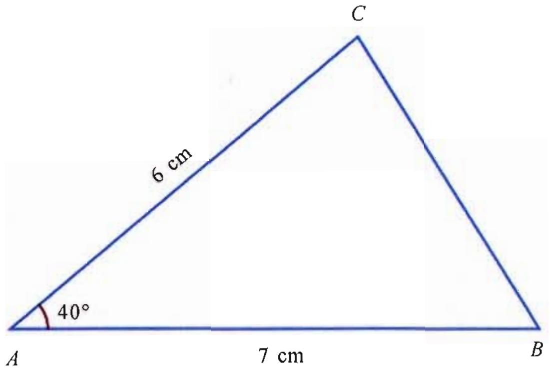
From the tables, we can see that $\sin 70° \approx 0.9397$ and then we can compute

$$BD = \frac{4}{0.9397} \approx 4.3$$

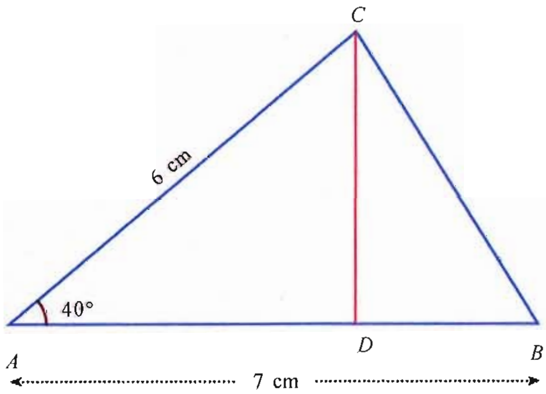
using a calculator. Thus the diameter of the cicumcircle is about 4.3 centimetres.

Can you compute the circum radius if the angle is 110° instead of 70° in this problem?

- Two sides of a triangle are 7 and 6 centimetres long and the angle between them is 40°. What is the length of the third side?



To find the length of BC, the trick is to draw the perpendicular from C to AB.



Now from the right angled triangle BCD, we get

$$BC^2 = BD^2 + DC^2$$

Now let's see how we can compute BD and DC.

From the right angled triangle ACD, we get

$$DC = AC \sin 40° \approx 6 \times 0.6428 \approx 3.86$$

Again from the same triangle,

$$AD = AC \cos 40° \approx 6 \times 0.7660 \approx 4.60$$

This gives

$$BD = AB - AD \approx 7 - 4.6 = 2.4$$

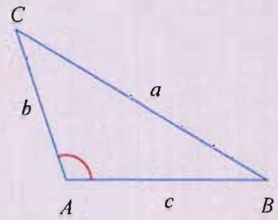
Now we can compute

$$BC = \sqrt{BD^2 + DC^2} \approx \sqrt{3.86^2 + 2.4^2} = 4.54$$

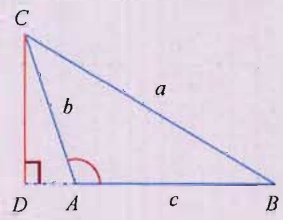
Thus the length of BC is about 4.5 centimetres. What would be the length of BC, if the angle at A is $110°$ instead?

Large angles

What is the area of this triangle?



If we draw the perpendicular from C, we get the figure below:



Here also,

$$\text{area} = \frac{1}{2} \times AB \times CD$$

But we cannot write CD as $b \sin A$ (why not?)

However, from the right angled triangle ADC, we have

$$\angle CAD = 180° - \angle CAB$$

so that

$$CD = b \sin (180° - \angle CAB)$$

Now let's write $\angle A$ instead of $\angle CAB$ (this is anyway the interior angle at A). Then we can write

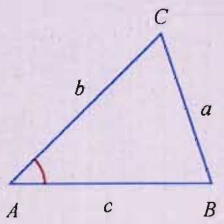
$$\text{area} = \frac{1}{2} bc \sin (180 - A)$$

In general, if $\angle A < 90°$ in ΔABC, then its area is $\frac{1}{2} bc \sin A$; if $\angle A > 90°$, then the area is $\frac{1}{2} bc \sin (180 - A)$.

What if $\angle A = 90°$?

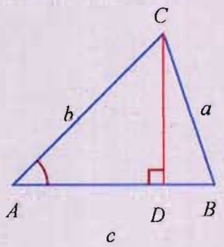
The third side

See this triangle:



Suppose we know the lengths b, c and the angle A. How do we find the length a?

For this also, we need only draw the perpendicular from C.



From the right angled triangle ADC, we get

$$AD = b \cos A \text{ and } CD = b \sin A$$

Now from the right angled triangle BDC, we get

$$a^2 = b^2 \sin^2 A + (c - b \cos A)^2$$

In this, if we write

$$(c - b \cos A)^2 = c^2 + b^2 \cos^2 A - 2bc \cos A$$

and also see that,

$$b^2 \sin^2 A + b^2 \cos^2 A = b^2 (\sin^2 A + \cos^2 A) = b^2$$

then we get

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Now some problems for you:

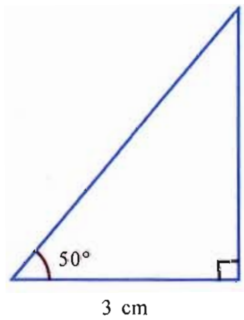
- Without actually drawing figures or looking up tables,can you arrange the numbers below in ascending order?

  $$\sin 1°, \cos 1°, \sin 2°, \cos 2°$$

- The lengths of two sides of a triangle are 6 centimetres and 4 centimetres; and the angle between them is 130°. What is its area?
- One angle of a triangle is 110° and the side opposite to it is 4 centimetres long. What is its circumradius?
- Two sides of a triangle are 7 and 6 centimetres long and the angle between them is 140°. What is the length of the third side?
- Two sides of a parallelogram are of length 6 centimetres and 4 centimetres and the angle between them is 35°. What are the lengths of its diagonals?

Another measure

We want to draw a right angled triangle with one of the shorter sides 3 centimetres long and an angle on it 50°.

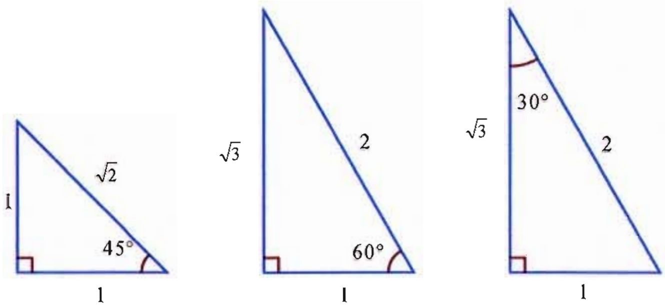


Not a difficult job, right? What is the length of the other short side?

If we look up cos 50° in the table, then we can compute the hypotenuse and then the third side using Pythagoras Theorem.

We can use another table to compute this directly. The numbers got by dividing the opposite side by the adjacent side of an angle in various right angled triangles have also been tabulated.

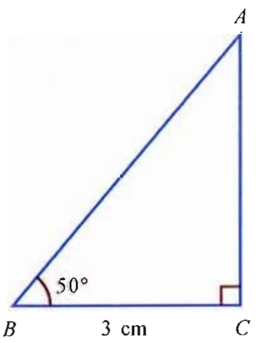
The number thus got is called the *tangent* of the angle and is shortened as tan. As examples, let's look at some of the triangles seen earlier.



We can then see that

$\tan 45° = 1 \qquad \tan 60° = \sqrt{3} \qquad \tan 30° = \frac{1}{\sqrt{3}}$

Let's return to our original problem:



As mentioned just now, we can see that

$$\frac{AC}{BC} = \tan 50°$$

and then compute

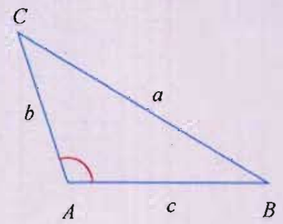
$$AC = BC \times \tan 50° \approx 3 \times 1.1918 = 3.5754 \approx 3.6$$

using the figure and the tables. Thus the required length of the side is about 3.6 centimetres.

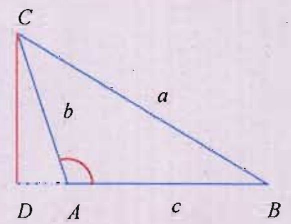
Here's another instance of using the tan measure.

We want to find out how high the man in the picture stands above the ground.

Large angles again



Suppose in the above figure, we know the lengths b, c and the angle A. Does the earlier method of computing the length a of the third side work for this also?



What all things are changed?

From the right angled triangle ADC, what we get here are

$$AD = b\cos\ (180 - A)$$

$$CD = b\sin\ (180 - A)$$

Also, what we get from the right angled triangle BDC becomes

$$BD = c + b\cos(180 - A)$$

With these changes, our earlier technique gives here

$$a^2 = b^2 + c^2 + 2bc\cos(180 - A)$$

What we saw here is a triangle with $\angle A > 90°$. What if $\angle A = 90°$?

New meanings

To compute the sin, cos or tan of an angle, we have to first draw a right angled triangle with this as one of its angles. This is possible only if the angle at hand is less than 90°. Thus so far as we have seen, trigonometric measures are possible only for angles less than 90°.

We have seen that because of this limitation, we need different formulas depending on the size of the angle in various computations, such as finding the area of a triangle or the lengths of its sides. To overcome this, we make new definitions of sin and cos for angles greater than 90°. They are as follows:

$$\sin(180° - x) = \sin x$$

$$\cos(180° - x) = -\cos x$$

We extend the definitions also as

$$\sin 90° = 1$$

$$\cos 90° = 0$$

(Visualize what happens to the opposite side and adjacent side of an angle in a right angled triangle, when it gets closer and closer to 90°)

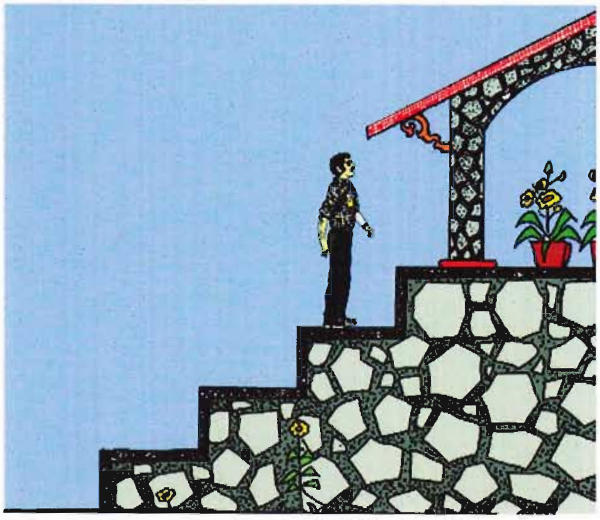
With these new definitions, we can use the single formula

$$\text{area} = \frac{1}{2} bc \sin A$$

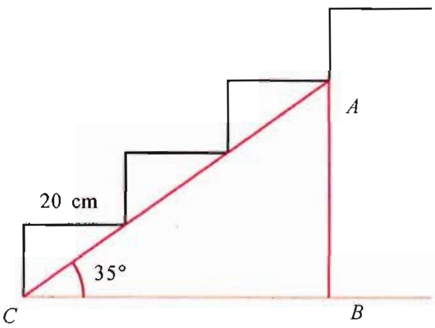
and

$$a^2 = b^2 + c^2 - 2bc \cos A$$

in any triangle with angles A, B, C and sides a, b, c.



The dimensions of the steps are as shown below:



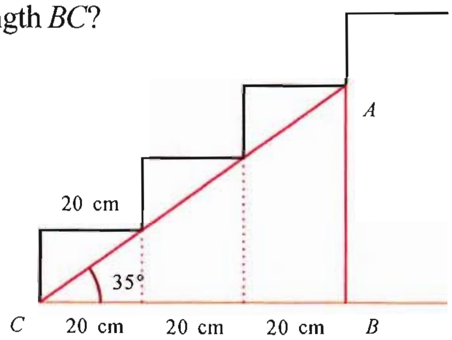
What we are asked to compute is the height AB. From the above figure, we get

$$AB = BC \times \tan 35°$$

and from the tables we get

$$\tan 35° \approx 0.7002$$

What about the length BC?



From the above figure, we can see that the length of BC is 60 centimetres. Thus

$$AB = BC \times \tan 35° \approx 60 \times 0.7002 = 42.012$$

So, the height is about 42 centimetres.

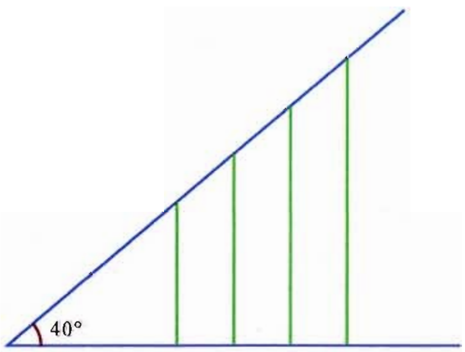
Now some problems for you to do on your own:

- How many rhombuses can we draw with one diagonal 5 centimetres long and one angle 50° ? What are their areas?
- A ladder leans against a wall with its foot 2 metres away from the wall and it makes a 40° angle with the ground. How high is the top of the ladder from the ground?
- Three rectangles are cut along their diagonals and the triangles so got are rearranged to form a regular pentagon as shown below:

30 cm

Find the dimensions of the rectangles.

- The vertical lines in the figure below are drawn 1 centimetre apart.

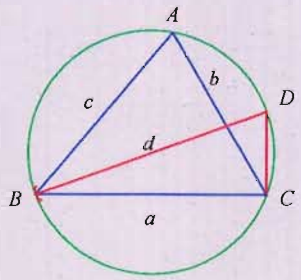


Prove that their heights are in arithmetic sequence.

What is the common difference?

Triangles and circles

See this picture:



BD is a diameter of the circle. So, $\angle BCD = 90°$ and $\angle D = \angle A$. Taking the diameter of the circle as d, we get

$$a = d \sin D = d \sin A$$

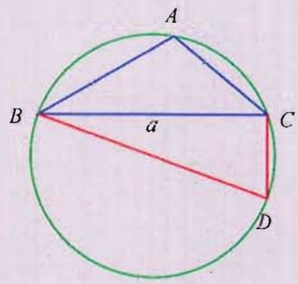
from the right angled triangle BCD. Similarly, we can see that

$$b = d \sin B, \ c = d \sin C.$$

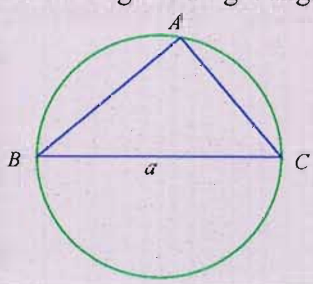
So,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d$$

Would this be right if an angle of the triangle is larger than 90°?

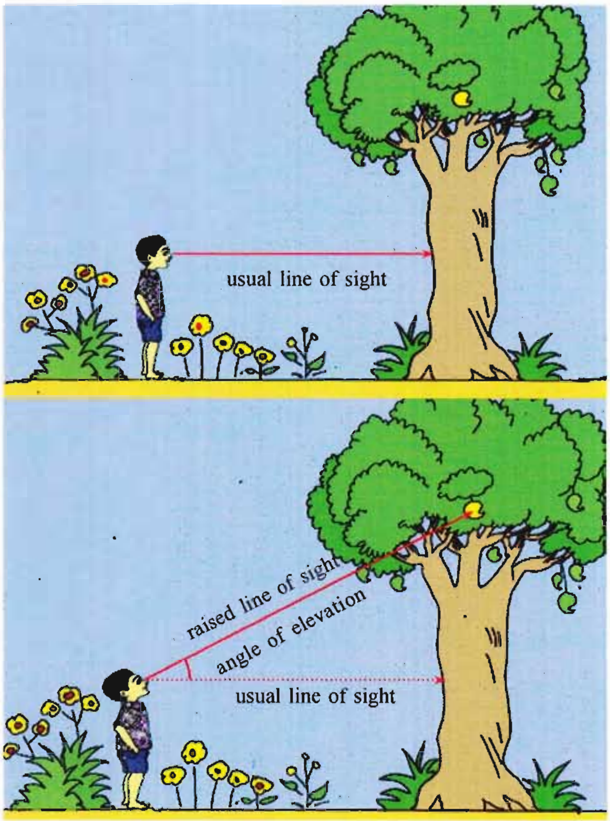


What if one angle is a right angle?



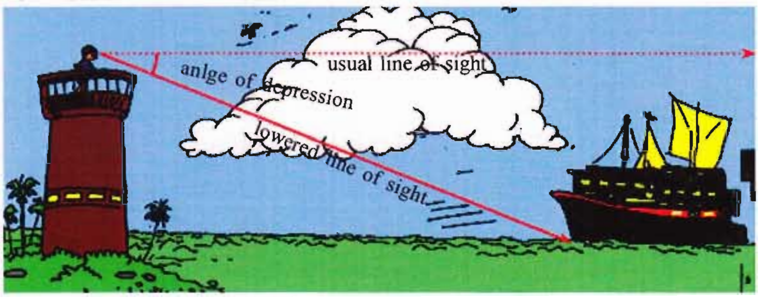
Heights and distances

To see things high up, we have to raise our heads. See these pictures



Our usual line of sight is parallel to the ground and when we look up at something high, this is raised. The angle between these lines is called the *angle of elevation.*

Similarly when we look down from a height, we have to lower our line of vision.



The angle so formed is called the *angle of depression.*

Such angles are measured using an instrument called *clinometer*. Heights and distances which cannot be directly measured are computed by measuring angles using a clinometer and doing calculations using sin and cos.

Problem solved!

We saw that in any triangle (whatever type it is) with angles A, B, C and the lengths of sides opposite them a, b, c, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In other words,

$$a : b : c = \sin A : \sin B : \sin C$$

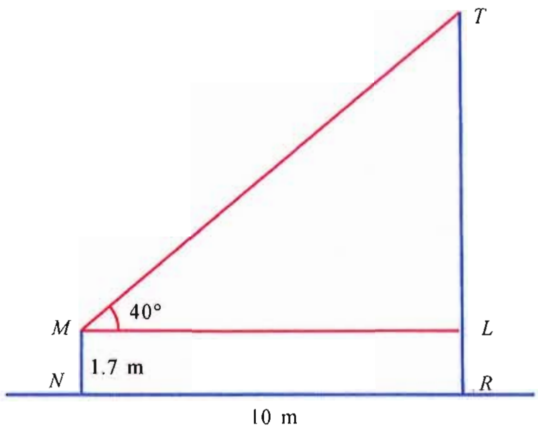
That is, in any triangle, the ratio of sides is equal to the ratio of the sines of the angles opposite them.

We started by observing that in all the various triangles having the same three angles, the ratio of the lengths of sides is the same; and then set out to find this unchanging ratio. Now we have the answer.

Let's look at some examples:

- A man 1.7 metres tall stands 10 metres away from the foot of a tree; and he sees the top of the tree at an angle of elevation 40°. How tall is the tree?

  In the figure below, the line MN denotes the man and TR, the tree.



From the figure and with the help of tables, we get

$$TL = ML \tan 40° \approx 10 \times 0.8390 = 8.39$$

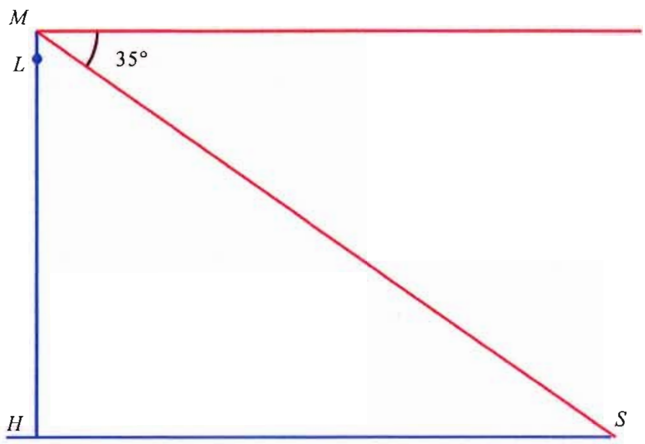
So,

$$TR = TL + LR = TL + MN \approx 8.39 + 1.7 = 10.09$$

Thus the height of the tree is about 10.09 metres.

- A man 1.8 metres tall looks down from the top of a lighthouse 25 metres high and sees a ship at an angle of depression 35°. How far is the ship from the foot of the lighthouse?

  Let's draw a figure first:



In this, LH is the lighthouse and LM is the man standing on top. The point S denotes the ship. Thus we want to find the length HS.

Congruency

We have seen in the lesson **Congruent Triangles** of the Class 8 textbook, that in a triangle if any of the three measures, (i) the lengths of all three sides, (ii) the lengths of two sides and the angle between them or (iii) the length of one side and the two angles on it, are specified, then all the other measures are fixed.

How do we compute them?

Suppose we know the lengths a, b, c of a triangle. Then the angles A, B, C can be computed using relations like

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

If we know the lengths a, b of two sides and the included angle C, then we can first find the length c of the remaining side using the equation

$$c^2 = a^2 + b^2 - 2ab \cos C$$

and then compute the remaining angles as in the first case.

If we know the length a of one side and the two angles B and C on it, then we first find the third angle A by

$$A = 180 - (B + C)$$

and then find b and c using the equations

$$b = \frac{a \sin B}{\sin A}, \; c = \frac{a \sin C}{\sin A}$$

Thus we complete the determination of triangles using trigonometry.

Using the facts given in the problem, we can find

$$MH = ML + LH = 25 + 1.8 = 26.8$$

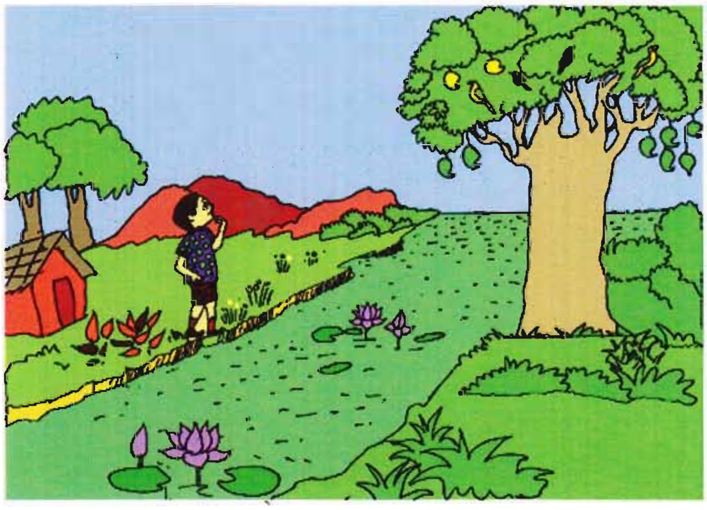
and $\angle HMS = 55°$

So, from the right angled triangle MHS, we get

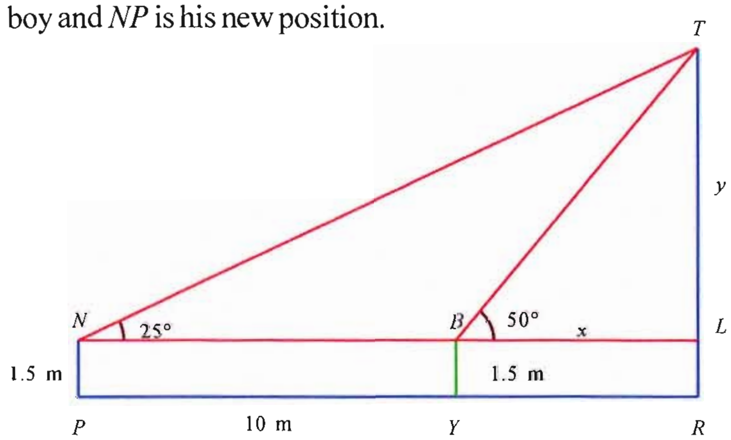
$$HS = MH \tan 55° \approx 26.8 \times 1.4281 \approx 38.27$$

This means the ship is about 38.27 metres away from the foot of the light house.

- A boy, 1.5 metres tall, standing at the edge of a river bank, sees the top of a tree on the the edge of the other bank at an elevation of 50°. Standing back by 10 metres, he sees it at an elevation of 25°. How wide is the river and how tall is the tree?



In the figure below, TR is the tree, BY is the first position of the boy and NP is his new position.



What we want to compute are the lengths of YR and TR.

Slant and spread

We have noted that the trigonometric measures sin and cos arose out of the need to see angle as spread. The measure tan is the result of connecting this method with the need to consider angle as a slant. (Its definition is nothing but the old way of measuring slant as the quotient of rise by run.)

Such a connection was first made by the ninth century Arab mathematician Ahmed ibn Abdallah al Mervazi. He also gave a table of tan measures.

The name tangent for this measure originated only in the sixteenth century. (see the section, **Name and meaning** of the lesson **Tangents**)

From the figure, we get

$YR = BL$ and $TR = TL + LR = TL + 1.5$

So, we need only find BL and TL. If we take $BL = x$ and $TL = y$, then from the right angled triangle BTL, we get

$$y = x \tan 50° \approx 1.1918x$$

and from the right angled triangle NTL,

$$y = (x + 10) \tan 25° \approx 0.4663\,(x + 10) = 0.4663x + 4.663$$

These equations show that

$$1.1918x \approx 0.4663x + 4.663$$

from which we can find

$$x \approx \frac{4.663}{0.7255} \approx 6.427$$

using a calculator. Then we can find

$$y \approx 1.1918 \times 6.427 \approx 7.659$$

Thus the width of the river is about 6.43 metres and the height of the tree is about $7.66 + 1.5 = 9.16$ metres.

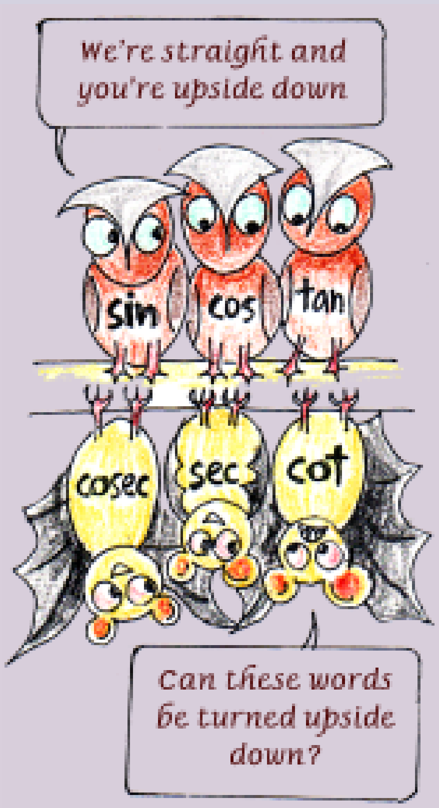
Now you can try these problems:

- The length of the shadow of a tree is 18 metres, when the sun is at an elevation of 40°. What is the height of the tree?
- A man 1.75 metres tall, standing at the foot of a tower sees the top of a hill 40 metres away at an elevation of 60°. On climbing to the top of the tower, he sees the top of the hill at an elevation of 50°. Compute the heights of the hill and the tower.
- A boy 1.5 metres tall, sees the top of a building under construction at an elevation of 30°. The building is completed, adding 10 more metres to its height; and then the boy sees the top at an elevation of 60° from the same spot. What is the total height of the completed building?
- A man 1.8 metres tall, looking down from the top of a telephone tower sees the top of a building 10 metres high at an angle of depression 40° and the foot of the building at an angle of depression 60°. What is the height of the tower? How far is it away from the building?

Other measures

We saw how measures such as sin, cos and tan are defined, by drawing a right angled triangle including the angle and then forming the quotients of the lengths of its sides in various ways. There are some more quotients remaining, which we have not so far considered. These too have special names.

The reciprocals of the sin, cos and tan of an angle are called its *cosecant*, *secant* and *cotangent*; and they are abbreviated as cosec, sec and cot.



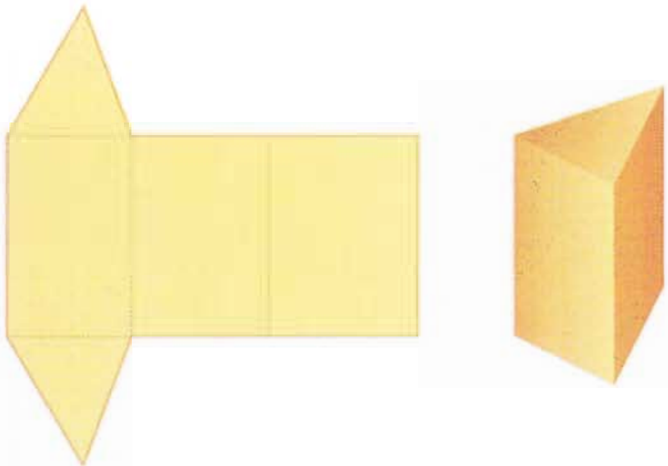
Trigonometric Tables

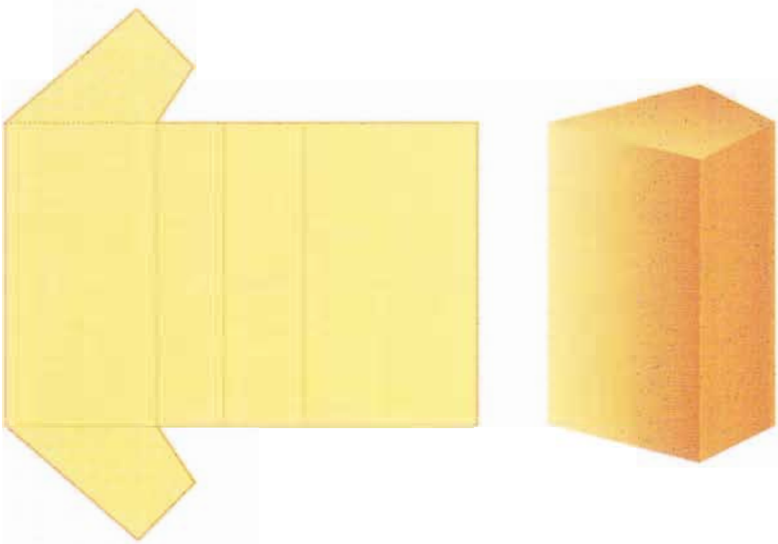
Angle	sin	cos	tan	Angle	sin	cos	tan
0°	0.0000	1.0000	0.0000	**46°**	0.7193	0.6947	1.0355
1°	0.0175	0.9998	0.0175	**47°**	0.7314	0.6820	1.0724
2°	0.0349	0.9994	0.0349	**48°**	0.7431	0.6891	1.1106
3°	0.0523	0.9986	0.0524	**49°**	0.7547	0.6561	1.1504
4°	0.0698	0.9976	0.0699	**50°**	0.7660	0.6428	1.1918
5°	0.0872	0.9962	0.0875	**51°**	0.7771	0.6293	1.2349
6°	0.1045	0.9945	0.1051	**52°**	0.7880	0.6157	1.2799
7°	0.1219	0.9925	0.1228	**53°**	0.7986	0.6018	1.3270
8°	0.1392	0.9903	0.1405	**54°**	0.8090	0.5878	1.3764
9°	0.1564	0.9877	0.1584	**55°**	0.8192	0.5736	1.4281
10°	0.1736	0.9848	0.1763	**56°**	0.8290	0.5592	1.4826
11°	0.1908	0.9816	0.1944	**57°**	0.8387	0.5446	1.5399
12°	0.2079	0.9781	0.2126	**58°**	0.8480	0.5299	1.6003
13°	0.2250	0.9744	0.2309	**59°**	0.8572	0.5150	1.6643
14°	0.2419	0.9703	0.2493	**60°**	0.8660	0.5000	1.7321
15°	0.2588	0.9659	0.2679	**61°**	0.8746	0.4848	1.8040
16°	0.2756	0.9613	0.2867	**62°**	0.8829	0.4695	1.8807
17°	0.2924	0.9563	0.3057	**63°**	0.8910	0.4540	1.9626
18°	0.3090	0.9511	0.3249	**64°**	0.8988	0.4384	2.0503
19°	0.3256	0.9455	0.3443	**65°**	0.9063	0.4226	2.1445
20°	0.3420	0.9397	0.3640	**66°**	0.9135	0.4067	2.2460
21°	0.3584	0.9336	0.3839	**67°**	0.9205	0.3907	2.3559
22°	0.3746	0.9272	0.4040	**68°**	0.9272	0.3746	2.4751
23°	0.3907	0.9205	0.4245	**69°**	0.9336	0.3584	2.6051
24°	0.4067	0.9135	0.4452	**70°**	0.9397	0.3420	2.7475
25°	0.4226	0.9063	0.4663	**71°**	0.9455	0.3256	2.9042
26°	0.4384	0.8988	0.4877	**72°**	0.9511	0.3090	3.0777
27°	0.4540	0.8910	0.5095	**73°**	0.9563	0.2924	3.2709
28°	0.4695	0.8829	0.5317	**74°**	0.9613	0.2756	3.4874
29°	0.4848	0.8746	0.5543	**75°**	0.9659	0.2588	3.7321
30°	0.5000	0.8660	0.5774	**76°**	0.9703	0.2419	4.0108
31°	0.5150	0.8572	0.6009	**77°**	0.9744	0.2250	4.3315
32°	0.5299	0.8480	0.6249	**78°**	0.9781	0.2079	4.7046
33°	0.5446	0.8387	0.6494	**79°**	0.9816	0.1908	5.1446
34°	0.5592	0.8290	0.6745	**80°**	0.9848	0.1736	5.6713
35°	0.5736	0.8192	0.7002	**81°**	0.9877	0.1564	6.3138
36°	0.5878	0.8090	0.7265	**82°**	0.9903	0.1392	7.1154
37°	0.6018	0.7986	0.7536	**83°**	0.9925	0.1219	8.1443
38°	0.6157	0.7880	0.7813	**84°**	0.9945	0.1045	9.5144
39°	0.6293	0.7771	0.8098	**85°**	0.9962	0.0872	11.4301
40°	0.6428	0.7660	0.8391	**86°**	0.9976	0.0698	14.3007
41°	0.6561	0.7547	0.8693	**87°**	0.9986	0.0523	19.0811
42°	0.6691	0.7431	0.9004	**88°**	0.9994	0.0349	28.6363
43°	0.6820	0.7314	0.9325	**89°**	0.9998	0.0175	57.2900
44°	0.6947	0.7193	0.9657	**90°**	1.0000	0.0000	....
45°	0.7071	0.7071	1.0000				

5 Solids

Pyramids

We have seen how we can make various kinds of prisms from paper sheets, by cutting, folding and pasting.





We have also learnt much about prisms.

Now let's make something different. First cut out a figure like the one given below from a thick sheet of paper.

Shapes

Some geometrical shapes such as triangles, rectangles and circles lie flat in a plane; and some such as prisms and cylinders rise up and cannot be confined to a plane.

Prisms and cylinders are seen around us as boxes and blocks and pillars:

We also see solids, which are not prisms.

MATHEMATICS 10

Pyramids of Egypt

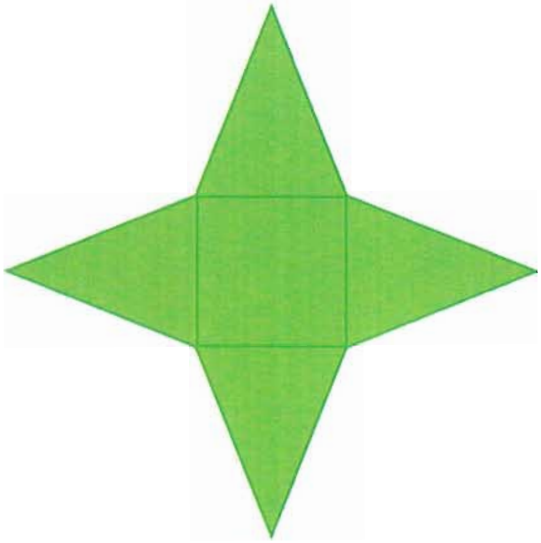
The very name pyramid brings to mind pictures of the great pyramids of Egypt.

About 138 pyramids have been found from various parts of Egypt. Most of them have been built around 2000 BC.

The largest among them is the Great Pyramid of Giza.

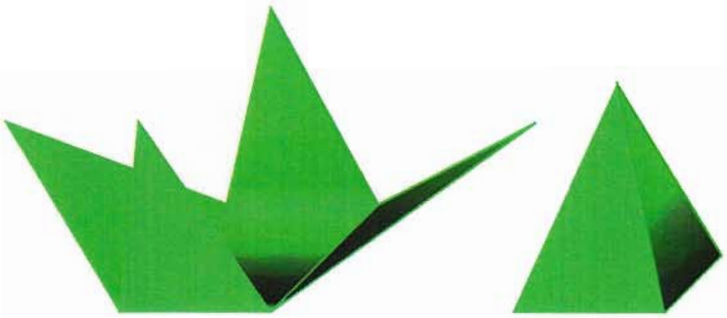
Its base is a square of almost half a lakh square metres; and the height is about 140 metres. It is reckoned that it took almost twenty years to build this.

These royal tombs, starting from a perfect square and rising to a point, erected with gigantic stone blocks, stand as living symbols of human endeavor, engineering skill and mathematical expertise.



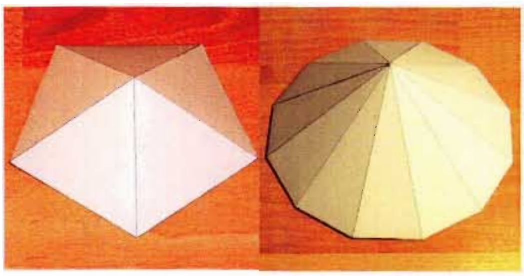
A square in the middle and four triangles around it; and they are (congruent) isosceles triangles.

Fold and paste its edges as shown below:



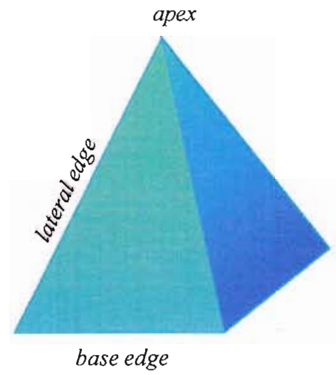
What kind of a solid is this? We cannot call it a prism; a prism has two identical bases and rectangles on its sides. In what we have made, there is a square at the bottom, a point at the top and triangles on the sides.

Instead of a square at the bottom, we can have a rectangle or a triangle or some other polygon. Try! (It looks nicer if the base is a regular polygon.)

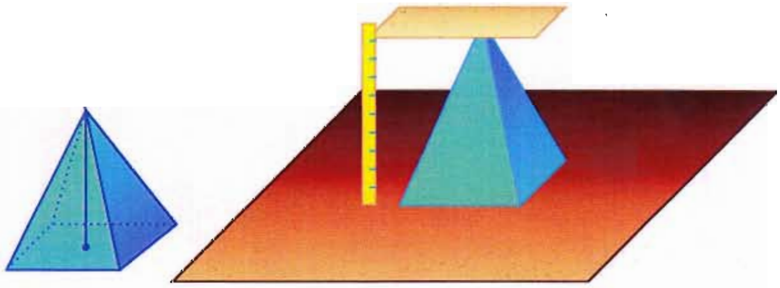


The general name for such a solid is *pyramid*.

The sides of the polygon forming the base of a pyramid are called its *base edges* and the other sides of the triangles are called *lateral edges*. The top end of a pyramid is called its *apex*.



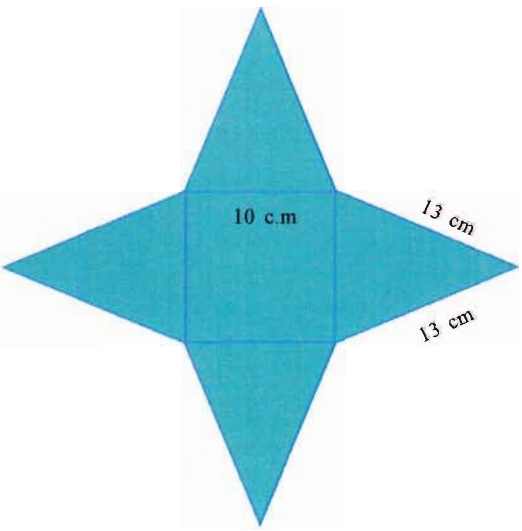
By the height of a prism, we mean the distance between its bases; by the *height* of a pyramid, we mean the perpendicular distance from its apex to its base.



Areas

What is the surface area of a square pyramid with base edges 10 centimetres and lateral edges 13 centimetres?

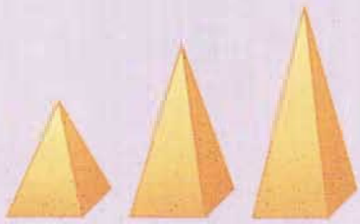
Surface area means the area of the paper needed to make this. How would it look if we cut the pyramid and spread it flat?



Angle and height

To make a square pyramid, we must first fix the base. This also fixes the base of the isosceles triangles on the sides. If we decide on the angle at the top vertex of these triangles also, then the pyramid is completely determined.

As we make this angle smaller, the pyramids become thinner; and we get tall, slender pyramids.



What if we make the angles larger? We get pyramids that are squat and fat:

How large can we make this angle? Can it be 90°?

How large can this angle be for a hexagonal pyramid? And for other pyramids?

We can quickly say that the area of the square in it is 100 square centimetres. What about the triangles?

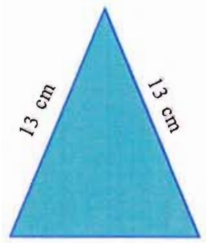
The sides of this triangle are 10, 13 and 13 centimetres. And to find the area using this, Heron comes to our aid: subtract the length of each side from half the perimeter and ...

$$\sqrt{18 \times 8 \times 5 \times 5} = \sqrt{9 \times 16 \times 5 \times 5} = 60$$

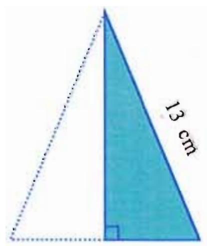
Thus the area of each triangle on the side is 60 square centimetres. So the surface area of the pyramid is

$$100 + (4 \times 60) = 340 \text{ sq.cm}$$

There is another way to compute the area of the triangle.



We need only the height of this triangle, right? Since it is isosceles, the perpendicular from the vertex bisects the base.



Now using Pythagoras Theorem, the length of this perpendicular can be found as

$$\sqrt{13^2 - 5^2} = 12 \text{ cm}$$

And using this, we can find the area of the triangle as $5 \times 12 = 60$ square centimetres.

When the paper is made into a pyramid, what becomes of this height?

Pyramidal numbers

In the section **Triangular numbers,** of the lesson **Square Numbers** in the Class 7 textbook, we saw how dots can be arranged as triangles to get what are called triangular numbers.

In the sequence of numbers we get from this, the n^{th} term is

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$$

as seen in the lesson **Arithmetic Sequences.**

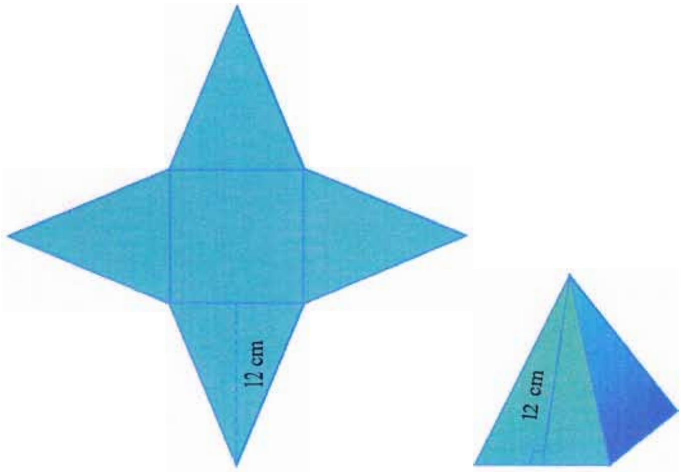
Like this, we can stack small balls in the shape of square pyramids to get a sequence of numbers:

The numbers 1, 5, 14, . . . in this sequence are called *pyramidal numbers*. The n^{th} term of this sequence is

$$1^2 + 2^2 + 3^2 + \dots + n^2$$

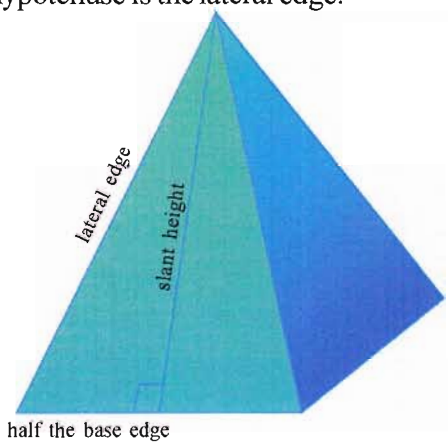
$$= \frac{1}{6}n(n+1)(2n+1)$$

as seen in the section **Sum of squares** of the lesson **Arithmetic Sequences.**



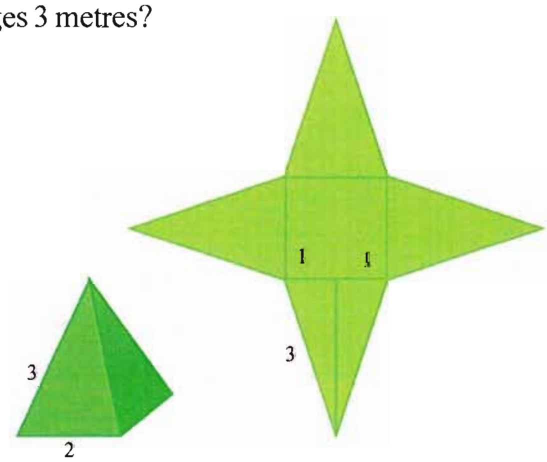
This length is called the *slant height* of the pyramid.

In the problem just done, we saw a relation connecting the base edge, lateral edge and the slant height of a square pyramid. There is a right angled triangle like the one shown below in each face of a square pyramid–its perpendicular sides are half the base edge and the slant height; hypotenuse is the lateral edge.



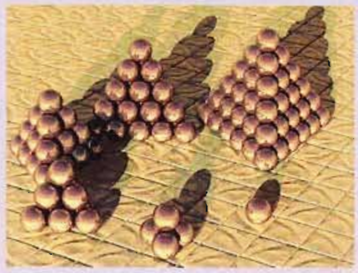
Now can't you do this problem?

What is the surface area of a square pyramid of base edge 2 metres and lateral edges 3 metres?



Tetrahedral numbers

We can also stack small balls as pyramids with equilateral triangle bases.



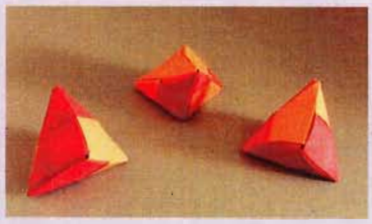
This gives the number sequences 1, 4, 10, ..., that is, each term is a sum of consecutive triangular numbers. It can be proved that the n^{th} term is

$$1 + 3 + 6 + ... + \frac{1}{2}\, n(n + 1)$$

$$= \frac{1}{6}\, n(n + 1)(n + 2)$$

Numbers got thus are called *tetrahedral* numbers.

Tetrahedron is the general name for a solid with four triangular faces:



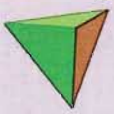
A pyramid based on an equilateral triangle is just a special case of a tetrahedron.

Polyhedrons

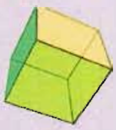
We noted that a solid with four triangular faces is called a tetrahedron. The general name for a solid with all its faces polygons is *polyhedron*.

Prisms and pyramids are special cases of polyhedrons. Cylinders and cones are not polyhedrons.

A polyhedron in which the faces are all regular polygons and in which the number of faces coming together at each vertex is the same, is called a *regular polyhedron*. Euclid has proved that there are only five such:



Tetrahedron



Cube

Octahedron

Dodecahedron

Icosahedron

The base area is 4 square metres. To compute the area of the lateral faces, we need the slant height. In the right angled triangle we just referred to, one short side is half the base edge, which is 1metre and the hypotenuse is the lateral edge, which is 3metres; so the slant height, which is the third side of this right angled triangle is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2} \text{ m}$$

Using this, the area of each triangular face can be found as

$$\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2} \text{ sq.m}$$

Thus the surface area of the pyramid is

$$4 + (4 \times 2\sqrt{2}) = 4 + 8\sqrt{2} \text{ sq.m}$$

If you are not satisfied with this, then you can use a calculator (or recall an approximate value of $\sqrt{2}$) to compute this as about 15.31 square metres.

Now do these problems on your own:

- A square of side 5 centimetres and four isosceles triangles, each of one side 5 centimetres and the height to the opposite vertex 8 centimetres; these are to be joined to make a square pyramid. How much paper is needed for the job?
- A toy in the shape of a square pyramid has base edge 16 centimetres and slant height 10 centimetres. 500 of these are to be painted and the cost is 80 rupees per square metre. What would be the total cost?
- The lateral faces of a square pyramid are equilateral triangles of side 30 centimetres. What is its surface area?

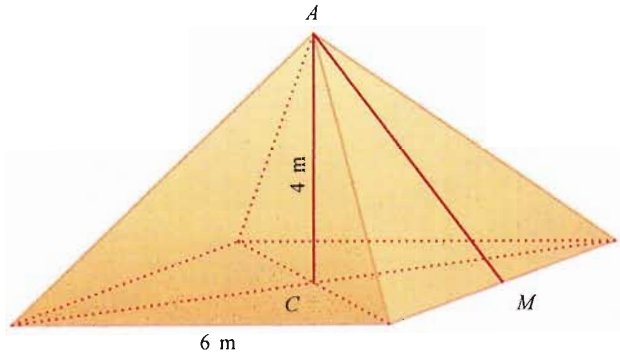
Height and slant height

The height of a pyramid is often an essential measure. See this problem:

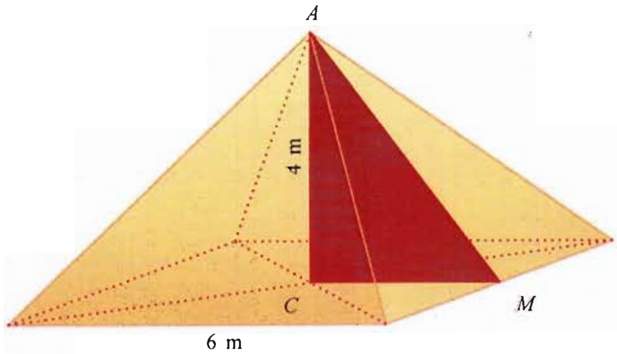
A tent is to be made in the shape of a square pyramid. The sides of its base should be 6 metres and its height should be 4 metres. How much canvas would be needed for this?

To compute the areas of the triangles forming the sides of the tent, we need the slant height. How do we find it from the given facts?

See this picture:



The slant height we need is the length AM. Joining CM, we get a right angled triangle. What is the length of CM?



We can see from the picture that

$$AM = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

So, to make the tent, we need four triangles each of base 6 metres and the height from it 5 metres. Their total area is

$$4 \times \frac{1}{2} \times 6 \times 5 = 60 \text{ sq.m}$$

And this is the area of the canvas needed to make the tent.

What we saw in this problem is true for all square pyramids. Inside any square pyramid, we can visualize a right angled triangle with the slant height as hypotenuse; its perpendicular sides are the height of the pyramid and half the base edge.

Lateral surface area

As in the case of prisms, the sum of the areas of the faces on the sides of a pyramid is called its *lateral surface area.*

If the base of a pyramid is a regular polygon, then the triangles on the sides are all congruent. So, in this case, to find the lateral surface area, we need only multiply the area of one triangle by the number of sides of the base.

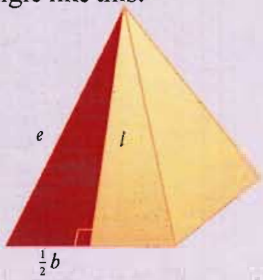
Let's put this in algebra. Suppose the base is a regular polygon of n sides and that each of its sides is of length a. If the slant height of the pyramid is taken as l, then the lateral surface area is

$$n \times \frac{1}{2} \times a \times l$$

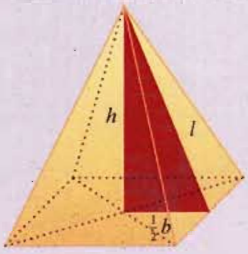
In this, $n \times a$ is the perimeter of the base. Thus the lateral surface area is half the product of the base perimeter and the slant height.

Triangular relations

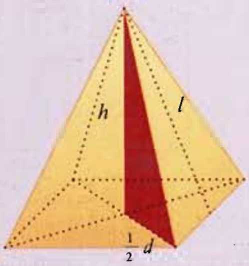
We noted that on each triangular face of a square pyramid, we have a right angled triangle like this:



Also, within the pyramid, another right angled triangle like this:



There's a third right angled triangle within the pyramid, as shown in this picture:



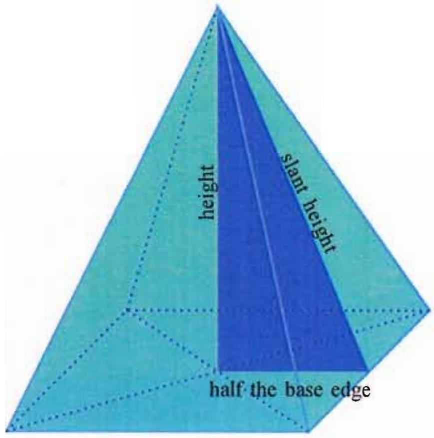
Using these, we get certain relations connecting the length b of the base edge, the length e of the lateral edge, the slant height l, the height h and the length d of the base diagonal:

$$e^2 = l^2 + \frac{1}{4}b^2$$

$$l^2 = h^2 + \frac{1}{4}b^2$$

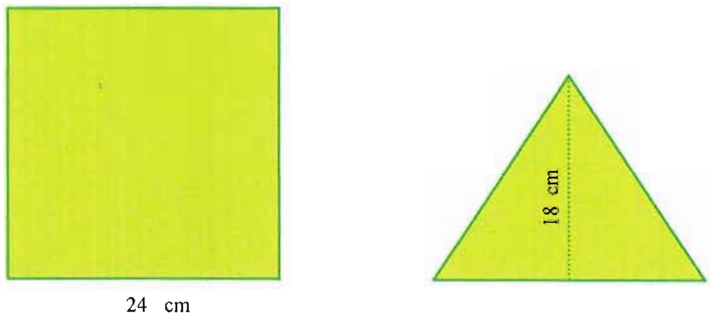
$$e^2 = h^2 + \frac{1}{4}d^2$$

Note that from any two of these equations, we can algebraically derive the third.



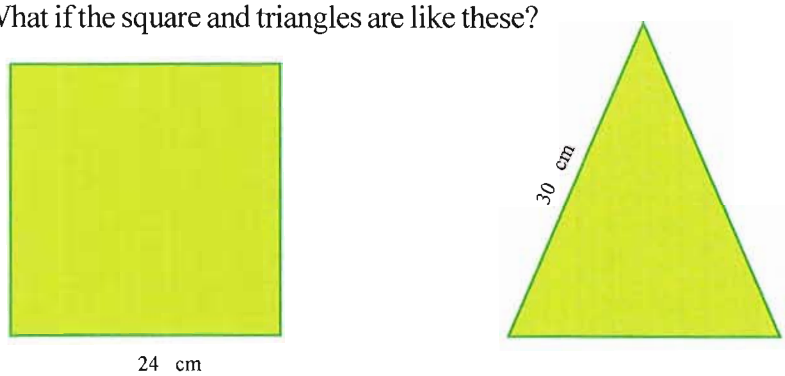
Now can't you do these problems?

- A square pyramid is made using a square and four triangles with dimensions as shown below.



What would be the height of the pyramid?

What if the square and triangles are like these?



- We want to make a paper pyramid with base a square of side 10 centimetres and height 12 centimetres. What should be the lengths of the sides of the triangles?
- Prove that in any square pyramid, the squares of the height, slant height and lateral edge are in arithmetic sequence.

Volume of a pyramid

We have seen that the volume of any prism is the product of its base area and height. What about the volume of a pyramid?

Let's take a square pyramid. First an experiment. Make a square pyramid using stiff paper. Also, a square prism with the same base and height.

Fill the pyramid with sand and pour it into the prism. We would have to do this thrice to fill the prism. Thus we can see that the volume of the pyramid is a third of the volume of the prism. (A mathematical proof of this is given in the **Appendix** at the end of this lesson.)

The volume of the prism is the product of its base area and height. So, what can we say about the volume of the pyramid?

The volume of a square pyramid is a third of the product of its base area and height

For example, the volume of a square pyramid of base edge 10 centimetres and height 8 centimetres is

$$\frac{1}{3} \times 10^2 \times 8 = 266\frac{2}{3} \text{ cu.cm.}$$

Look at this problem:

Each edge of a metal cube is 15 centimetres. It is melted and recast into a square pyramid of base edge 25 centimetres. What would be its height?

The volume of the cube is 15^3 cubic centimetres.

Truncated pyramid

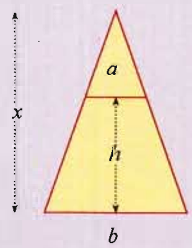
If we cut a square pyramid parallel to its base, we get a smaller square pyramid from the top:

What remains at the bottom?

Such a shape is called a *frustum*; more precisely, here we get a frustum of a square pyramid.

If in such a frustum, we know the lengths of the sides of the squares at the top and bottom and also the height of the frustum, can we compute the height of the original pyramid from which it was cut out?

Look at the triangle got by slicing the whole pyramid vertically through the apex:



From the two similar triangles in this figure, we find

$$\frac{a}{b} = \frac{x-h}{x}$$

from which we get

$$x = \frac{bh}{b-a}$$

Can you fill in the details?

This is also the volume of the pyramid made. We know that the volume of the pyramid is a third of the product of its height and base area. Since the base area of the pyramid is given to be 25^2 square centimetres, we can find a third of the height as $\frac{15^3}{25^2}$ and then the height as

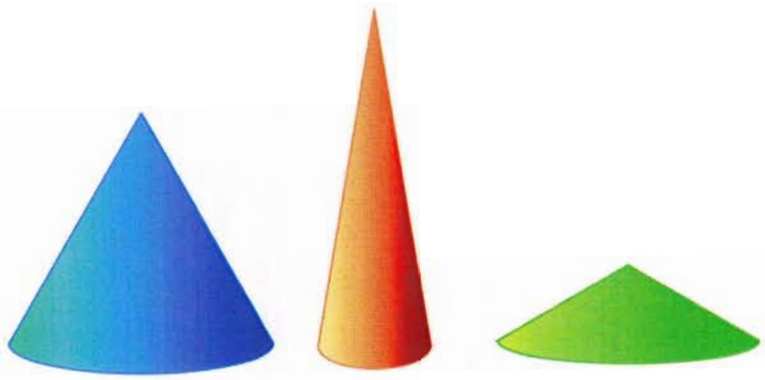
$$3 \times \frac{15^3}{25^2} = 16.2 \text{ cm}$$

Now you can try some problems:

- What is the volume of a square pyramid of base edge 10 centimetres and slant height 15 centimetres?
- In two square pyramids of the same volume, the base edge of one is half the base edge of the other. How many times the height of the pyramid with larger base is the height of the other?
- The base edges of two square pyramids are in the ratio 1 : 2 and their heights are in the ratio 1 : 3. The volume of the first pyramid is 180 cubic centimetres. What is the volume of the second?

Cones

Like cylinders, which can be thought of as prisms with circular base, we have solids like these, which can be thought of as pyramids with circular base.

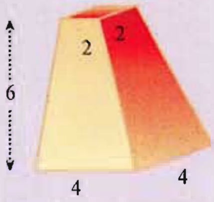


Such a solid is called a *cone*.

Just as we can make a cylinder by rolling up a rectangle, we can make a cone by rolling up a sector of a circle. (If we want a closed cone, then we need another small circle also.)

Volume of a frustum

There is an ancient Egyptian papyrus, dated around BC 1850, in the Pushkin Museum at Moscow. The fourteenth problem in this is to find the volume of a frustum of a square pyramid. Its two square faces are said to have lengths 2 and 4 and the height 6.



The method of computing the volume, given in the papyrus is like this:

Adding the squares of 4 and 2 and twice 4 gives 28; multiplying this by a third of 6, we get 56. This is the volume of the frustum

Let's look at this using algebra, taking a and b for the sides of the squares at the top and bottom, and h for the height. Taking the height of the original pyramid, from which the frustum was cut out as x, its volume would be $\frac{1}{3} b^2x$ and the volume of the smaller pyramid cut out would be $\frac{1}{3} a^2(x - h)$. So that the volume of the frustum is $\frac{1}{3}(b^2 - a^2(x - h))$

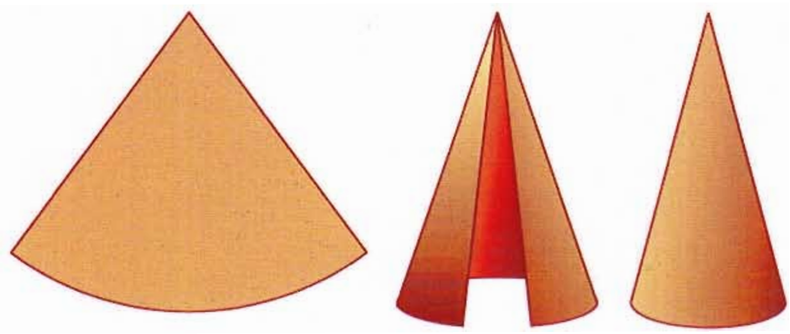
In this, we have already seen that

$$x = \frac{bh}{b - a}$$

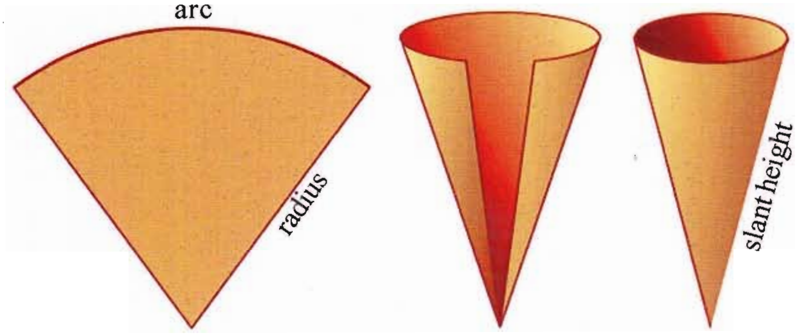
Using this in the expression above and simplifying, we get the volume as

$$\frac{1}{3} h(b^2 + ab + b^2)$$

But this is precisely what the papyrus says.



In this, what's the relation between the dimensions of the sector rolled up and those of the cone made?



The radius of the sector becomes the slant height of the cone; the arc length of the sector becomes the base-circumference of the cone.

We often specify the size of a sector in terms of its central angle. See this problem:

A sector of central angle 45° is cut out from a circle of radius 12 centimetres, and it is rolled into a cone. What is the radius of the base of this cone and what is its slant height?

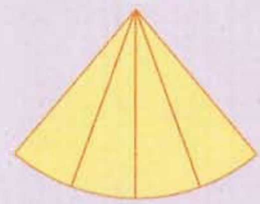
The slant height of the cone is just the radius of the circle, which is given to be 12 centimetres. What about the radius of the base?

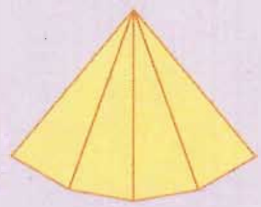
The central angle 45° is $\frac{1}{8}$ of 360°; and the length of the arc of a sector is proportional to the central angle. So, the arc length of this sector is $\frac{1}{8}$ of the circumference of the full circle.

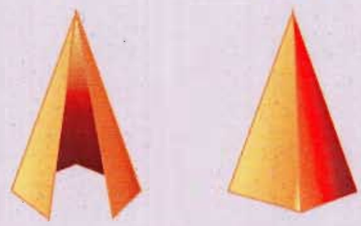
The arc of the circle becomes the circumference of the base circle of the cone. Thus the circumference of the base circle of the cone is $\frac{1}{8}$ the circumference of the large circle from which the sector is cut

Sectors and pyramids

We cannot make a cone by pasting triangles around a circle, as we do for pyramids; but we can make pyramids by bending a sector as we do for cones. See how we make a square pyramid.



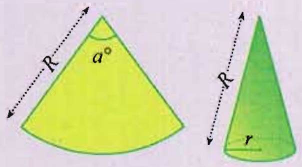




By dividing the sector into more equal parts, can't we make other pyramids also?

Radius and slant height

Suppose a sector of radius R and central angle $a°$ is rolled into a cone.



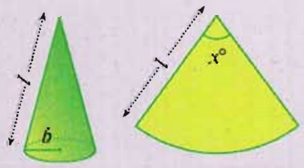
Then the slant height of the cone is R. To find the base-radius, note that the arc-length of the sector is $\frac{a}{360} \times 2\pi R$; and this is the circumference of the base of the cone. So, if we take the base-radius of the cone as r, then

$$2\pi r = \frac{a}{360} \times 2\pi R$$

and this gives

$$r = \frac{a}{360} \times R$$

On the other hand, suppose that a cone of base-radius b and slant height l is cut open and spread flat as a sector.



The radius of the sector is l. And if we take the central angle as $x°$, then we get

$$\frac{x}{360} \times 2\pi l = 2\pi b$$

and from this

$$x = \frac{b}{l} \times 360$$

out. Since the radius of a circle is proportional to the circumference, the radius of the small circle is $\frac{1}{8}$ times the radius of the large circle.

Thus the radius of the base of the cone is $\frac{1}{8} \times 12 = 1.5$ centimetres.

How about a question in reverse?

How do we make a cone of base radius 5 centimetres and slant height 15 centimetres?

To make a cone, we need a sector of a circle. Since the slant height is to be 15 centimetres, the sector must be cut out from a circle of radius 15 centimetres. What about its central angle?

The radius of the small circle forming the base of the cone is to be $\frac{5}{15} = \frac{1}{3}$ of the radius of the large circle from which the sector is to be cut out. So, the circumference of the small circle is also $\frac{1}{3}$ of the circumference of the large circle.

The circumference of the small circle is the arc length of the sector.

Thus the arc of the sector should be $\frac{1}{3}$ of the circle from which it is cut out. So, its central angle must be $360 \times \frac{1}{3} = 120°$.

Now try these problems:

- What is the base-radius and slant height of the cone made by rolling up a sector of radius 10 centimetres and central angle 60°?
- What is the central angle of the sector needed to make a cone of base-radius 10 centimetres and slant height 25 centimetres?
- What is the ratio of the base-radius and slant height of a cone made by rolling up a semicircle?

Curved surface area

Like a cylinder, a cone also has a *curved surface;* the slanted surface rising up.

The area of this curved surface is the area of the sector used to make the cone. (In the case of a cylinder also, the curved surface area is the area of rectangle rolled up to make it, isn't it?)

Look at this problem:

What is the area of the paper needed to make a conical hat of base-radius 8 centimetres and slant height 30 centimetres?

What we need is the area of the circular sector needed to make such a hat. Since the slant height is to be 30 centimetres, the sector must be cut out from a circle of this radius.

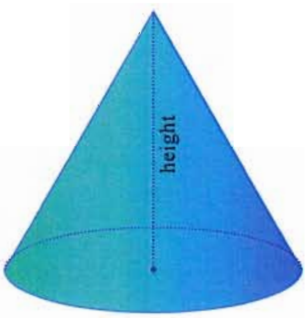
Also, the radius of the small circle forming the base of the cone is to be 8 centimetres; that is, $\frac{8}{30} = \frac{4}{15}$ of the radius of the large circle from which the sector is to be cut out. So, the circumference of the small circle is the same fraction of the circumference of the large circle. But the circumference of the small circle is the arc-length of the sector.

Thus the sector must be $\frac{4}{15}$ of the full circle. So, its area is also this fraction of the area of the circle; that is

$$\pi \times 30^2 \times \frac{4}{15} = \pi \times 2 \times 30 \times 4 = 240\pi$$

So, to make the hat, we need 240π square centimetres of paper. (Doing further computations, we can get this as about 754 square centimetres.)

As in a square pyramid, the *height* of a cone is the perpendicular distance from the apex to the base; and it is the distance between the centre of the base-circle and the apex.



Curved surface area

The curved surface area of a cone is the area of the sector used to make it. If we take the base-radius of the cone as r and the slant height as l, then the radius of the sector is l and its central angle is $\frac{r}{l} \times 360$.

So, its area can be computed as

$$\frac{1}{360} \times \left(\frac{r}{l} \times 360\right) \times \pi l^2 = \pi r l$$

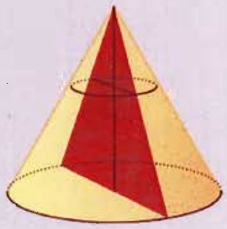
(Recall the computation of the area of a sector, done in Class 9)

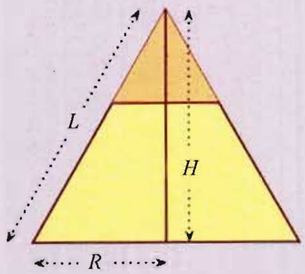
Thus the curved surface area of a cone is half the product of the base circumference and the slant height.

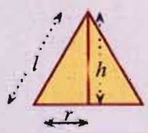
Small and large

If we slice a cone parallel to its base, we get a small cone from the top:

Is there any relation between the dimensions of this small cone and the original large cone?



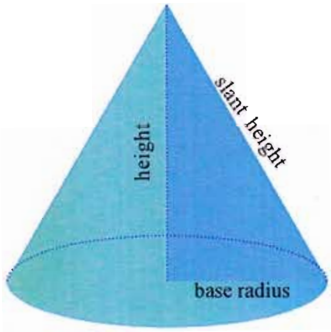




If we denote the base-radius, height and slant height of the large cone as R, H, L and those of the small cone as r, h, l then from the pictures above, we can see that

$$\frac{r}{R} = \frac{h}{H} = \frac{l}{L}$$

Again, as in a square pyramid, there is a relation between base-radius, height and slant height in a cone also, effected by a right angled triangle.



For example, in a cone of base-radius 5 centimetres and height 10 centimetres, the slant height is $\sqrt{5^2+10^2} = 5\sqrt{5}$ centimetres.

Now some problems for you:

- What is the curved surface area of a cone of base radius 12 centimetres and slant height 25 centimetres?
- What is the curved surface area of a cone of base diameter 30 centimetres and height 40 centimetres?
- A cone shaped firework is of base-diameter 10 centimetres and height 12 centimetres. 10000 such fireworks are to be wrapped in colour paper. The price of paper is 2 rupees per square metre. What is the total cost of wrapping?
- Prove that for a cone made by bending a semicircle, the curved surface area is double the base area.

Volume of a cone

Remember the experiment done to find out the volume of a square pyramid? We can do a similar experiment with a cone and a cylinder of the same base–radius and height. And see that the volume of the cone is a third of the volume of the cylinder. That is

The volume of a cone is a third of the product of its base area and height

(A mathematical proof of this also is given in the **Appendix** at the end of the lesson.)

For example, the volume of a cone with base-radius 4 centimetres and height 6 centimetres is

$$\frac{1}{3} \times \pi \times 4^2 \times 6 = 32\pi \text{ cu.cm.}$$

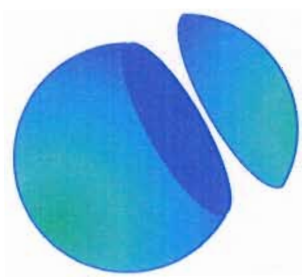
The problems below are for you:

- The base-radius of a cylindrical block of wood is 15 centimetres and its height is 40 centimetres. What is the volume of the largest cone that can be carved out from this?
- A solid metal cylinder is of base-radius 12 centimetres and height 20 centimeters. By melting and recasting, how many cones of base-radius 4 centimetres and height 5 centimetres can be made?
- A sector of central angle 216° is cut out from a circle of radius 25 centimetres and it is rolled up into a cone. What is the base-radius and height of this cone? What is its volume?
- The ratio of the base-radii of two cones is 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?
- Two cones have the same volume and their base - radii are in the ratio 4 : 5. What is the ratio of their heights?

Spheres

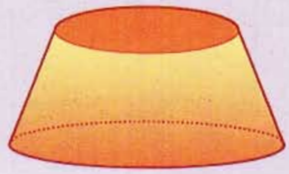
We have watched with thrill the soaring flight of a football or a cricket balls and tasted with relish small round sweets like *laddus*. Now let's look at the mathematics of such round things little or large, called *spheres*.

If we slice a cylinder or cone parallel to its base, we get a circle. In whatever way we slice a sphere, we get a circle.

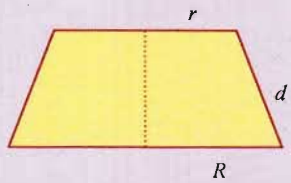


Frustum of a cone

Suppose we slice a cone parallel to its base and cut off a small cone from the top. The part remaining at the bottom is called a *frustum of a cone.*



If we know the radii of the top and bottom circles of such a frustum and also its slant height, can we compute its curved surface area?



If the slant heights of the original large cone and the small cone cut off are denoted L and l, then $d = L - l$ in the figure above. So, the curved surface area of the frustum is

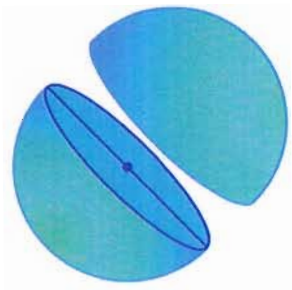
$$\begin{aligned}\pi RL - \pi rl &= \pi(RL - rl)\\ &= \pi\big(R(l+d) - rl\big)\\ &= \pi(Rl + Rd - rl)\end{aligned}$$

Also, we have seen earlier that $\frac{r}{R} = \frac{l}{L}$, which gives $Rl = rL$. Using this, the expression for the curved surface area becomes

$$\begin{aligned}\pi(rL + Rd - rl) &= \pi\big(r(L-l) + Rd\big)\\ &= \pi(rd + Rd)\\ &= \pi(r+R)d\end{aligned}$$

The distance of any point on a circle from the centre of the circle is the same, isn't it? A sphere also has a centre; and the distance of any point on the sphere from this centre is the same. This distance is called the radius of the sphere and double the radius is called its diameter.

If we cut a sphere into two identical halves, we get a circle; and the centre, radius and diameter of this circle are the centre, radius and diameter of the sphere.



We cannot cut open a sphere and spread it flat to compute its area, as with other solids. The fact is that we cannot cut and spread out a sphere without some folds or without some stretching.

But we can prove that the surface area of a sphere of radius r is $4\pi r^2$. (This is given in the **Appendix** at the end of this lesson.)

In other words,

The surface area of a sphere is the product of the square of its radius by 4π.

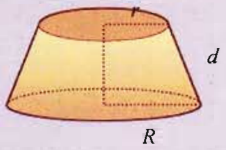
Also, we can prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. (This also is given in the **Appendix**.)

Let's look at some examples:

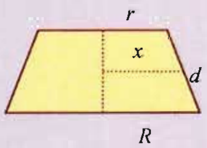
- What is the surface area of the largest sphere that can be carved out from a cube of side 8 centimetres?

Frustum and cylinder

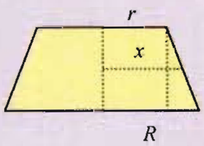
As we have seen, the curved surface area of the frustum shown below is $\pi(r + R)d$



Let's denote by x, the radius of the circle at the middle, as shown in the figure below:



Let's draw one more line as shown below:



From the two similar right angled triangles on the right, we get

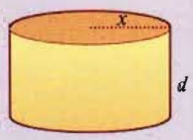
$$\frac{x-r}{R-r} = \frac{1}{2}$$

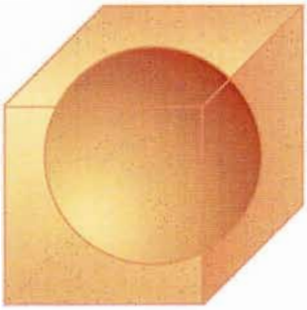
Simplifying this, we get

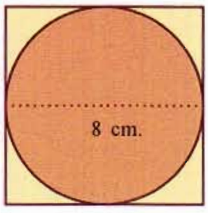
$$x = \frac{1}{2}(R+r)$$

So, the curved surface area of the frustum is $2\pi xd$.

But this is the curved surface area of the cylinder with base-radius x and height d, isn't it?







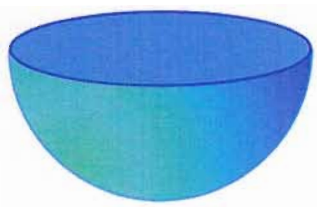
From the pictures above, we see that the diameter of the sphere is the length of an edge of the cube. So, the surface area of the sphere is

$$4\pi\times4^2 = 64\pi \text{ sq.cm.}$$

Let's look at another problem

- A solid sphere of radius 12 centimetres is cut into two equal halves. What is the surface area of the hemisphere so got?

The surface of a hemisphere consists of half the surface of the sphere and a circle.



Since the radius of the sphere is 12 centimetres, its surface area is

$$4\pi\times 12^2 = 576\pi \text{ sq.cm}$$

The surface area of the hemisphere is half of this added to the area of the circle. Since the radius of the circle is also 12 centimetres, its area is

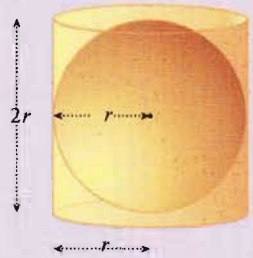
$$\pi \times 12^2 = 144\pi \text{ sq.cm.}$$

Thus, the surface area of the hemisphere is

$$\frac{1}{2} \times 576\pi + 144\pi = 432\pi \text{ sq.cm.}$$

Sphere and cylinder

Consider a cylinder which just covers a sphere:



The base-radius of the cylinder is the radius of the sphere and the height of the cylinder is the diameter of the sphere.

That is, if we denote the radius of the sphere by r, then the base - radius of the cylinder is also r and the height of the cylinder is $2r$. So, the surface area of the cylinder is

$$(2\pi r \times 2r) + (2 \times \pi r^2) = 6\pi r^2$$

The surface area of the sphere is $4\pi r^2$. Thus the ratio of these surface areas is 3 : 2

Again, the volume of the cylinder is

$$\pi r^2 \times 2r = 2\pi r^3$$

and the volume of the sphere is $\frac{4}{3}\pi r^3$. So, the ratio of the volumes is also 3 : 2.

Thus we see that ratio of the surface areas and the ratio of the volumes of a sphere and the cylinder which just covers it, are both equal to 3 : 2.

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Archimedes

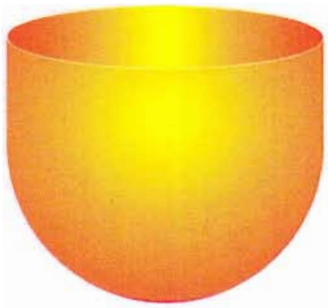
It was the great Archimedes who discovered the fact that a sphere and the cylinder just covering it has the ratio 3 : 2 for surface areas and volumes. It is said that this idea was so dear to him that he asked for this to be engraved on his tomb.

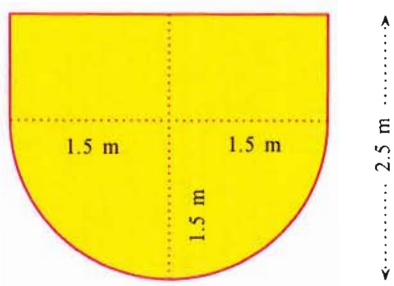
We have seen in Class 8, the story of Archimedes's defence of his home country Syracuse against the invading Roman army. The Romans did conquer Syracuse in BC 212. A soldier, whose name no one remembers, killed Archimedes.

About a hundred and fifty years later, the Roman scholar Cicero, discovered Archimedes's tomb. What helped him was the sight of a stone cylinder and sphere rising above thorns and bush. As an act of penitence, Cicero had the area cleared and paid his respects to one of the greatest scientists ever.

One more example:

A water tank is in the shape of a hemisphere attached to the bottom of a cylinder. Its radius is 1.5 metres and total height is 2.5 metres. How many litres of water can it contain?





The volume of the hemispherical part of the tank is

$$\frac{2}{3}\pi \times 1.5^3 = 2.25\pi \text{ cu.m}$$

and the volume of the cylindrical part is

$$\pi \times 1.5^2(2.5 - 1.5) = 2.25\pi \text{ cu.m}$$

So, the total volume of the tank is

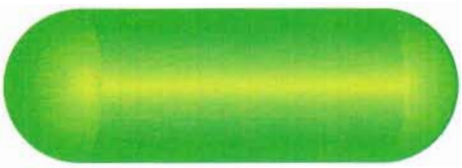
$$2.25\pi + 2.25\pi = 4.5\pi \approx 14.13 \text{ cu.m}$$

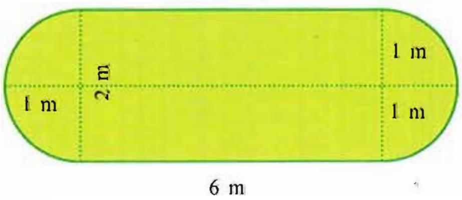
Since a cubic metre is 1000 litres, the tank can contain about 14130 litres.

Now some problems for you:

- The volumes of two spheres are in the ratio 27 : 64. What is the ratio of their radii?

- A metal cylinder of base-radius 4 centimetres and height 10 centimetres is melted and recast into spheres of radius 2 centimetres. How many such spheres are got?
- The picture below shows a petrol tank.





How many litres does it hold?

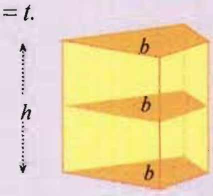
Catch-all

The formulas to compute the volumes of prisms, pyramids and sphere are all different, aren't they? But there is a single formula which works for all these.

Let b be the area of the bottom, m be the area at the middle, t be the area at the top and h be the height of any of these kinds of solids. Then the volume is

$$\frac{1}{6}h(b+4m+t)$$

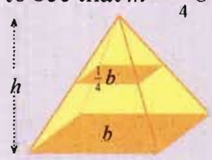
For prisms, the area at the bottom, middle and top are all the same; that is $b=m=t$.



So, by this formula, the volume of a prism is

$$\frac{1}{6}h(b+4b+b)=\frac{1}{6}h\times 6b=bh$$

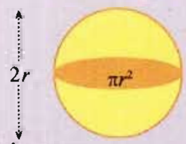
What about pyramids? It is not difficult to see that $m=\frac{1}{4}b$ and $t=0$



So, the volume of a pyramid is

$$\frac{1}{6}h(b+b+0)=\frac{1}{6}h\times 2b=\frac{1}{3}bh$$

Now what about a sphere? If we take the radius as r, then $m=\pi r^2$, $b=t=0$, $h=2r$



So, the volume is

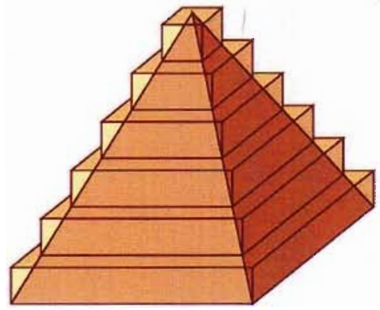
$$\frac{1}{6}\times 2r\times 4\pi r^2=\frac{4}{3}\pi r^3$$

Appendix

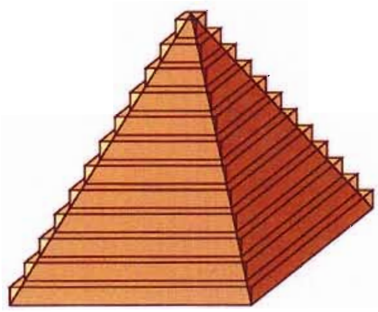
In the lesson, we have mentioned only the techniques of finding the volume of pyramids and cones, as also the surface area and volume of spheres. For those who are interested in knowing how these formulas are actually got, we give below the proofs.

Volume of a square pyramid

We can think of a stack of square plates of decreasing size as an approximation to a square pyramid.



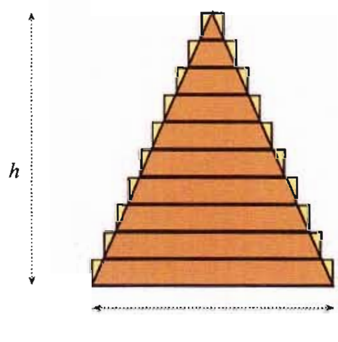
As we decrease the thickness of plates and increase their number, we get better approximations.



So, as we do this, the sum of the volumes of these plates give better and better approximations to the volume of the pyramid.

To start with, suppose we use 10 plates. Each plate is a short square prism. Let's take them to be of equal height. So, if the height of the pyramid we are trying to approximate is h, then the height of each plate is $\frac{1}{10}h$. Now how do we find the base-edge of each plate?

If we slice the pyramid and the enveloping plates vertically through the apex, we get a figure like this:



Starting from the top, we have a sequence of isosceles triangles of increasing size. Their heights increase at the rate of $\frac{1}{10}h$ for each plate. These triangles are all similar (why?) and so their bases also increase at the same rate. Thus if we take the base-edge of the pyramid as b, the bases of the triangles, starting at the top, are $\frac{1}{10}b, \frac{2}{10}b, \ldots., b$

So, the volumes of these plates are

$$\left(\frac{1}{10}b\right)^2 \times \frac{1}{10}h,\ \left(\frac{2}{10}b\right)^2 \times \frac{1}{10}h, \ldots b^2 \times \frac{1}{10}h$$

What about their sum?

$$\frac{1}{10}b^2h\left(\frac{1}{10^2}+\frac{2^2}{10^2}+\ldots+\frac{9^2}{10^2}+\frac{10^2}{10^2}\right) = \frac{1}{1000}b^2h\left(1^2+2^2+3^2+\ldots+10^2\right)$$

We have seen how such sums can be computed in the section **Sum of squares** of the lesson **Arithmetic Sequence**.

$$1^2 + 2^2 + 3^2 + \ldots + 10^2 = \frac{1}{6} \times 10 \times (10 + 1) \times (2 \times 10 + 1)$$

So, the sum of the volumes is

$$\frac{1}{1000}b^2h \times \frac{1}{6} \times 10 \times 11 \times 21 = \frac{1}{6}b^2h \times \frac{10}{10} \times \frac{11}{10} \times \frac{21}{10} = \frac{1}{6}b^2h \times 1.1 \times 2.1$$

Now imagine of 100 such plates. (We cannot draw it anyway.)

The thickness of the plates become $\frac{1}{100}h$; the base-edges become $\frac{1}{100}b, \frac{2}{100}b, \frac{3}{100}b, \ldots$, and the sum of the volumes would be

$$\frac{1}{100^3}b^2h\left(1^2+2^2+3^2+\ldots+100^2\right) = \frac{1}{100^3}b^2h \times \frac{1}{6} \times 100 \times 101 \times 201$$

$$= \frac{1}{6}b^2h \times \frac{100}{100} \times \frac{101}{100} \times \frac{201}{100}$$

$$= \frac{1}{6}b^2h \times 1.01 \times 2.01$$

What if the number of plates is made 1000? Without actually computing, we can see that it is

$$\frac{1}{6}b^2h \times 1.001 \times 2.001$$

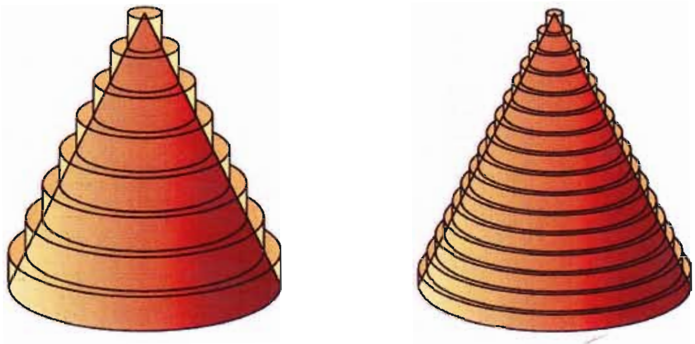
What is the number to which these numbers get closer and closer?

It is the volume of the pyramid. That is,

$$\frac{1}{6}b^2h \times 1 \times 2 = \frac{1}{3}b^2h$$

Volume of a cone

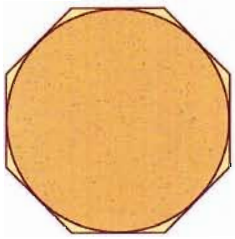
Just as we stacked square plates to approximate a pyramid, we can stack circular plates to approximate a cone.



And in much the same way, we can compute the volume of the cone also. (Try!)

Surface area of a sphere

First consider a circle through the middle of the sphere and a polygon which just contains it. (In more mathematical language, our circle is the incircle of the polygon.)

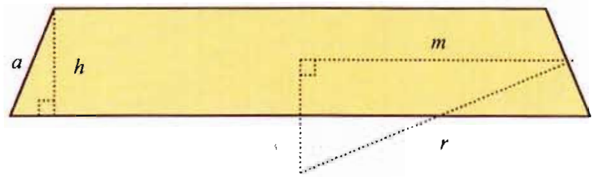


Now if we think of a rotation of these, then we get a sphere and a solid which just covers it.

In the figure above, the solid covering the sphere can be divided into two frustums of cones and a cylinder in the middle.

As the number of sides of the polygon covering the circle increases, the solid generated by rotation approximates the sphere better.

To find the surface area of these frustums, let's consider one of them separately. Let's take the radius of its middle circle as m and its height as h. If we also take the radius of the sphere as r and the length of a side of the covering polygon as a, we get a figure like this, showing all these:



Since the two right angled triangles in it are similar, we get

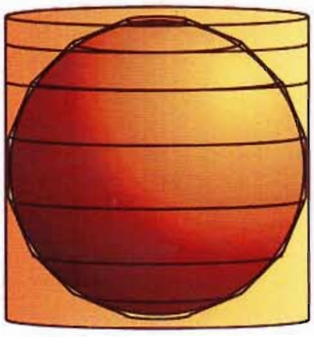
$$\frac{m}{r} = \frac{h}{a}$$

from which we get

$$am = rh$$

The surface area of the frustum got by rotating this is $2\pi ma$, as seen in the section, **Frustum and cylinder**. Using the equation above, we see that this also equal to $2\pi rh$; that is the curved surface area of a cylinder with base - radius r and height h.

So, what do we find? In the solid approximating the sphere, the surface area of each frustum is equal to the surface area of a cylinder with the same height and base-radius equal to that of the sphere. So, the surface area of this solid is the sum of the surface areas of all these cylinders. And what do we get on putting together all these cylinders? A large cylinder that just covers the sphere.



As the number of sides of the covering polygon circle increases, it becomes more and more circle like; and the solid got by rotation becomes more and more sphere like. As seen just now, the surface area of any such shape is equal to the surface area of the cylinder just covering the sphere. So, the surface area of the sphere is also equal to the surface area of the enveloping cylinder. Since the base-radius of the cylinder is r and its height is $2r$, its surface area is

$$2\pi \times r \times 2r = 4\pi r^2$$

and this is the surface area of the sphere also.

Volume of a sphere

See this picture:

The sphere is divided into cells by means of horizontal and vertical circles. If we join the corners of such a cell with the centre of the sphere, we get a pyramid like solid.

The sphere is the combination of these solids and so the volume of the sphere is the sum of the volumes of these solids. Now if we change each of these cells into an actual square which touches the sphere, then we get a solid which just covers the sphere; and this solid is made up of actual square pyramids. All these pyramids have height equal to the radius r of the sphere. If we take the base area of each such pyramid as a, then its volume is $\frac{1}{3}ar$. The volume of the solid covering the sphere is the sum of the volumes of these pyramids. The bases of all the pyramids together make up the surface of this solid. So, if we take the surface area of this solid as s, then its volume is $\frac{1}{3}sr$.

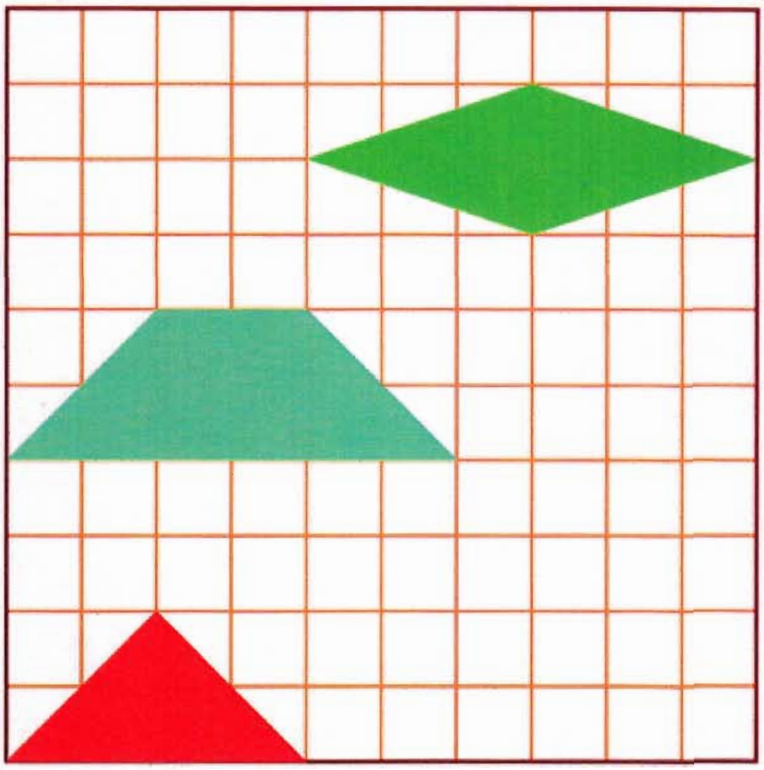
As the size of the cells decreases and their number increases, the solid covering the sphere gets closer to the sphere; and its surface area s to the surface area of the sphere, which is $4\pi r^2$. Thus the volume of the sphere is

$$\frac{1}{3} \times 4\pi r^2 \times r = \frac{4}{3}\pi r^3$$

6 Coordinates

Pictures and numbers

See the picture below.



We have drawn such pictures in Class 9 (see the section **Drawing Polygons**, of the lesson **Polygons**).

How do you make a copy of this picture?

First draw a 10 centimetre square; then draw lines across and down, 1 centimetre apart, and thus divide it into small squares. What next?

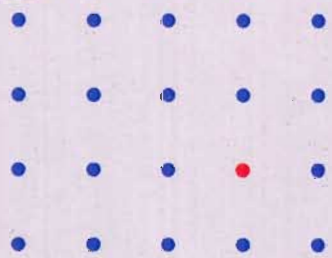
Let's first draw the rhombus at the top. For this, we must mark its four vertices. Where do we put the left vertex?

It is at a point where a line across and a line down meet. Which lines?

Rows and columns

From things arranged in rows and columns, how do we indicate a thing at a specific position? For example, from the books in a shelf, we can say that the one we want is, "the one in the third row from the bottom, fifth from the left", or something similar.

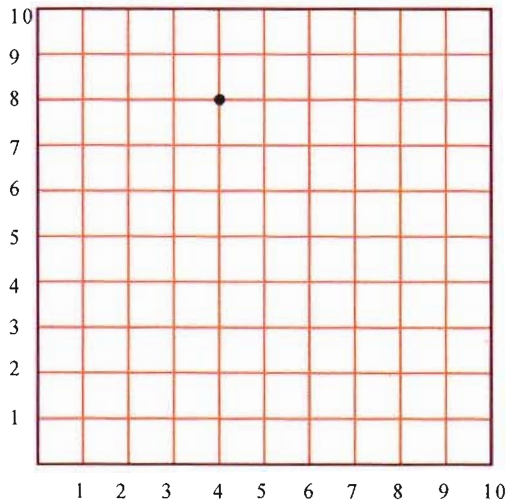
See this picture:



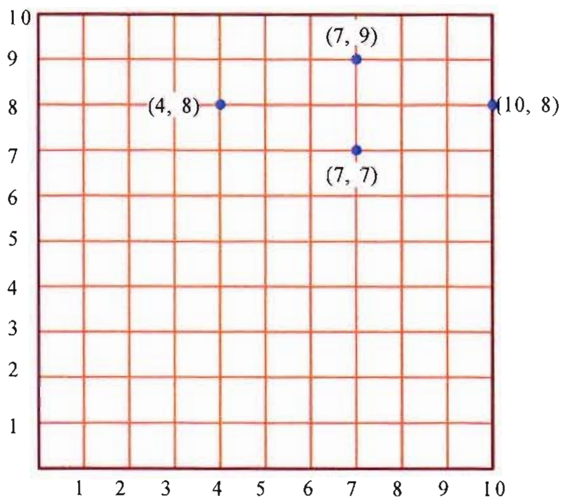
How do we indicate the position of the red dot in this?

We can say, "the dot in the second row from the bottom, fourth from the left". Any other ways of specifying this position?

The line 4 centimetres to the right from the left of the large square and the line 8 centimetres up from its bottom.



Similarly, we can specify the positions of the other vertices, in terms of the distances from the left and the bottom of the large square. For convenience of reference, let's write these numbers alongside the points.



Now we can draw the rhombus. Moreover, we can easily tell the others where to draw it.

Likewise, how do we specify the vertices of the trapezium in the first picture?

Its bottom-left corner is on the left line itself. Let's indicate this line and the bottom line of the square as 0 (Why?)

Position in a table

A table consists of cells arranged in rows and columns. How do we specify a certain cell?

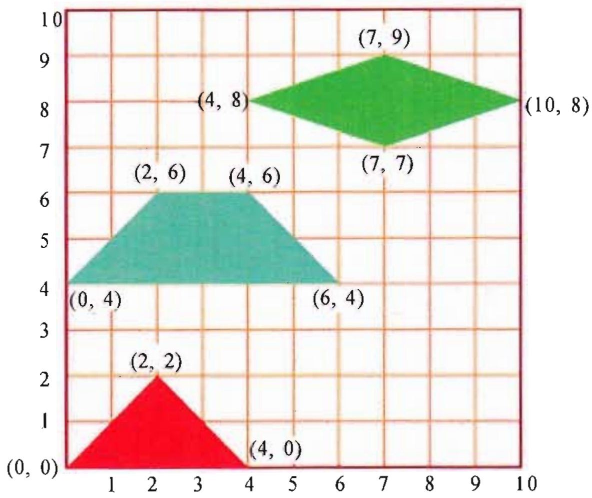
Aren't you familiar with spreadsheet programs such as Open Office Calc? How do you specify cells in a spreadsheet?

The rows of the table are indicated by the numbers 1,2,3 ,.. on the left, running from the top to the bottom; and the columns are indicated by the letters A, B, C, ... at the top, running left to right. Using these two, we can specify any cell.

For example, in the picture above, the number 100 is in the cell F6.

So, how do we write the bottom-left vertex of our trapezium? What about its other vertices?

What numbers give the vertices of the triangle in the picture?

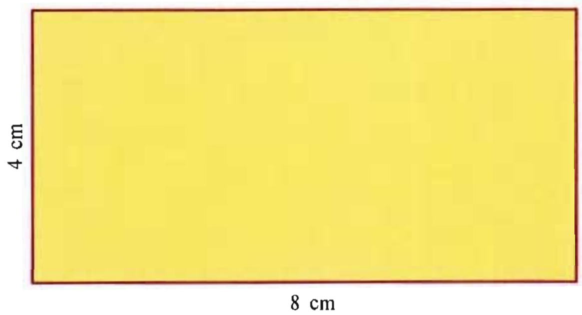


Now make a new grid like this and draw the figures listed below; also indicate the positions of their vertices using numbers as done above:

- a non-isosceles triangle
- a parallelogram which is not a rhombus
- a non-isosceles trapezium
- a pentagon
- a hexagon

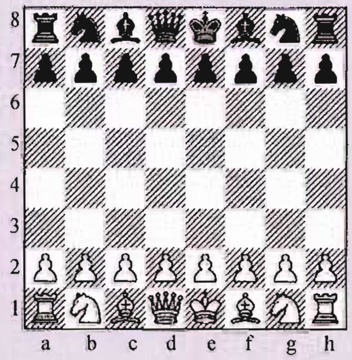
More on numbers and pictures

From the middle of a rectangle 8 centimetres wide and 4 centimetres high, we want to cut out a rectangle 4 centimetres wide and 2 centimetres high.



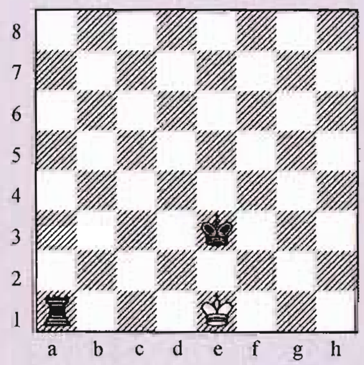
Chess positions

Have you noticed how the positions on the board are specified in descriptions of chess matches?

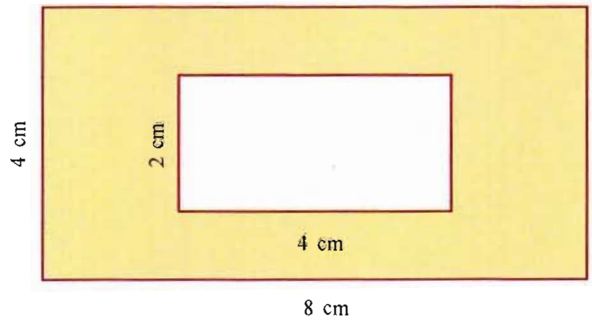


The rows are named with numbers and the columns with letters, as shown above.

Now see this picture:



The position of the White King is e1 and the position of the Black King is e3. The Black Rook is at a1.

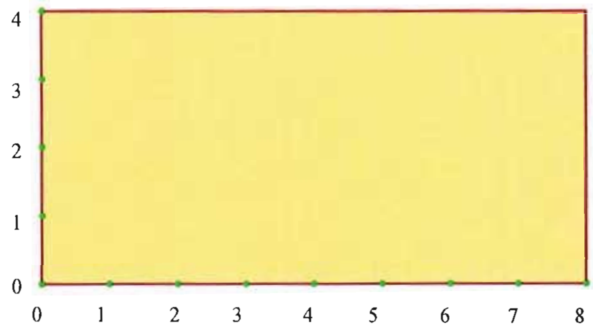


We first mark the vertices of the rectangle to be cut out in the large rectangle.

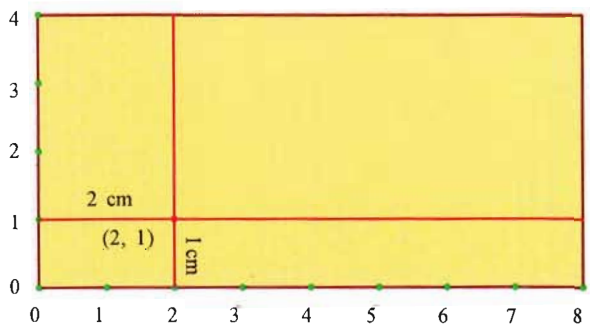
Since the small rectangle is to be right in the middle of the large rectangle, its left and right sides must be at the same distance from the left and right sides of the large rectangle; the same goes for the top and bottom sides.

At what distance?

Let's mark distances along the left and bottom lines of the large rectangles, one centimetre apart.



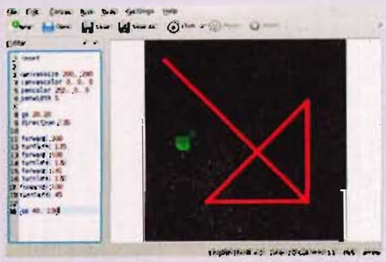
Now how do we mark the bottom-left corner of the rectangle to be cut off?



We can mark the other vertices also like this.

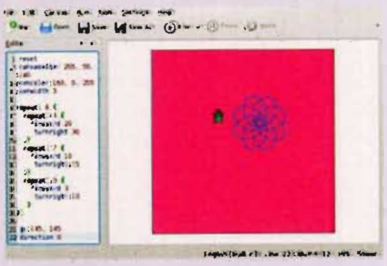
Computer pictures

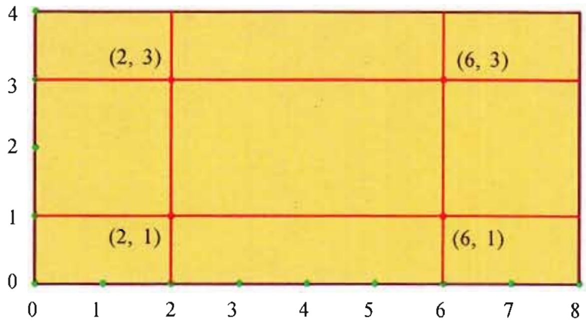
There's a simple program called KTurtle in Gnu/Linux which helps us to draw geometrical shapes with a computer. It is done by specifying positions on the screen in terms of numbers.



The left pane in the picture shows the code used to draw the picture in the right pane.

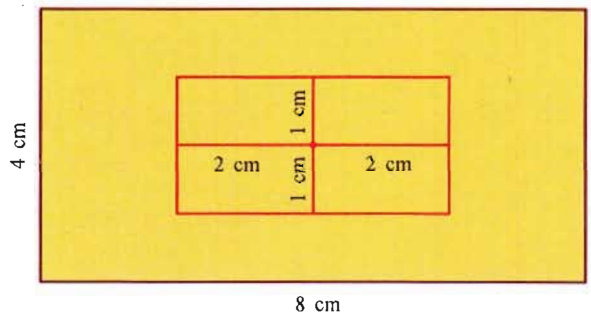
With a little effort, complex diagrams can also be drawn with it.



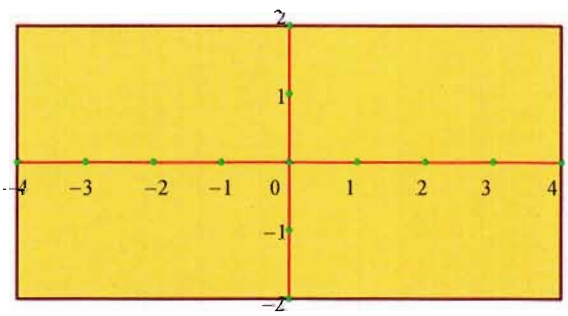


Now we can draw the rectangle and cut it out.

We can think about this in a slightly different way. The left and right sides of the rectangle to be cut out are 2 centimetres to the left and right of the centre of the large rectangle; its top and bottom are 1 centimetre up and down this point.



To specify the vertices of the small rectangle in this manner, let's draw a pair of lines across and down the large rectangle, through its centre. To distinguish between left-right and up-down directions, we write distances towards the left and distances down as negative numbers. (Remember how we marked the positions of numbers on the number line?)



How do we specify with respect to these lines, the vertices of the rectangle to be cut out?

Typesetting

Computers are now extensively used in typesetting–that is, the design of pages to be printed, specifying where in the page each letter or picture should be placed.

A language used for this purpose is PostScript. It uses numbers to specify the various locations in a printed page.

Let's look at an example. Use some text editor such as gedit in Gnu/ Linux and type the lines below:

```
newpath
20 20 moveto
40 20 lineto
40 40 lineto
20 40 lineto
closepath
fill
showpage
```

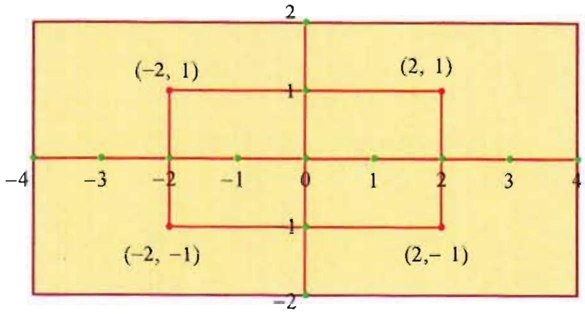
This is a sample of the PostScript language. To see what has been drawn, we can use a program such as gv. Save the file as **test.ps** and then run the command

gv test.ps

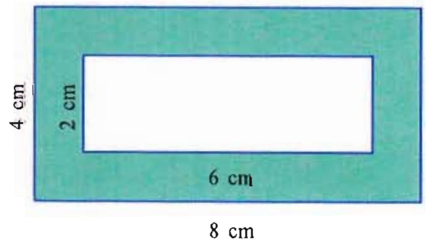
in a terminal. A small black square at the bottom left of a white screen will be displayed.

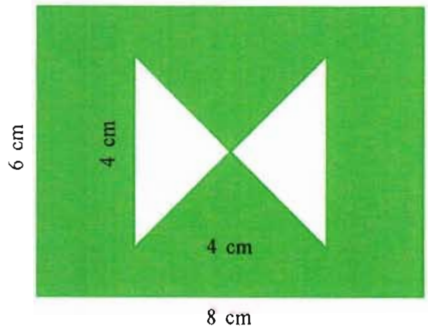
The number pairs used in this code are all distances from the left and bottom sides of the page. Distances are measured in point, which is the usual unit in printing. One point is about 0.035 centimetres.

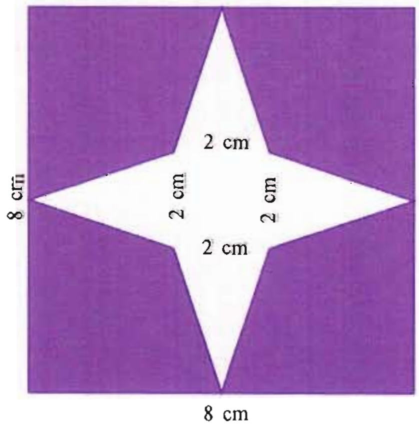
In most DTP applications, PostScript works invisibly in the background.



Now write the points to be marked for cutting out the figures shown below, using measurements from the centre of the rectangle, as done in the last example above.



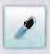




Colours and numbers

In a computer, not only positions on the screen, but the colours used are also specified by numbers. Various colours are produced by mixing different amounts of the basic colours red, green and blue.

This can be easily understood using the application Gcolor2 in GNU/Linux.

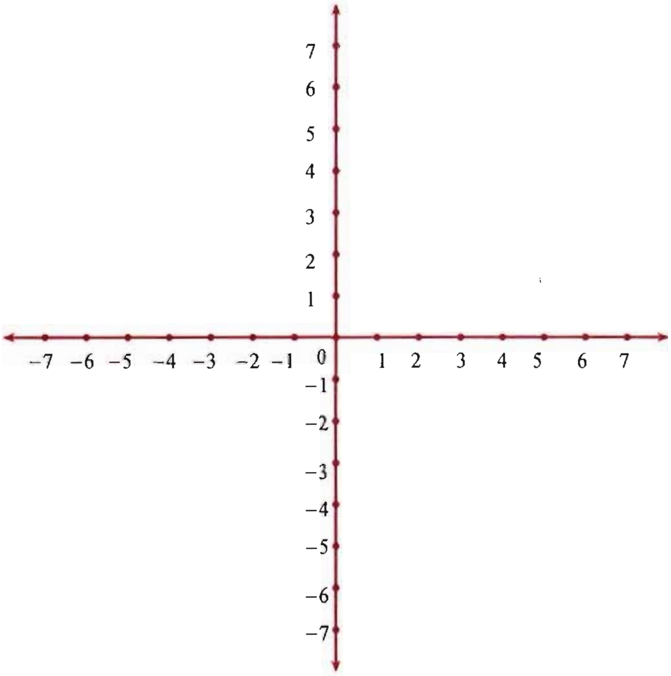
By first clicking on the tool  and then clicking on any part of the screen, we get the RGB values of the colour at that point.

Positions and numbers

We have seen some examples of how the positions of certain points in a plane can be specified by means of a pair of numbers. What does each pair of numbers represent?

The distances of the point from a pair of perpendicular lines, right?

See the figure below:



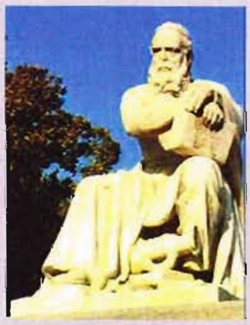
A pair of perpendicular lines, with distances from the point of intersection marked along each. Positive and negative numbers are used, to distinguish between right-left and up-down directions. The markings need not be 1 centimetre apart; the only requirement is that the distance between adjacent points should be the same throughout. In other words, we can use any unit for measuring distance. (We can think of these as two number lines drawn perpendicular to each other and sharing a common zero.)

Using the distances from these lines, we can specify any point in the plane.

A bit of history

The technique of specifying the positions of points using numbers was used as early as the second century BC by the Greek mathematician Appollonius, in his solution of certain geometric problems. These numbers were the distances of the points from some fixed lines.

In the eleventh century AD, the Persian mathematician and poet, Omar Khayyam used the method of representing number pairs as points to convert certain algebraic problems to geometry.



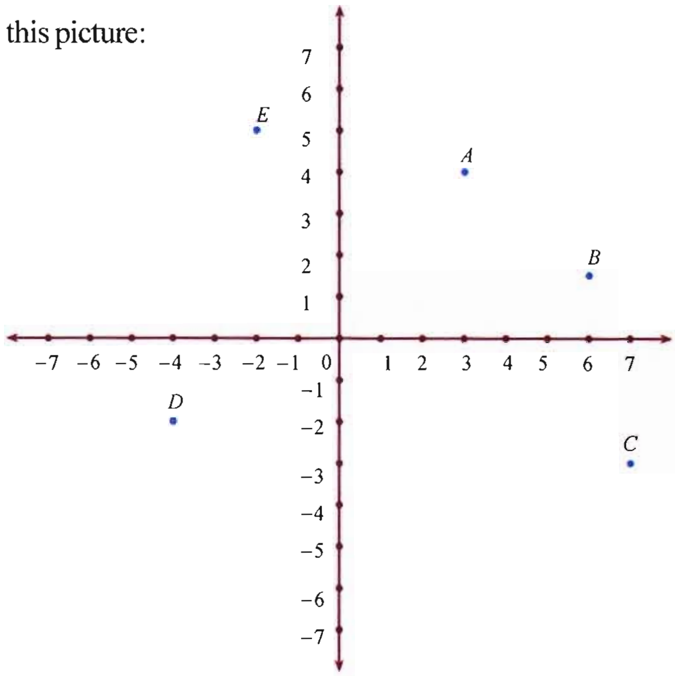
This relation between geometry and algebra developed as a definite branch of mathematics after the French mathematician and philosopher, Ren´e Descartes published his "Geometry", in the seventeenth century.

Geometry and a fly

There's an intersesting story told about Descartes discovery of his new method in geometry. Lying on the bed and thinking, he notices a fly crawling on the ceiling. He is hit by the idea that to trace its path, he need only know how its distances from two adjacent walls change. And this leads him to a method of reasoning in geometry.

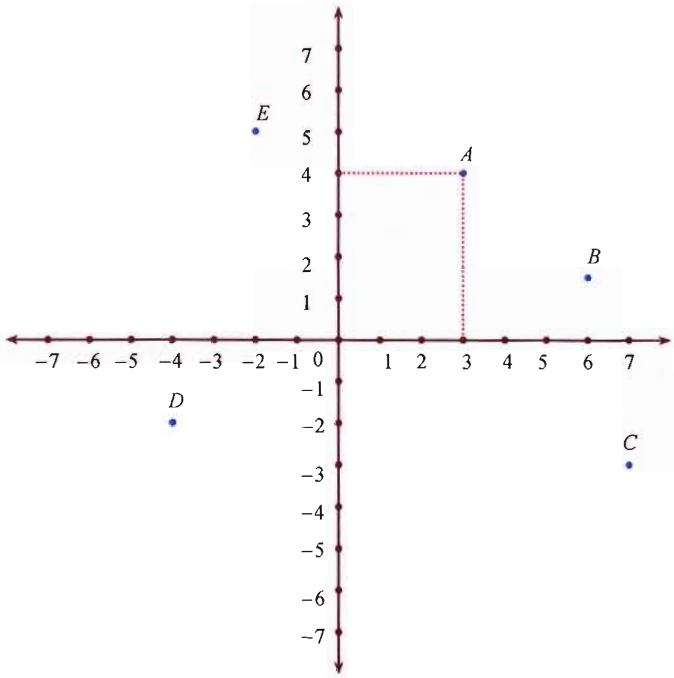
Whether this story is true or not, it is a nice illustration of the fact that using the distances from two reference lines, we can indicate the position of any point in the plane and that geometric figures can be described using such pairs of numbers.

See this picture:



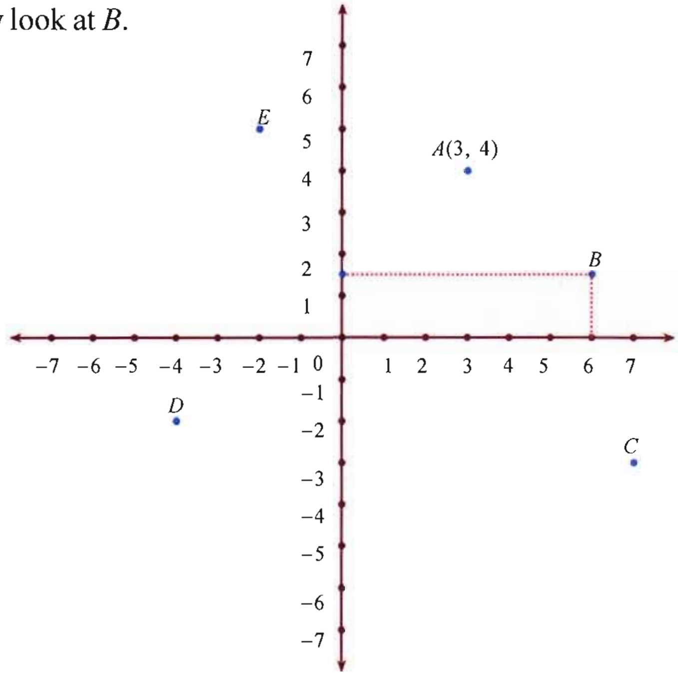
What are the number pairs of the points A, B, C, D, E?

Take A. Draw perpendiculars from A to the reference lines.



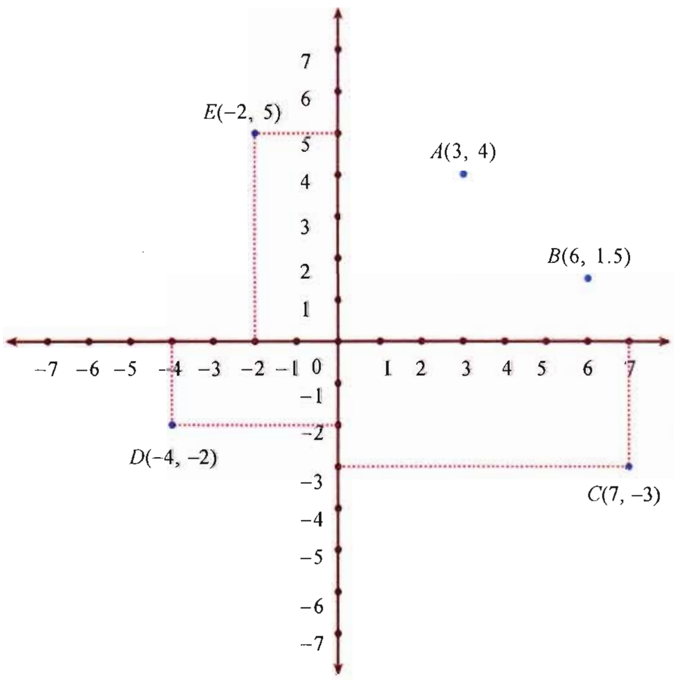
The perpendicular to the horizontal line meets it at the point marked 3, and the perpendicular to the vertical line meets at the point 4. So, as before, we can denote the point A by the pair (3, 4) of numbers.

Now look at *B*.



The perpendicular to the vertical line meets it exactly at the middle of 1 and 2. So, we can write *B* as $(6, \frac{3}{2})$

In the same way, we can also denote the remaining points *C*, *D* and *E* by number pairs.



Thus, using a pair of line perpendicular to each other and a suitable unit for measuring length, we can denote every point in a plane by means of pairs of numbers.

Dividing the Earth

We use latitudes and longitudes to locate places on the earth. What is their meaning?

First let's see how we place a grid over the earth, as we did on squares and rectangles.

The earth rotates; and in any rotating sphere, there are two points which don't move. These are the *poles* and the line joing them is the *axis of rotation*. By a *great circle* on a sphere, we mean a circle on the sphere with the same centre as that of the sphere. The great circle on the earth equidistant from the poles is *the equator*. Other circles on the earth parallel to the equator are the lines of *latitudes*. Great circles through the poles are the lines of *longitudes* (also called *meridians*). Of these, the one that passes through Greenwhich village in England is the *prime meridian*.

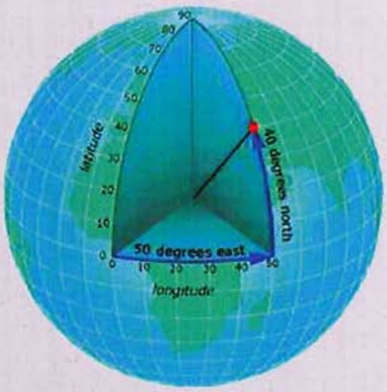
Thus we can imagine the grid of latitudes and longitudes covering the earth:

Locations on the Earth

We saw how the earth is divided into cells by the latitudes and the longitudes. We use this to indicate any position on the earth.

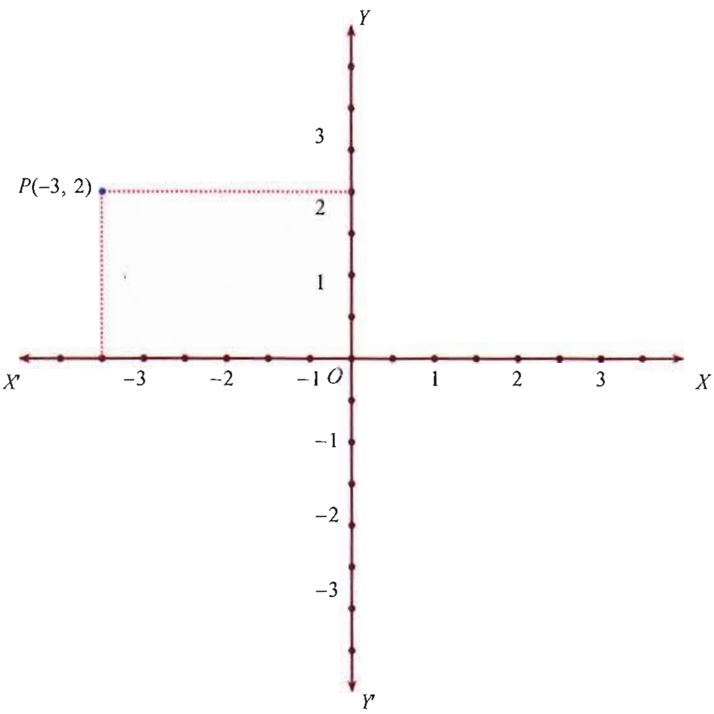
For this, we first suppose a line joining the centre of the earth with a point of intersection of the equator and the Greenwhich meridian. For this point to move to another latitude, it must move north or south; and so the line joining the point to the centre of the earth must move up or down through a certain angle. The latitudes are specified in terms of these angles (and also the adjectives north or south). On the other hand, what if our original point is to move to another longitude? It must move east or west; and the line should move left or right through a certain angle. These angles are the labels for the longitudes.

We can specify any position on earth using these angles.



The two reference lines are called the *axes of coordinates*; the horizontal line is called the *x-axis* and the vertical line is called the *y-axis*. The *x-axis* usually named XX' and the *y-axis*, YY'. Their point of intersection is called the origin and is usually named O.

We have seen the practice of drawing perpendiculars from a point to the coordinate axes, to indicate its position. The pair of numbers thus got are called the *coordinates* of the point; the first number is called the *x-coordinate* and the second number is called the *y-coordinate*.

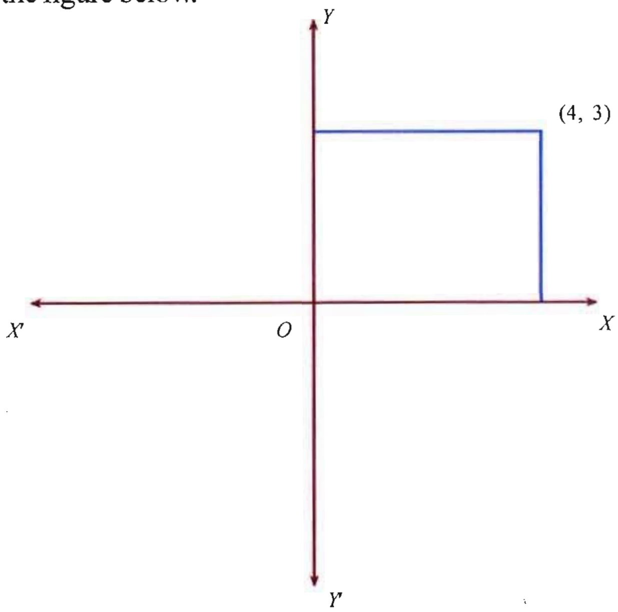


For example, in the figure above, the *x-coordinate* of the point P is −3 and the *y-coordinate* is 2.

Now some problems:

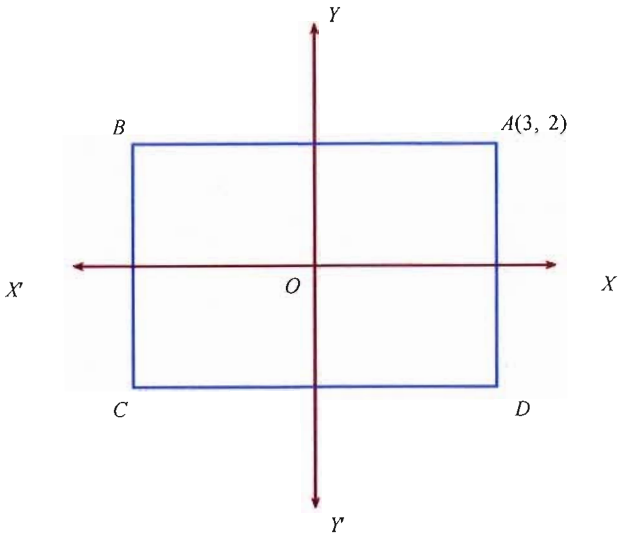
- What is the speciality of the y-coordinates of points on the *x-axis*?
- What is the speciality of the x-coordinates of points on the *y-axis*?
- What are the coordinates of the origin?

- Find the coordinates of the other three vertices of the rectangle in the figure below.



The unit of length used in this $\frac{3}{4}$ centimetres. What is actual width and height of this rectangle?

- In the figure below, *ABCD* is a rectangle with the origin *O* as its centre and sides parallel to the axes.



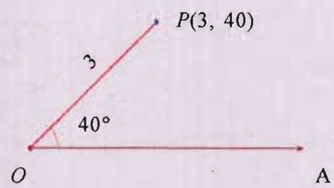
What are the coordinates of *B*, *C*, *D*?

Distance and direction

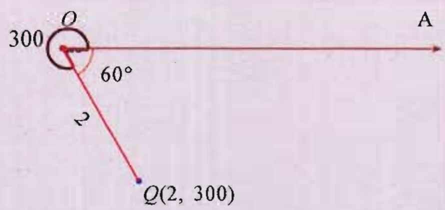
Instead of using the distances from two perpendicular lines to specify positions of points, we can use the distance from a fixed point and the angle made with a fixed line.

For this, take a point *O* and a line *A* through it:

Now for any point *P* in this plane, we can specify its position by means of the length *OP* and the angle *POA*.

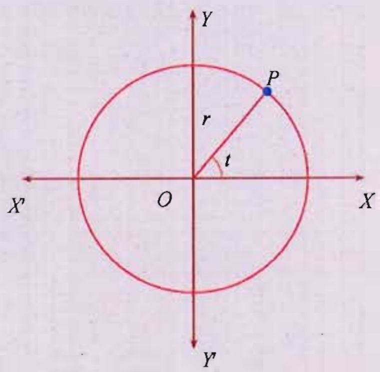


How do we specify the position of *Q* in this method?



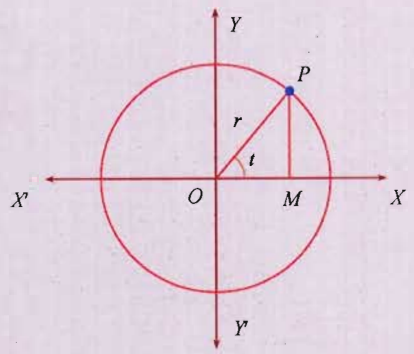
A little trigonometry

We draw a circle of radius r centred at the origin, and take a point P on it, as shown blelow:



Taking $\angle POX = t$, what are the coordinates of P?

Drawing the perpendicular PM from P to the x-axis, we get the right angled triangle POM.



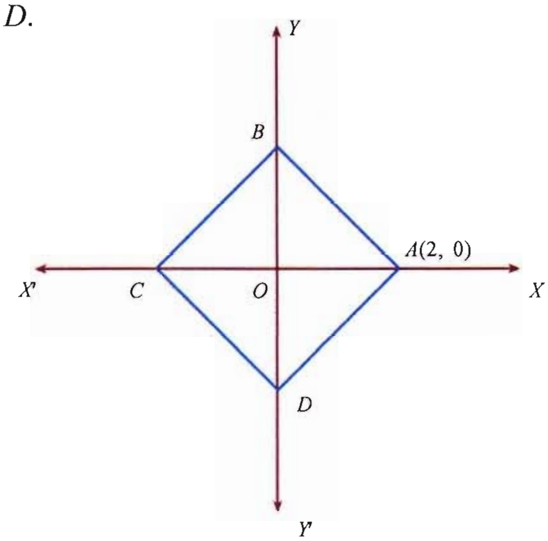
From the figure, we see that

$$OM = r \cos t. \quad PM = r \sin t$$

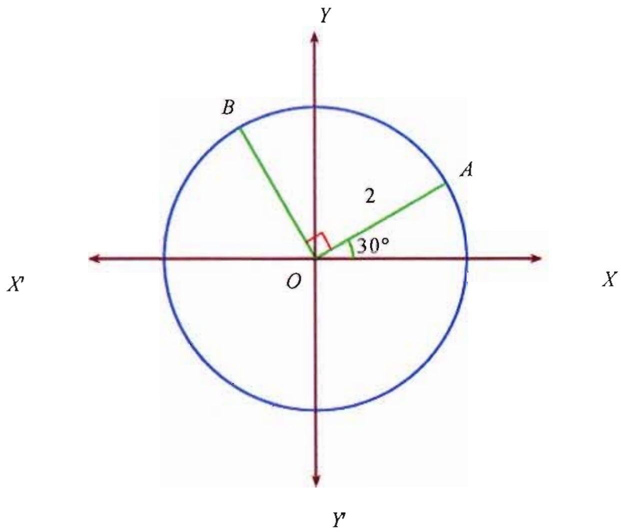
This means the coordinates of P are $(r \cos t, r \sin t)$.

What if $\angle POX$ is a right angle or larger?

- In the figure below, $ABCD$ is a square. Find the coordinates of B, C, D.



- What are the coordinates of the points A and B in the figure below?

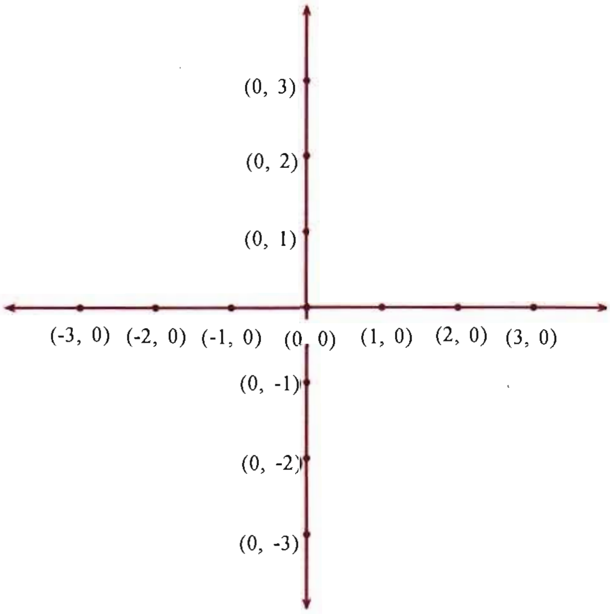


- With the axes of coordinates chosen along two adjacent sides of a rectangle, two opposite vertices have coordinates $(0, 0)$ and $(4, 3)$. What are the coordinates of the other two vertices?

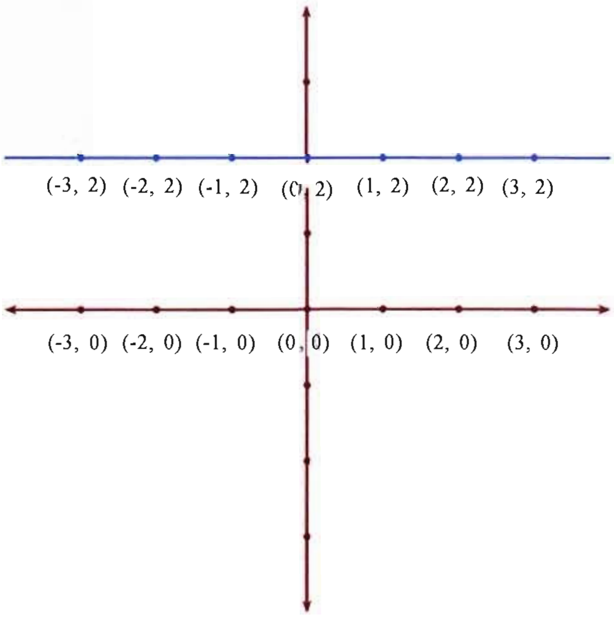
Parallels

We noted that all points on the x-axis have their y-coordinate 0; on the other hand all points with their y-coordinate 0 are on the x-axis. In other words, the x-axis can be described as the collection of all points whose coordinates are of the form $(x, 0)$.

Likewise, the y-axis is the collection of all points of the form $(0, y)$.

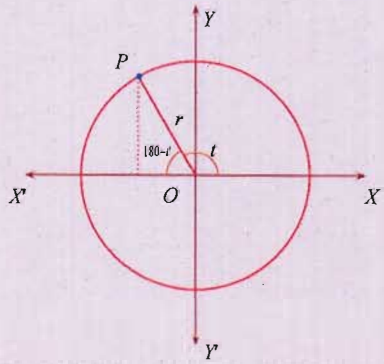


What if we take some other number instead of 0? For example, what about points of the form $(x, 2)$? All these are at a distance 2 (with respect to the unit of length chosen) from the x-axis. So, all these are on the line parallel to the x- axis, at a distance 2 from it.



Some more trigonometry

See this figure:



It can be directly seen from the figure itself that the x-coordinate of P is $r\cos(180 - t)$ and its y-coordinate is $r\sin(180 - t)$.

Also, t is an angle between 90 and 180, and we have defined

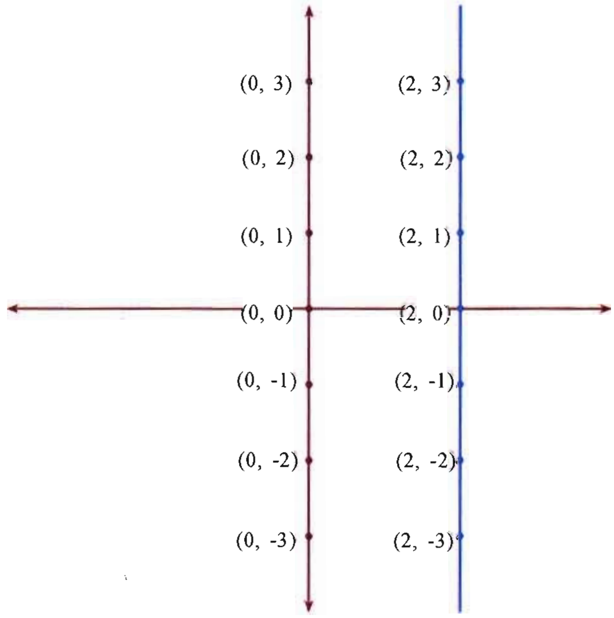
$$\cos(180 - t) = -\cos t$$

$$\sin(180 - t) = \sin t$$

in the lesson **Trigonometry**. So, here also, the coordinates of P are $(+\cos t, +\sin t)$

And all points with y-coordinate 2 are on this line.

What about the collection of points of the form $(2, y)$?



In short, whatever number we take, the collection of all points of the form (x, a) is the line parallel to the x-axis, at a distance a from it; and the collection of all points of the form (a, y) is the line parallel to the y-axis, at a distance a from it.

In one sense, all these are number lines. We use labels like (x, a) or (a, y) to denote the points instead of single numbers like x or y.

So then, a question: what's the distance between $(1, 2)$ and $(3, 2)$?

What about the distance between $(1, 2)$ and $(-3, 2)$?

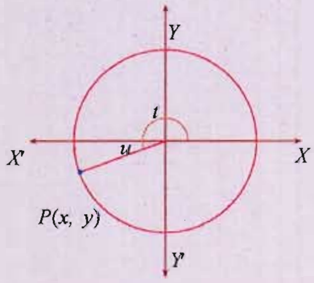
As in a number line, we need only subtract the smaller x-coordinate from the larger, right?

In short, to find the distance between (x_1, a) and (x_2, a), we need only subtract the smaller of x_1, x_2, from the larger; in algebraic language, the distance is $|x_1 - x_2|$.

Similarly, to find the distance between (a, y_1) and (a, y_2), we need only subtract the smaller of y_1, y_2, from the larger; in algebraic language, the distance is $|y_1 - y_2|$.

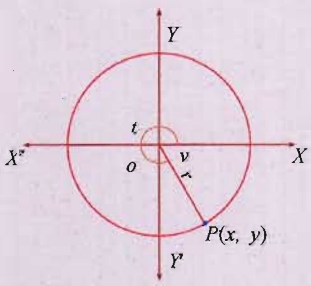
Trigonometry through circles

See this figure:



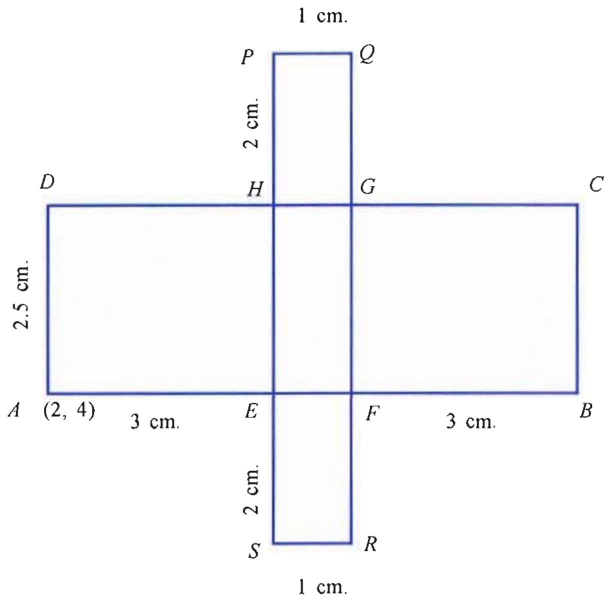
Here we can see that the coordinates of P are $(-r \cos u, -r \sin u)$; also $u = t - 180$. So, if we define, $\cos (180 + u) = -\cos u$ and $\sin (180 + u) = -\sin u$, then the coordinates of P would be $(r \cos (180 + u), r \sin (180 + u))$; and thus the coordinates of P would be $(r \cos t, r \sin t)$ in this case also.

What about the figure below?



Here, the coordinates of P are $(r \cos v, -r \sin v)$ and $v = 360 - t$. So, if we define $\cos(360 - v) = \cos v$ and $\sin(360 - v) = -\sin v$, then the cordinates of P would be $(r \cos t, r \sin t)$ in this case also.

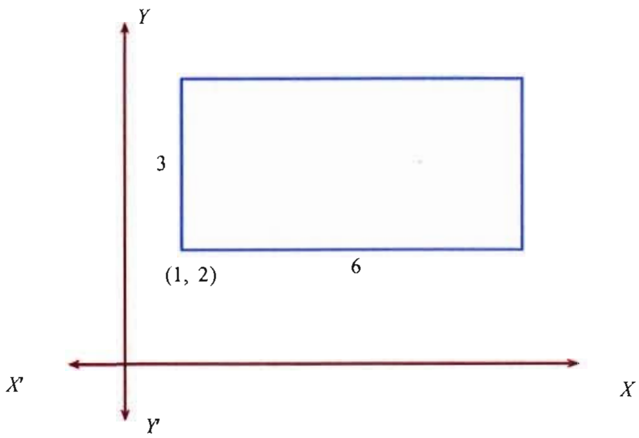
In short, we extend the definitions of $\sin t$ and $\cos t$ in such a way that the coordinates of P on the circle centred at O with radius r is $(r \cos t, r \sin t)$, where t is the central angle of the arc from the x-axis to P, however large t be.



In the figure, the sides of the rectangle $ABCD$ and $PQRS$ are parallel to the axes. Find the coordinates of the vertices of all the rectangles in it.

Rectangles

Look at this figure:

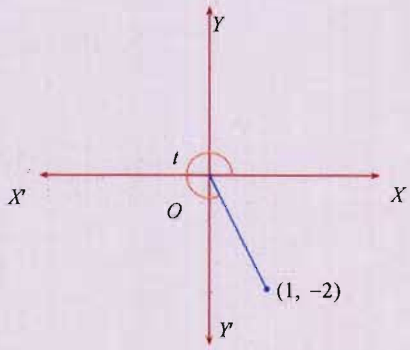


We want to draw a rectangle like this, with sides parallel to the axes. What should be the coordinates of the other three vertices?

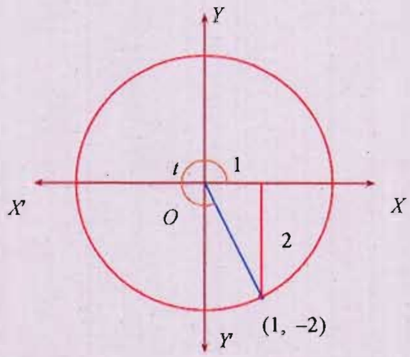
All points on the bottom side of the rectangle would be of the form $(x, 2)$ (why?) Among these, the bottom-right corner of the rectangle is 6 units away from $(1, 2)$. So, what are its coordinates?

Without circles

What is the sin and cos of the angle t in the figure below?



We can imagine a circle through $(1, -2)$, centred at the origin:



We can see from the figure that the radius of the circle must be

$$\sqrt{1^2 + 2^2} = \sqrt{5}$$

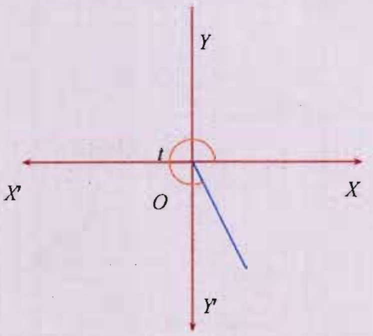
So, by definition

$$\cos t = \frac{1}{\sqrt{5}} \quad \sin t = -\frac{2}{\sqrt{5}}$$

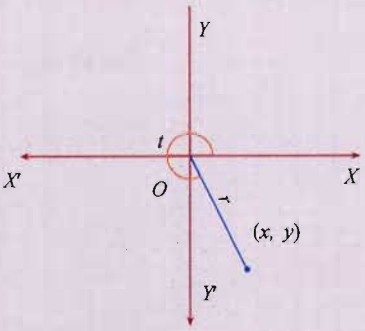
Coordinates and angles

We saw how the trigonometric measures of angles of any size are defined. How do we find them for angles larger than 180°?

First we draw a line through the origin, making this angle with the x-axis:



We then choose a point on this line and find its coordinates and its distance from the origin:



Then by definition,

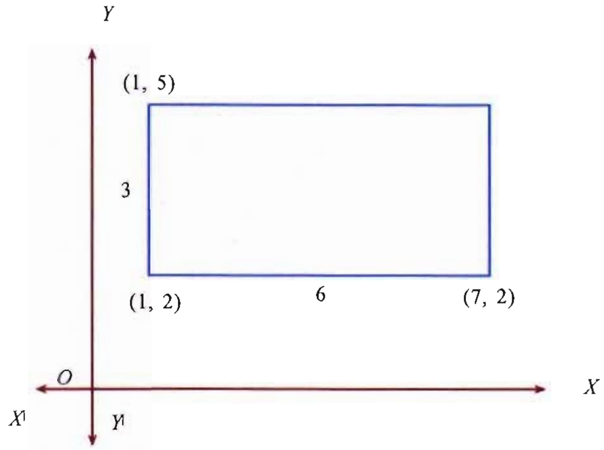
$$\cos t = \frac{x}{r} \qquad \sin t = \frac{y}{r}$$

We can also see that

$$\tan t = \frac{y}{x}$$

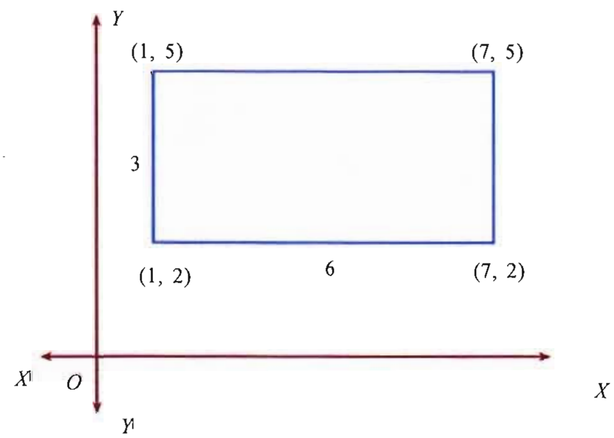
Thus to find the tan measure, we need only the coordinates.

Likewise, what can we say about the general form of the points on the left side of the rectangle? Among these, what are the coordinates of the top-left corner of the rectangle?



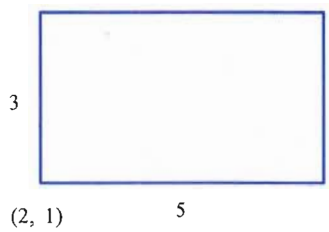
Finally, what about the fourth vertex?

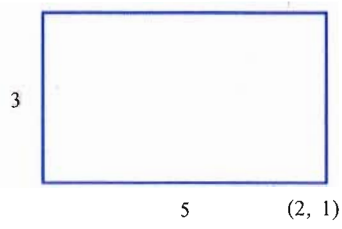
We can think along different lines. Since this point is on the right side of the rectangle, its x-coordinate is 7 (why?) and since it is on the top side of the rectangle also, its y-coordinate is 5.

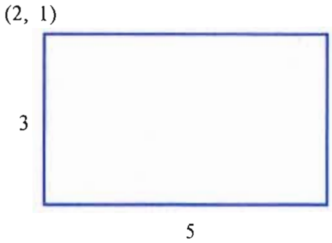


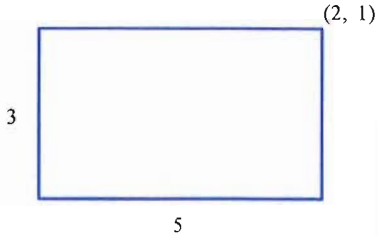
(What are the other ways of getting this?)

Some rectangles are shown below. The width and height of each is shown; also the coordinate of one vertex with respect to a pair of axes (not shown) parallel to the sides. Find the coordinates of the other vertices.

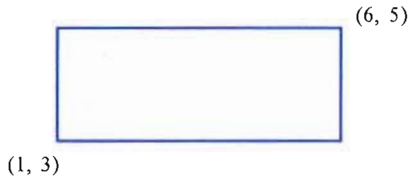








Now look at this rectangle;



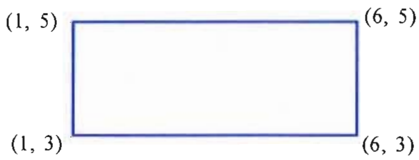
It also has its axes parallel to the axes of coordinates. Can you find out the coordinates of the other vertices?

First let's consider the bottom-right corner. This vertex is on the right side of the rectangle; this side is parallel to the y-axis; one point on it is (6, 5). So, the x-coordinate of the point we seek is also 6.

What about the y-coordinate?

The bottom side is parallel to the x-axis and a point on it is (1, 3). So, the y-coordinate of the bottom-right vertex is also 3.

Similarly, can't we find the top-left vertex also?



Now what are the lengths of the sides of this rectangle?

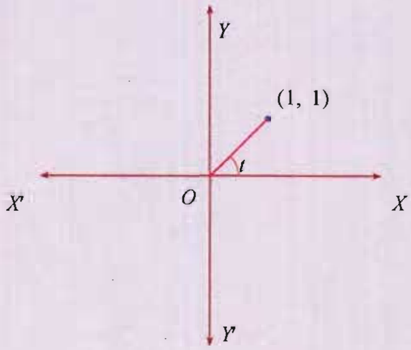
Here's another rectangle with sides parallel to the axes:

Can you find the other vertices? And the lengths of sides?

Can we have a rectangle with any two given points as opposite vertices and sides parallel to the axes?

Lines and points

Can you say how much angle t is in the figure below?



By what we have seen earlier,

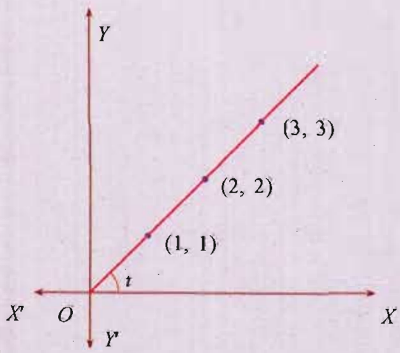
$$\tan t = \frac{1}{1} = 1$$

So,

$$t = 45°$$

We would get the same angle, if we take (2, 2) instead of (1, 1), isn't it so? How about (3, 3)?

So what do we see here? The points (1, 1), (2, 2), (3, 3), . . . are all on the same line.



Have we seen this before? Take another look at the section, **Geometry of arithmetic sequence** in the lesson, **Arithmetic Sequences.**

The line joining these points should not be parallel to either axis, anyhow; that is, their x-coordinates should be different and so must be their y-coordinates.

For example, let's take (–2, 3) and (6, 5). What are the positions of these points?

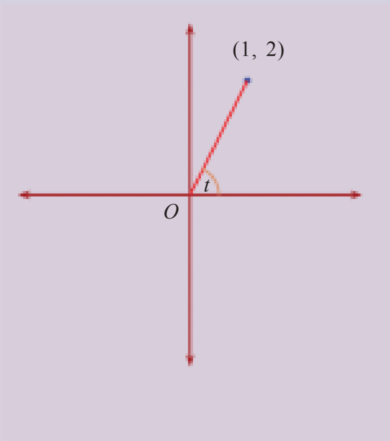
What are the other vertices of the rectangle?

How about (–2, 3) and (6, 1)?

The coordinates of some pairs of points are given below. Without drawing the axes of coordinates, mark these points with the left-right, up-down positions correct. Draw rectangles with these as opposite vertices. Find the coordinates of the other two vertices and the lengths of the sides of these rectangles:

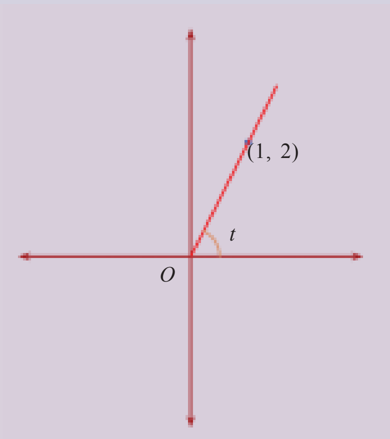
- (3, 5), (7, 8)
- (–3, 5), (–7, 1)
- (6, 2), (5, 4)
- (–1, –2), (–5, -4)

Another line



Can you compute tan t in the figure above?

Suppose we extend this line.



Can you say the coordinates of a few more points on this line?