Section 1.1: Pythagorean Tuning

Of course, we would like to be able to make use of a wide variety of tones and intervals, rather than being limited to just octaves. However, note that for every frequency we elect to add to a tuning system, it will be reasonable to include frequencies that fall one or more octaves above or below that frequency, as illustrated in the preceding example. As a result, it turns out that, to fully develop a tuning system, all we need to explicitly work out is the set of frequencies between the base frequency and the tone one octave above the base frequency (i.e., the base frequency times two). Once this is accomplished, all other frequencies can be found, if needed, by raising/lowering each of the frequencies in this specific range by octaves.

Now, let's look at some musical intervals other than just the octave. One of the most fundamental intervals in music is the "perfect fifth:"

Definition: A <u>perfect fifth</u> is an interval with a frequency ratio of exactly 3/2, or 1.5.

In other words, two tones form a perfect fifth if the higher frequency divided by the lower frequency gives us a quotient, or ratio, of exactly 3/2. For example, the interval formed by the frequencies 440 Hz and 660 Hz is a perfect fifth, since 660/440 = 3/2, or 1.5. (Note: to find the higher frequency from the lower frequency, we can simply multiply the lower frequency by the desired frequency ratio; e.g., $440 \times 3/2 = 660$.)

Example: Let's revisit the example from section 1.0 in which we begin to design a tuning system with a base frequency of 450 Hz. We've already decided to include other frequencies that lie one or more octaves above or below 450 Hz (which we found numerically by repeatedly multiplying or dividing by 2).

Let's now also add a tone with a frequency of 675 Hz. The motivation for this specific tone is that, together with the base tone (450 Hz), we now have a "perfect fifth" interval to work with, since the frequency ratio formed by these two tones is $675/450$, which reduces to $3/2$ (or 1.5 on a calculator). Since perfect fifths are desirable intervals, it makes sense to include 675 Hz in our tuning system.

Now that we've decided to introduce 675 Hz, it follows that we should include tones that are one or more octaves above or below 675 Hz as well – this gives us, for example, 1350 Hz and 2700 Hz when we raise by octaves, and 337.5 Hz and 168.75 Hz when we lower by octaves. As before, the frequencies found by multiplying or dividing by 2 repeatedly are "automatically" included in our tuning system as a result of our decision to include 675 Hz.

Because of our convention of automatically including tones that lie one or more octaves above or below each tone we decide to include in a tuning system, we can actually give a complete description of a tuning system by simply listing the frequencies that lie within the range between the base frequency and one octave above the base frequency, as stated earlier. So, for our current example, all that is necessary is to state that our tuning system contains the frequencies 450 Hz (base), 675 Hz, and 900 Hz;

from this, we can extrapolate the rest! (In fact, even explicitly listing 900 Hz is a bit redundant, since its inclusion is implied by doubling the base 450 Hz frequency.)

Notation: R$\boldsymbol{n}$ and L$\boldsymbol{n}$: Given a "base" tone with frequency, f, we define "Rn" to be the frequency of the tone which is found by raising the "base" tone by a perfect fifth n times, then lowering by octaves (as necessary) to obtain a result between f and $2f$. Similarly, we define "Ln" to be the frequency of the tone which is found by *lowering* the "base" tone by a perfect fifth n times, then raising by octaves (as necessary) to obtain a result between f and $2f$.

Examples: If we select a base tone with a frequency of 450 Hz, then R1 would be found by raising the base tone by a perfect fifth; numerically, this means we multiply by 3/2, or 1.5: $R1 = 450 \times \frac{3}{2} = 675\ Hz$.
To find R2, we'd multiply by 1.5 again: $675 \times \frac{3}{2} = 1012.5\ Hz$; however, since 1012.5 is greater than twice the base frequency, we'd lower it by an octave; so, $R2 = 1012.5 \times \frac{1}{2} = 506.25\ Hz$.
We'll discuss L1, L2, etc. a little bit later.

<u>Pentatonic tuning system</u>
A pentatonic tuning system is found by first selecting a base tone, and then adding the tones whose frequencies are **R1, R2, R3 and R4** (based on the definition on the preceding page). These five notes form the "pentatonic scale;" the tuning system then includes these five tone, along with any other tone obtained through raising/lowering by octaves.

Example: If we select a base tone with frequency $f = 320\ Hz$, then our "pentatonic scale" includes the following frequencies:
R1: We raise the base frequency by a perfect fifth: $320 \times \frac{3}{2} = \mathbf{480}$ **Hz.** (Note: 480 is less than twice the base frequency, so it's part of our scale)
R2: We start by raising R1 by a perfect fifth: $480 \times \frac{3}{2} = 720$. Since this is more than twice the base frequency, we need to lower it by an octave. So, R2 = $720 \times \frac{1}{2} = \mathbf{360}$ **Hz.**
R3: We raise R2 by a perfect fifth: $360 \times \frac{3}{2} = 540$ Hz. This is less than double the base frequency, so **R3 = 540 Hz.**
R4: We raise R3 by a perfect fifth: $540 \times \frac{3}{2} = 810$ Hz. This is more than twice the base frequency, so we lower it by an octave: $810 \times \frac{1}{2} = \mathbf{405}$ **Hz.**

Thus, our "pentatonic scale" includes the frequencies: 320 Hz, 360 Hz, 405 Hz, 480 Hz, 540 Hz. (Optional: the "top" of the scale is usually taken to be the base tone raised by an octave; so, we could list 640 Hz as the "top" of the scale.)

Note: the calculation of R1, R2, R3, etc. is a repetitive process; practice it to make sure you get the hang of it. For example, if we continue raising by perfect fifths from the preceding example, we'd get the following results (make sure you can duplicate these results on your own):
R5: 607.5 Hz; R6: 455.625 Hz; R7: 341.71875 Hz; R8: 512.578125 Hz; R9: 384.43359375 Hz

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Pythagorean Tuning System
The "Pythagorean" system is the original basis of the "twelve-tone scale," which is commonly used in western music. The standard piano keyboard is based on a twelve-tone scale (as opposed to a pentatonic scale). The Pythagorean tuning system is based on the twelve-tone scale consisting of the following frequencies:

The "base" frequency; **R1, R2, R3, R4, R5** and **R6**; and also **L1, L2, L3, L4** and **L5**.

Example: Let's develop a Pythagorean tuning system based on a base frequency of 320 Hz (as used in the preceding example). We've already worked out several of the frequencies (see above); all we still need to do is find L1, L2, L3, L4 and L5.

To find L1, we first lower the base frequency by a perfect fifth. That is, we must find a frequency, call it x, such that the frequency ratio $320 \div x$ equals 3/2. That is, $\frac{320}{x} = \frac{3}{2}$. There are (at least) two ways to solve for x:
1. Cross-multiply: $320 \cdot 2 = 3x$, so $x = \frac{640}{3} \approx 213.333$.
2. Divide (rather than multiply) by the frequency ratio 3/2. The rationale here is that lowering is the opposite of raising; thus, if we multiply to raise by a perfect fifth, then we should divide in order to lower by a perfect fifth.
This gives us: $x = 320 \div \frac{3}{2}$. Recall now that dividing by a fraction is equivalent to multiplying by its reciprocal; therefore, $x = \frac{320}{1} \div \frac{3}{2} = \frac{320}{1} \times \frac{2}{3} = \frac{640}{3} \approx 213.33$ Hz.

(Note: in practice, it's easier to just "invert and multiply" when dividing by a fraction, which is the second method shown above. That's what we'll usually do, rather than the "cross-multiplication" method shown previously. We showed two methods here, though, to make the point that dividing by a frequency ratio – in this case, 3/2 – does actually solve the problem we're interested in.)

This isn't L1 yet – remember, we want frequencies between the base frequency (320 in this example) and twice the base frequency (640). Our answer above, 640/3 is too low, so we must raise it by an octave:
$L1 = \frac{640}{3} \times \frac{2}{1} = \frac{1280}{3} \approx 426.67$ Hz.

Proceeding similarly, next we find L2. Note that in order to avoid introducing unnecessary rounding errors into your calculations, you should use fractions rather than decimals to calculate each frequency. Only round off when writing your final answer (usually two decimal places is fine) for each frequency, but use the fraction – not the decimal – to calculate additional frequencies. (Note: this wasn't as big of a deal for R1, R2, etc., because those frequencies worked out to exact values; we didn't need to round off any repeating decimals. Here, though, the decimals repeat, so we have to be more careful about rounding…)

So, to find L2, L3, etc., we proceed as follows.

L2:

$$\frac{1280}{3} \div \frac{3}{2} = \frac{1280}{3} \times \frac{2}{3} = \frac{2560}{9} = 284.44444 \ldots$$

Now, since this is less than the base frequency of 320 Hz, we need to raise it by an octave again:

$$\frac{2560}{9} \times \frac{2}{1} = \frac{5120}{9} = 568.8888 \ldots,$$

So, $L2 \approx 568.89\ Hz$.

Proceed similarly to find L3, L4 and L5. You should get the following results. (As before, make sure that you can obtain these results on your own.)

L3: Exact answer: $\frac{10240}{27}$. Decimal approximation: 379.26 Hz
L4: Exact answer: $\frac{40960}{81}$. Decimal approximation: 505.68 Hz
L5: Exact answer: $\frac{81920}{243}$. Decimal approximation: 337.12 Hz.

Note: don't be intimidated by the large numerators and denominators in these fractions; you can use a calculator to find each of these. (For example: in L3, the numerator 10240 is just 2560 multiplied by 2 twice.) Finding these fractions correctly is really just a matter of keeping your work organized! After that, use a calculator to find a decimal approximation to that fraction; but, again, remember to use the fraction, not the decimal, to find the next frequency in the scale. This is the only way to ensure answers that are exactly correct.

Thus, our Pythagorean "scale" consists of the following frequencies. These are the base frequency followed by R1 through R6 and L1 through L5 (rearranged from smallest to largest, and each rounded to two decimal places):

320 (base), 337.12, 360, 379.26, 405, 426.67, 455.63, 480, 505.68, 540, 568.89, 607.5, 640 (octave)

So, if a keyboard were tuned in such a way that one of the keys had a frequency of 320 Hz, then the next twelve keys would be tuned to approximately the frequencies shown in the list above. This is an example of a Pythagorean scale, or tuning system.