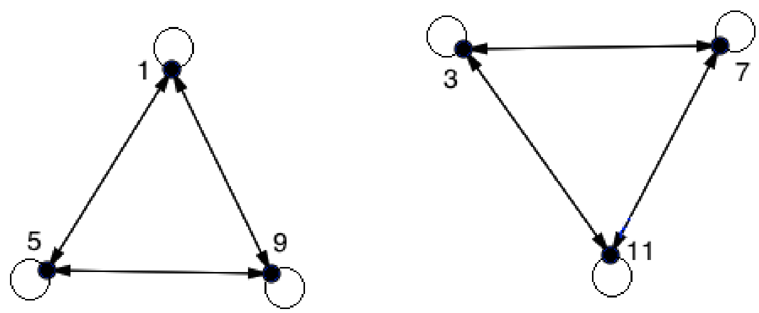
MATH 210, TEST #2
SOLUTIONS

1. Consider the relation $\propto$ on the set {1, 3, 5, 7, 9, 11}, which is defined as follows:

$$a \propto b \text{ iff } a + b \text{ is NOT a multiple of 4.}$$

a) Draw a directed graph (or "digraph") for this relation.

Solution: One way to draw a valid digraph for this relation is as follows:



b) Determine whether this relation is an equivalence relation. Explain your answer.

Solution: This relation is an equivalence relation because it partitions the set into two disjoint subsets, {1,5,9} and {3,7,11}. For this relation, we see that $x \propto y$ if and only if x, y are both in the same subset (or "equivalence class").

Equivalently, we can see that this relation is reflexive (every element of the set is related to itself), symmetric (whenever $x \propto y$, we also have $y \propto x$), and transitive (whenever $x \propto y$ and $y \propto z$, we also have $x \propto z$).

c) If $\propto$ is an equivalence relation, write down its equivalence classes. If $\propto$ is not an equivalence relation, give a specific example showing why it isn't.

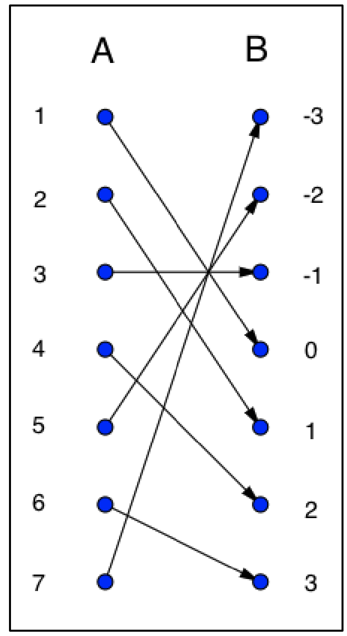
Solution: As observed above, the equivalence classes for this equivalence relation are {1,5,9} and {3,7,11}.

2. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, and let $B = \{-3, -2, -1, 0, 1, 2, 3\}$.

Define the function $f: A \to B$ as follows:

$$f(n) = \frac{n}{2}, \qquad \text{if } n \text{ is even}$$
$$f(n) = \frac{1-n}{2}, \qquad \text{if } n \text{ is odd}$$

a) Sketch an arrow diagram for this function.



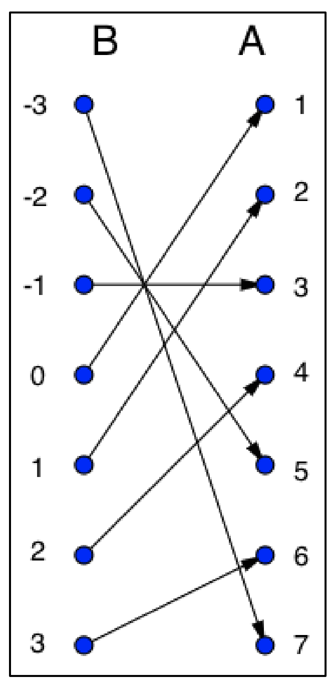
Solution:
The diagram to the right is an arrow diagram for $f: A \to B$

b) Determine whether this function is one-to-one, onto, both, or neither. Explain your answers.

Solution:
This function in one-to-one, since each codomain element is the image of at most one domain element.
This function is also onto, since each codomain element is the image of at least one domain element.

Shorter answer: Since each element of the codomain is the image of exactly one element of the domain, we see that f is both one-to-one and onto.
(In other words, f is a "bijection" between sets A and B.)

c) Draw an arrow diagram for $f^{-1}: B \to A$, and determine whether f^{-1} is a function. Explain your answer.



Solution:
The diagram to the right is an arrow diagram for $f^{-1}: B \to A$.
Since each element of B is mapped to exactly one element of A, this is a function. (Note: the inverse mapping of a function is again a function if, and only if, the original function was both one-to-one and onto, as is the case here.)

3. Define the relation $\prec$ on the set of natural numbers as follows:

$$x \prec y \text{ iff } y = kx \text{ for some } \textit{odd} \text{ integer, } k.$$

For example, $2 \prec 6$, since $6 = 2 \cdot 3$ and 3 is an odd integer.
On the other hand, $2 \nprec 8$, since $8 = 2 \cdot 4$, and 4 is *not* an odd integer.

a. Prove that $\prec$ is a partial ordering on $\mathbb{N}$.

Solution: We must show that $\prec$ is reflexive, antisymmetric, and transitive.

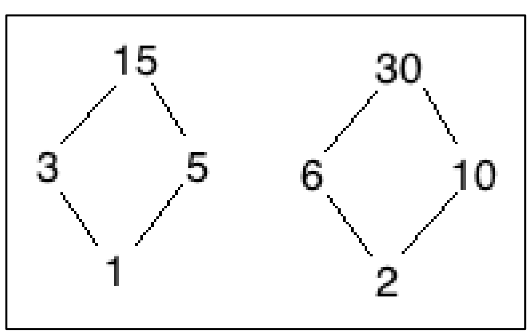
Reflexive: For each $x \in \mathbb{N}$, $x = 1 \cdot x$. Since 1 is an odd integer, it follows that $x \prec x$.

Antisymmetric: Let $x, y \in \mathbb{N}$.
Assume $x \prec y$ and $y \prec x$. (We want to show that $x = y$ follows from this assumption.)
Then, $y = kx$ for some odd integer k, and $x = jy$ for some odd integer, j.
Substituting jy for x in the first equation above gives us $y = k(jy) = (kj)y$.
Since $y = (kj)y$, it follows that $kj = 1$. Since k, j are both odd integers, this implies $k = j = 1$.
Since $k = 1$ and $y = kx$, it follows that $y = x$, which was to be shown.
Thus, we've shown that $(x \prec y \text{ and } y \prec x) \rightarrow x = y$; that is, $\prec$ is antisymmetric.

Transitive: Let $x, y, z \in \mathbb{N}$.
Assume $x \prec y$ and $y \prec z$. (We want to show that $x \prec z$ follows from this assumption.)
Then, $y = kx$ for some odd integer k, and $z = jy$ for some odd integer, j.
Substituting kx for y in the second equation above gives us $z = j(kx) = (jk)x$.
Since j, k are both odd integers, it follows that jk is an odd integer. Therefore, z is equal to an odd integer times x, which means $x \prec z$, which was to be shown.
Thus, we've shown that $(x \prec y \text{ and } y \prec z) \rightarrow x \prec z$; that is, $\prec$ is antisymmetric.

b. Draw a Hasse diagram for $\prec$ on the set of divisors of 30: {1, 2, 3, 5, 6, 10, 15, 30}.

Solution: One way to draw a Hasse diagram for $\prec$ on the set of divisors of 30 is as follows:



4. Prove the theorem: for $n \in \mathbb{Z}$, n is divisible by 3 if and only if n^2 is divisible by 3.

Solution: Recall that to prove an "if and only if" statement, we must show that the propositions are logically equivalent; that is, the first implies the second, and also the second implies the first. (In other "words," $p \leftrightarrow q$ is logically equivalent to $(p \to q) \wedge (q \to p)$.)

First, we will prove: if n is divisible by 3, then n^2 is divisible by 3.

Proof: Let $n \in \mathbb{Z}$, and assume n is divisible by 3.
Then, $n = 3q$ for some integer q.
It follows that $n^2 = (3q)^2 = 9q^2 = 3(3q^2)$. Note that $3q^2$ is an integer (since $q \in \mathbb{Z}$).
Since n^2 is 3 times an integer, we conclude that n^2 is divisible by 3.
Therefore, if n is divisible by 3, then n^2 is divisible by 3

Next, we wish to show: if n^2 is divisible by 3, then n is divisible by 3.
Since this is difficult to prove via direct proof, we will instead prove the (logically equivalent) <u>contrapositive</u>: if n is not divisible by 3, then n^2 is not divisible by 3.

Proof: Let $n \in \mathbb{Z}$, and assume n is not divisible by 3.
Then, $n = 3q + 1$ OR $n = 3q + 2$ for some integer q. (We must consider both cases!)

- If $n = 3q + 1$, then $n^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$.
  That is, n^2 has a remainder of 1 when it is divided by 3.
  Thus, n^2 is not divisible by 3 in this case.

- If $n = 3q + 2$, then $n^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1$.
  That is, n^2 has a remainder of 1 when it is divided by 3.
  Thus, n^2 is not divisible by 3 in this case.

In both cases, we can conclude that n^2 is not divisible by 3.
Thus, if n is not divisible by 3, it follows that n^2 is not divisible by 3.
Therefore (by the contrapositive law), if n^2 is divisible by 3, then n is divisible by 3.

Conclusion: We have proved both of the following theorems for $n \in \mathbb{N}$:

If n is divisible by 3, then n^2 is divisible by 3
If n^2 is divisible by 3, then n is divisible by 3

Thus, for $n \in \mathbb{N}$, the following proposition is a theorem as well:

n is divisible by 3 if and only if n^2 is divisible by 3.

5. Consider the proposition (for $x, y \in \mathbb{Z}$):

$$x \equiv y \ (mod\ 6) \rightarrow 3x \equiv 3y \ (mod\ 9)$$

a) Prove that the proposition is a theorem.

Proof: Let $x, y \in \mathbb{Z}$.
Assume $x \equiv y \ (mod\ 6)$. That is, $x - y$ is divisible by 6. Then, $x - y = 6q$ for some integer q.

The following are equivalent:

$$x - y = 6q$$
$$3(x - y) = 3(6q)$$
$$3x - 3y = 18q$$
$$3x - 3y = 9(2q)$$

Since q is an integer, clearly $2q$ is an integer as well. Thus, $3x - 3y$ is divisible by 9.
That is, $3x \equiv 3y \ (mod\ 9)$.

Therefore, for all $x, y \in \mathbb{Z}$: if $x \equiv y \ (mod\ 6)$, then $3x \equiv 3y \ (mod\ 9)$.

b) Show that the converse of the proposition is not a theorem.

First, note that the converse of the original proposition is as follows:

$$3x \equiv 3y (mod\ 9) \rightarrow x \equiv y \ (mod\ 6)$$

The negation of $p \rightarrow q$ is $p \wedge \neg q$, which in this case would be:

$$3x \equiv 3y \ (mod\ 9) \wedge x \not\equiv y \ (mod\ 6);$$

that is, we just need to find integers x, y such that $3x - 3y$ is divisible by 9 but $x - y$ is not divisible by 6. It turns out that any pair of integers x, y which differ by an odd multiple of 3 will serve as a counterexample. For example, if we let $(x, y) = (4,1)$, then $3(4) \equiv 3(1) \ (mod\ 9)$ is true but $4 \equiv 1 \ (mod\ 6)$ is false.

6. Suppose g is a function with domain D and codomain C, and suppose the first line of a proof reads as follows:

"Assume (for a contradiction) that there exist $x, y \in D$ such that $x \neq y$ and $g(x) = g(y)$."

Which of the following would be the objective of a proof that starts out like this?

a) **To prove that g is one-to-one**
b) To prove that g is not one-to-one
c) To prove that g is onto
d) To prove that g is not onto

Explain your choice. You *must* give a written explanation to receive credit for this problem.

Solution: The correct choice is (a) To prove that g is one-to-one.

In a contradiction proof, we start by assuming that the proposition we wish to prove is false. In this case, the proof starts with the statement that function g is *not* one-to-one. Thus, we can infer that the writer's intent is to show that this assumption leads to a contradiction and is therefore false; this would imply that function g is one-to-one.