Math 210, Fall 2015

Collected Homework #8 Solutions

1. Find the greatest integer which *cannot* be written in the form $11a+4b$, where a and b are non-negative integers. (This may take some trial-and-error on your part.) Once you've come up with your answer, *prove* it using induction.

   Solution: The greatest such integer turns out to be 29. Since 29, 29-11=18, and 29-2(11)=7 are not integer multiples of 4, we cannot write 29 in the form $11a + 4b$ where a, b are non-negative integers.

   So, our theorem is as follows: for all $n \geq 30$, there exist non-negative integers such that $11a + 4b = n$.

   Proof: To help with notation, we'll let p_n denote the proposition that there exist non-negative integers a, b such that $n = 11a + 4b$.

   For our initial step, we will verify that p_{30}, p_{31}, p_{32}, and p_{33} are true:

   - p_{30} is true, since $11(2) + 4(2) = 22 + 8 = 30$
   - p_{31} is true, since $11(1) + 4(5) = 11 + 20 = 31$
   - p_{32} is true, since $11(0) + 4(8) = 0 + 32 = 32$
   - p_{33} is true, since $11(3) + 4(0) = 33 + 0 = 33$

   At this point, we can see that p_n will be true for any number greater than 33 as well, since any such number can be obtained from 30, 31, 32, or 33 by adding some multiple of 4. Formally, the inductive step of the proof goes as follows:

   Inductive step: Assume p_n is true for all values of n from $n = 30$ up to $n = k$, where $k \geq 33$. Then, in particular, this implies p_{k-3} is true; that is, there exist non-negative integers a, b such that $11a+4b = k-3$. Thus, we have:

$$\begin{aligned} 11a + 4b &= k - 3 \\ 11a + 4b + 4 &= k - 3 + 4 \\ 11a + 4(b + 1) &= k + 1 \end{aligned}$$

   Thus, there exist non-negative integers – specifically, a and $b + 1$ – such that $11a + 4(b + 1) = k + 1$. Thus, p_{k+1} is true.

We've shown that p_{k+1} follows from the assumption that p_n is true for all values of n from 30 up to k. This completes the inductive step of the proof.

We've verified the initial step and the inductive step of the proof. Therefore, we can conclude that, for all integers $n \geq 30$, there exist non-negative integers a, b such that $n = 11a + 4b$.

Comment: Since $29 = 11(-1) + 4(10)$ and $29 = 11(3) + 4(-1)$, it follows that any integer solution a, b for $11a + 4b = 29$ requires either $a \leq -1$ or $b \leq -1$. In particular, it turns out that the general solution to the equation $29 = 11a+4b$ is $a = 3+4t, b = -1+11t, t \in \mathbb{Z}$. This is an example of a "Diophantine equation" (an equation for which only integer-valued solutions are considered), which is an important topic in number theory.

2. Define the sequence $\{a_n\} = \{a_1, a_2, a_3, a_4, ...\}$ as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + 2a_{n-2}, \quad \text{when } n \geq 3 \end{aligned}$$

That is, after the first two terms, each new term is equal to the preceding term plus twice the term before that one.

The first few terms of this sequence are 1, 1, 3, 5, 11, 21, 43, ...

Prove (using induction): For all $n \geq 1$,

$$a_n = \frac{2^n - (-1)^n}{3}.$$

Proof: Let p_n denote the proposition $a_n = \dfrac{2^n - (-1)^n}{3}$. For our initial step, we must verify (at least) p_1 and p_2:

- $n = 1$: $a_1 = 1$ by definition, and $\frac{2^1-(-1)^1}{3} = 3/3 = 1$. So, p_1 is true
- $n = 2$: $a_2 = 1$ by definition, and $\frac{2^2-(-1)^2}{3} = 3/3 = 1$. So, p_2 is true

Inductive step: Assume p_n is true for all values of n from $n = 1$ up to $n = k$, where $k \geq 3$. (We want to show: $a_{k+1} = \frac{2^{k+1}-(-1)^{k+1}}{3}$.)

Then, in particular, p_k and p_{k-1} are true; that is,

$$a_k = \frac{2^k - (-1)^k}{3},$$
$$\text{and } a_{k-1} = \frac{2^{k-1} - (-1)^{k-1}}{3}$$

By the two-term recursion for $\{a_n\}$, it follows that

$$\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= \frac{2^k - (-1)^k}{3} + 2 \cdot \frac{2^{k-1} - (-1)^{k-1}}{3} \\
&= \frac{1}{3}\left((2^k - (-1)^k + 2(2^{k-1} - (-1)^{k-1})\right) \\
&= \frac{1}{3}\left(2^k - (-1)^k + 2^k - 2(-1)^{k-1}\right) \\
&= \frac{1}{3}\left(2^k + 2^k - (-1)^k - 2(-1)^{k-1}\right) \\
&= \frac{1}{3}\left(2^{k+1} - (-1)^k - 2(-1)^{k-1}\right)
\end{aligned}$$

Since $(-1)^n$ alternates between -1 and 1 every time n is increased by 1, we can rewrite $(-1)^k$ as $-(-1)^{k+1}$, and $(-1)^{k-1}$ as $(-1)^{k+1}$. So, the above equation can be rewritten as:

$$\begin{aligned}
a_{k+1} &= \frac{1}{3}\left(2^{k+1} - (-1)^k - 2(-1)^{k-1}\right) \\
&= \frac{1}{3}\left(2^{k+1} + (-1) \cdot (-1)^k - 2 \cdot (-1)^2 \cdot (-1)^{k-1}\right) \\
&= \frac{1}{3}\left(2^{k+1} + (-1)^{k+1} - 2(-1)^{k+1}\right) \\
&= \frac{1}{3}\left(2^{k+1} - (-1)^{k+1}\right) \\
a_{k+1} &= \frac{2^{k+1} - (-1)^{k+1}}{3}
\end{aligned}$$

This is exactly the statement of proposition p_{k+1}, which was to be shown.

We've verified the initial step and the inductive step of the proof; therefore, we can conclude, that, for all $n \in \mathbb{N}$, $a_n = \dfrac{2^n - (-1)^n}{3}$.