**Math 105: Music & Mathematics**
**Test #1 Solutions & Comments**

1. For each of the following, the correct choice is highlighted in **bold** font.

a. This interval has a width of four semitones.

**Major Third** Perfect Fourth Perfect Fifth Major Sixth

b. This is the ideal (just intonation) frequency ratio of a perfect fifth.

2/1 **3/2** 5/3 5/4

c. This interval's ideal (just intonation) frequency ratio is 4/3.

Major third **Perfect Fourth** Perfect Fifth Major Sixth

d. This tuning system has consistent semitones.

Pythagorean Just Intonation **Equal Temperament** ALL of these

e. In this tuning system, the frequency ratio of an octave is always exactly 2/1.

Pythagorean Just Intonation Equal Temperament **ALL of these**

f. This describes a standard, modern piano keyboard.

i) A total of 36 keys, including three octaves and five perfect fifths

ii) A total of 64 keys, including five octaves and nine perfect fifths

**iii) A total of 88 keys, including seven octaves and twelve perfect fifths**

iv) A total of 373 keys, including thirty-one octaves and fifth-three perfect fifths

2. Suppose an instrument is tuned using Pythagorean tuning, with a base note D:540 Hz. (Hint: you may wish to make use of the Circle of Fifths diagram at the bottom of this page.)

a. Find the correct frequency for the next higher E. (Your answer should be between 540 Hz and 1080 Hz.)

Solution: To move from D to E by perfect fifths, we need to raise by fifths two times. (See the circle diagram below – E is two places around the circle clockwise from D.) To raise by fifths twice, we need to multiply by 3/2 twice: $540 \times \frac{3}{2} \times \frac{3}{2} = 1215$. But, since 1215 is higher than 1080, we need to lower this result by an octave: $1215 \div 2 = 607.5$.

So, our answer is that the next higher E would be tuned to a frequency of **607.5 Hz.**

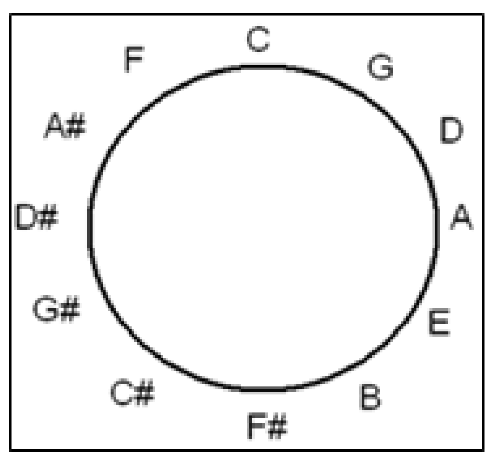
b. What note would be "L3" in this Pythagorean tuning system? Give the name of the note (A, B, C, etc.) and its frequency in Hertz (a number between 540 and 1080).

Solution: On the circle diagram, count three places counter-clockwise from D to lower D by fifths three times; the resulting note is an F. To lower by fifths three times, we need to divide by 3/2 three times, which is equivalent to multiplying by 2/3 three times:

$$540 \times \left(\frac{2}{3}\right)^3 = \frac{540 \times 8}{27} = 20 \times 8 = 160.$$

So, an F is tuned to 160 Hz. To find the frequency of the F in the 540-1080 Hz range, we need to raise 160 Hz by octaves twice: $160 \times 2^2 = 160 \times 4 = 640$.

So, our answer is that the next higher **F** would be tuned to a frequency of **640 Hz.**



CIRCLE OF FIFTHS

3. In a just intonation tuning system with base note C:300 Hz, find each of the following.

a. The correct frequency for the next higher E (between 300 Hz and 600 Hz).

Solution: The C-E interval is a major third (four semitones), so under just intonation it should have its ideal frequency ratio of 5/4. So, if the lower (base) note has frequency 300 Hz, the higher note's frequency must be $300 \times \frac{5}{4} = 375$ Hz.

b. The correct frequency for the next *lower* A (between 150 Hz and 300 Hz).

Solution: The C-A interval is a major sixth (nine semitones), so the frequency ratio should be 5/3. From a base note of C:150 Hz, the higher note's frequency must be $150 \times \frac{5}{3} = 250$ Hz.

(Alternately, you could first multiply 300 by 5/3 to get 500 Hz for the higher A, then lower by an octave to get $500 \div 2 = 250$ Hz.)

c. The note name (A, B, C, etc.) of the note whose frequency is 1000 Hz.

The trick here is to raise/lower by octaves until we find a familiar frequency ratio. If we lower 1000 Hz by an octave, we get 500 Hz. Since 500/300=5/3, which is the frequency ratio of a major sixth, we know that 500 Hz (and therefore 1000 Hz) corresponds to a note a major sixth above C:300 Hz. So, our answer is A.

(Note: Another way to get this result is based on the answer to part (b). Since there is an A with frequency 250 Hz, we can raise by octaves by doubling: $250 \times 2 = 500$, and $500 \times 2 = 1000$. This also shows us that the note with frequency 1000 Hz must be an A.)

4. The standard piano tuning uses 12-TET, with base note A:440 Hz. Based on this, find each of the following:

Note: recall that under 12-TET, all frequency ratios are $2^{n/12}$, where $n$ is the number of semitones separating the notes in the interval.

a. The correct frequency for the next higher D (between 440 Hz and 880 Hz).

Solution: A-D is a 5-semitone interval, so its frequency ratio is $2^{5/12}$. Thus, the frequency ratio of the next higher D is $440 \times 2^{5/12} \approx 587.33$ Hz.

b. The correct frequency for the next lower G (between 220 Hz and 440 Hz).

Solution: A-G is a 10-semitone inteval. So, if we start from A:220 and raise by 10 semitones to tune the next higher G, the correct frequency would be $220 \times 2^{10/12} \approx 392.00$ Hz.

Alternate solution: Starting from A:440, we would lower by 2 semitones to tune the next lower G. So, we can use the $2^{n/12}$ with $n = -2$ (negative to indicate lowering rather than raising). This would give us $440 \times 2^{-2/12} \approx 392.00$ Hz.

c. The note name (A, B, C, etc.) for the note whose frequency is (approximately) 2489 Hz.

Solution: The trick here is to figure out the value of $n$ for which $440 \times 2^{n/12} = 2489$. There are a few ways to do this; one is just trial and error (or "guess and check") until you find the whole number that works. Another approach would be to first jump up by a couple of octaves, from A:440 to A:880 to A:1760, and then solve $1760 \times 2^{n/12} = 2489$. (This is easier since, if you start in the correct octave, you just have to try whole numbers between 1 and 11.)

An alternative to trial and error would be to use logarithms:

$$
\begin{aligned}
1760 \times 2^{n/12} &= 2489 \\
2^{n/12} &= \frac{2489}{1760} \approx 1.4142 \\
\log\left(2^{n/12}\right) &\approx \log(1.4142) \\
\frac{n}{12}\log(2) &\approx \log(1.4142) \\
n &\approx \frac{\log(1.4142)}{\log(2)} \times 12 \approx 6
\end{aligned}
$$

So, the note we want is 6 semitones above A:1760, which makes it a D#.

5. In just intonation, a "minor sixth" interval has a frequency ratio of 8/5.

a. Find the width of a minor sixth in cents measurement. Round your answer to the nearest whole number of cents.

Solution: The formula to convert a frequency ratio, r, to cents, c, is

$$c = 1200 \times \frac{\log(r)}{\log(2)}.$$

Since $r = \frac{8}{5}$, this formula gives us

$$c = 1200 \times \frac{\log(8/5)}{\log(2)} \approx 1200 \times 0.678072 \approx 814 \text{ cents.}$$

b. Based on your result for (a), what is the width of a "minor sixth" in semitones? (Round your answer to the nearest whole number of semitones.) Briefly explain your answer.

Solution: Recall that, by definition, 100 cents equals one (12-TET) semitone. Therefore, 814 semitones is 8.14 semitones;, so the closest whole number equivalent would be <u>eight</u> semitones.

6. An "augmented fourth*" interval has a width of 427 cents.

a. Find the frequency ratio for an "augmented fourth."

Solution: Since 100 cents = 1 12-TET semitone, it follows that 427 cents corresponds to 4.27 12-TET semitones. So, we can use the 12-TET frequency ratio formula (also used in #4) to find the frequency ratio: $2^{4.27/12} \approx 1.28$

(Equivalently, use the $r = 2^{c/1200}$ version of this formula: $r = 2^{427/1200} \approx 1.28$.

b. If the lower note of an "augmented fourth" has frequency 640 Hz, find the frequency of the higher note.

Answer: If the lower note has frequency 640 Hz, we must multiply by the frequency ratio 1.75 to find the higher note's frequency: $640 \times: 2^{4.27/12} \approx 819.02$ Hz.

> Comment: Though it doesn't affect any of the work or answers for this problem, I should mention that my use of the term "augmented fourth" here was erroneous. (Sorry!) The actual music-theoretic name for the interval with width 427 cents (or with frequency ratio 1.28) is "diminished fourth," rather than "augmented fourth."
>
> There is also an interval called "augmented fourth," but its frequency ratio is 25/18 (about 1.389) rather than 1.28, and its width is 563 cents rather than 427 cents.
>
> For more information than you ever wanted about various interval names and width, Wikipedia has you covered: https://en.wikipedia.org/wiki/List_of_pitch_intervals

7. Use continued fractions to find a rational approximation for $\sqrt{7}$ (approx. 2.645751311). Use a list of five whole numbers to find your answer. After you've found your answer, *check* to make sure that it is a good approximation for $\sqrt{7}$.

Solution: Start by building the list of five whole numbers.
First we have $\sqrt{7} \approx 2 + 0.64575\ldots$
The reciprocal of 0.64575… = 1+0.54858… (Integer list: 2, 1)
The reciprocal of 0.54858… = 1+0.82287… (Integer list: 2, 1, 1)
The reciprocal of 0.82287… = 1+ 0.21525… (Integer list: 2, 1, 1, 1)
The reciprocal of 0.21525… = 4.64575…

The instructions asked for a list of five whole numbers, so we'll stop here. At this step, you can replace 4.64575 with 4 (its integer part) or with 5 (since 4.645… is closer to 5 than 4). Either way is fine; both give you a very good approximation (though using 5 is slightly better).

Using 2, 1, 1, 1, 4:

$$4 \to 1 + \frac{1}{4} = \frac{5}{4} \to 1 + \frac{4}{5} = \frac{9}{5} \to 1 + \frac{5}{9} = \frac{14}{9} \to 2 + \frac{9}{14} = \frac{37}{14} \approx 2.642857$$

Using 2, 1, 1, 1, 5:

$$5 \to 1 + \frac{1}{5} = \frac{6}{5} \to 1 + \frac{5}{6} = \frac{11}{6} \to 1 + \frac{6}{11} = \frac{17}{11} \to 2 + \frac{11}{17} = \frac{45}{17} \approx 2.647059$$

Both of these are very close approximations to $\sqrt{7}$, based on the decimal expansions.

Comments:

- Since we're trying to approximate the square root of 7, another good way to check our results would be to square them…

$$\left(\frac{37}{14}\right)^2 = \frac{37^2}{14^2} = \frac{1369}{196} \approx 6.9847$$
$$\left(\frac{45}{17}\right)^2 = \frac{45^2}{17^2} = \frac{2025}{289} \approx 7.0069$$

  Note that $\left(\frac{45}{17}\right)^2$ is slightly closer to 7 than $\left(\frac{37}{14}\right)^2$, which indicates 45/17 is a slightly better approximation to $\sqrt{7}$.

- The "continued fraction" version of the work for the list 2, 1, 1, 1, 5, above would look like this:

$$2 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{5}}}} = 2 + \frac{1}{1 + \frac{1}{1+\frac{1}{6/5}}} = 2 + \frac{1}{1 + \frac{1}{11/6}} = 2 + \frac{1}{17/11} = 2 + \frac{11}{17} = \frac{45}{17}$$

- If we had continued the whole number list, a pattern was about to start. Notice that the last "fractional part" we found, 0.64575…, is the same as the original fractional part of $\sqrt{7}$. This means we'd have started over from the second 1 in the list. The pattern of whole number parts goes: 2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4…