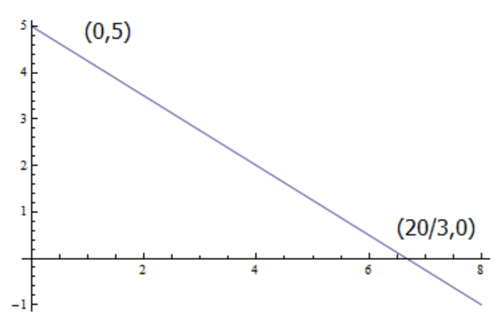
Math 160, Test #1B
Solutions

1. Let $L$ denote the line that passes through the points (4, 2) and (8, -1).

   (a) Find an equation for $L$. Write the equation in slope-intercept form.
   Solution: First, find the slope:
   $$m = \frac{2-(-1)}{4-8} = \frac{3}{-4} = -\frac{3}{4}.$$
   So, we have the equation
   $$y = -\frac{3}{4}x + b$$
   for some number $b$. To solve for $b$, substitute either (4,2) or (8,-1) for $(x, y)$:
   $$\begin{aligned} 2 &= -\frac{3}{4}(4) + b \\ 2 &= -3 + b \\ 5 &= b \end{aligned}$$
   Therefore, the line's equation is
   $$y = -\frac{3}{4}x + 5.$$

   (b) Sketch a graph of line $L$ in the space below. On your graph, label the $y$-intercept and the $x$-intercept with their exact coordinates.
   Solution: Note that to find the exact coordinates of the $x$-intercept, we need to substitute $y = 0$ into the equation for the line:
   $$\begin{aligned} 0 &= -\frac{3}{4}x + 5 \\ 5 &= \frac{3}{4}x \\ \frac{4}{3}(5) &= \frac{4}{3} \cdot \left(\frac{3}{4}x\right) \\ \frac{20}{3} &= x \end{aligned}$$




   Therefore, the $x$-intercept is at (20/3, 0), or equivalently $\left(6\frac{2}{3}, 0\right)$.

2. For all three parts of this problem, $r(x) = x^2$ and $s(x) = \sqrt{x+2}$.

   Find algebraic expressions for each of the following functions, and find each function's domain. Write the domain using interval notation.

   (a) $r + s$

   Solution:
$$(r+s)(x) = r(x) + s(x) = x^2 + \sqrt{x+2}.$$
   The only restriction on the domain of $s + r$ is that $x + 2$ must be non-negative, since the square root of a negative number is not a real number. So, we must have $x + 2 \geq 0$, which implies $x \geq -2$. Thus, our domain consists of the interval $[-2, \infty)$.

   (b) $s/r$

   Solution:
$$(s/r)(x) = \frac{s(x)}{r(x)} = \frac{\sqrt{x+2}}{x^2}.$$
   Here we have two restrictions on the domain. First, for the same reason as in part (a), we must have $x \geq -2$. However, the $x^2$ denominator gives us an additional restriction: $x \neq 0$ (since division by zero is undefined). Thus, $x$ must be at least -2, but $x$ can't be zero. Therefore, the domain of $r/s$ consists of all numbers greater than or equal to -2 *except* zero. This gives us two intervals: $-2 \leq x < 0$, and $x > 0$. In interval notation, the domain of $r/s$ is $[-2, 0) \cup (0, \infty)$.

   (c) $r \circ s$

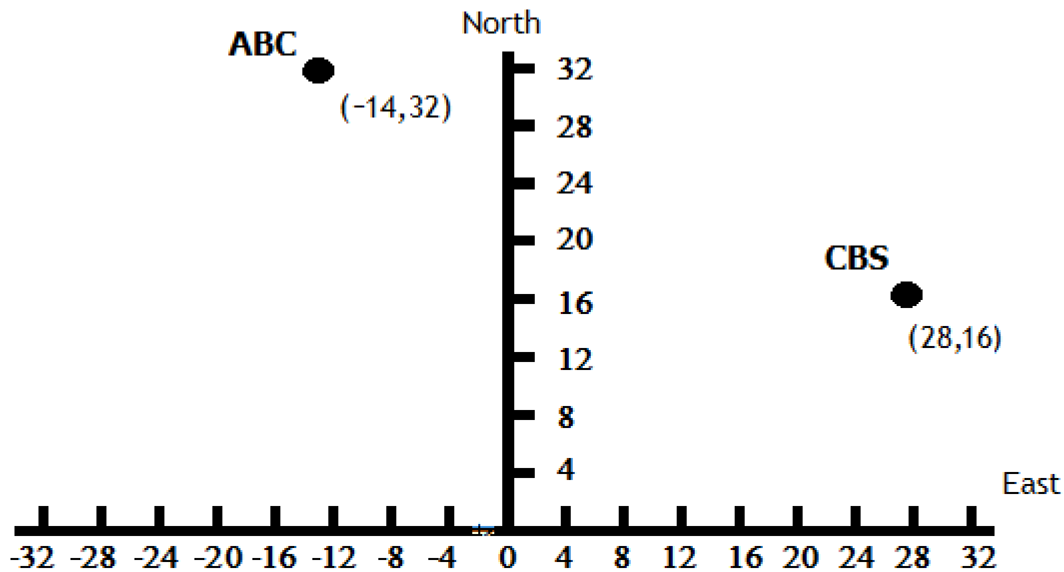
   Solution:
$$(r \circ s)(x) = r(s(x)) = r(\sqrt{x+2}) = (\sqrt{x+2})^2 = x + 2.$$
   We have to be careful here when we find the domain. Although the expression $x + 2$, on its own, is defined for all real numbers $x$, we have to account for the fact that it is a composition of functions, which means it's arrived at through a two-step process – first, find $\sqrt{x+2}$; second, square that result. The second step is valid for any input; however, the first step is defined only if $x \geq -2$. Therefore, $s \circ r$ has domain $[-2, \infty)$.

   In other "words," we could write:
$$(s \circ r)(x) = x + 2, x \geq -2$$
   to indicate at a glance that the expression $x+2$ describes the rule for the function $s \circ r$, and that the rule is valid when $x \geq -2$.

3. Joe is shopping for a new TV antenna to pick up "over-the-air" channels. The local CBS channel broadcasts from a location that is 16 miles north and 28 miles east of Joe's house. The local ABC channel broadcasts from a location that is 32 miles north and 14 miles west from Joe's house.

   (a) Sketch a graph, with Joe's house at the origin, showing the location of each channel's broadcast location relative to Joe's house. Clearly identify the coordinates $(x, y)$ of each location.

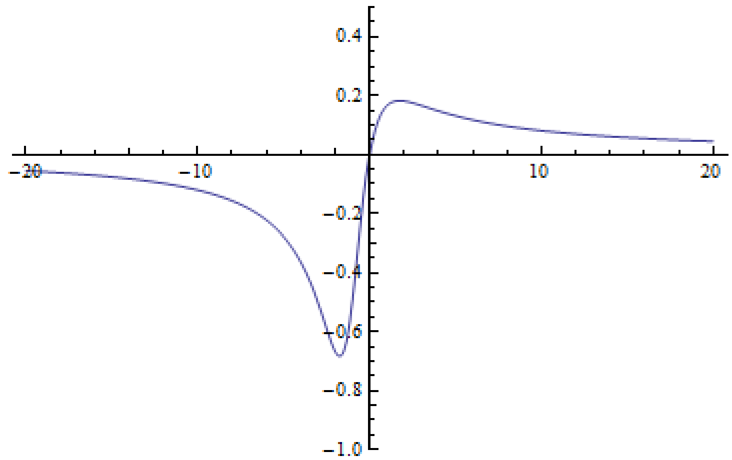
   Solution:




   (b) At the local Radio Shack, there's an inexpensive TV antenna that can pick up signals from up to 35 miles away. Would this antenna be able to pick up the local CBS and ABC broadcasts? Justify your answer.

   Solution: To answer this question, we need to find the distance from Joe's house to each of the broadcast locations. The distance to the CBS location is $\sqrt{28^2 + 16^2} = \sqrt{1040} \approx 32.25$ miles, and the distance to the ABC location is $\sqrt{32^2 + 14^2} = \sqrt{1220} \approx 34.93$ miles. Since both are within a 35 mile radius of Joe's house, the antenna should be able to pick up both broadcasts. (Although, as one student noted, ABC's signal might be a little fuzzy!)

4. The diagram below shows a graph of the function $f(t) = \dfrac{t}{t^2 + 2t + 3}$.




You will need to use the diagram to answer part (b) below.

(a) What is the *domain* of $f$? (Hint: the denominator, $t^2 + 2t + 3$, is never equal to zero.)

Solution: Due to the hint, it's clear that $f(t)$ is a real number for all values of $t$. Therefore, the domain of $f$ is the set of all real numbers.

(b) Based on the graph, what is the apparent *range* of $f$? (Your answer might not be exact; just give a reasonable estimate based on what you can see from the graph.) Write your answer using interval notation.

Solution: (Answers may vary slightly) From the graph, it appears that the maximum value of $f(t)$ is about 0.2 (when $t$ is approximately 2), and the minimum value of $f(t)$ is about -0.7 (when $t$ is approximately -2). So, the approximate range of this function would be the set of $y$-values between -0.7 and 0.2, or (-0.7, 0.2) in interval notation.

Comment: A little bit later this semester, we will learn to figure out the exact locations of the highest and lowest points on this graph, which would allow us to find the exact range of the function. It turns out that the high and low points occur at $t = \sqrt{3}$ and $t = -\sqrt{3}$, respectively, and the $y$-values at these points are $f(\sqrt{3}) = \frac{1}{4}(\sqrt{3} - 1) \approx 0.183$ and $f(-\sqrt{3}) = -\frac{1}{4}(\sqrt{3} + 1) \approx -0.683$.

5. Each of the following is a multiple-choice question. Circle the correct choice for each, and briefly (one or two sentences) explain your choice.

   (a) Walter owns and operates a car wash. Let $f(t)$ denote the number of cars washed on day $t$ of the week (where $t = 1$ on Sunday, $t = 2$ on Monday, etc.), and let $g(x)$ denote the profit earned from washing $x$ cars in a day. What does the function $(g \circ f)(t)$ represent? Circle the correct choice, and explain your answer.

      i. The total profit earned for washing $t$ cars.
      ii. The total profit earned on day $t$.
      iii. The number of cars that need to be washed to earn a profit of $t$ dollars.
      iv. The number of cars washed on day $t$.

   Solution: The correct choice is (ii) The total profit earned on day $t$.

   Note that $(g \circ f)(t)$ means $g(f(t))$. This indicates that, given $t$ (the day of the week), we are first finding the number of cars begin washed on that day ($f(t)$), and we are then passing that result along to function $g$ to find the total profit earned from washing $f(t)$ cars. So, $g \circ f$ tells us total profit, based on the input $t$, which is the day of the week; in other words, it tells us the total profit earned on day $t$.

   Comment: It's important to understand the distinction between a composition of functions and a product of functions. We are not multiplying $f(t)$ by $g(t)$ here – in fact, $g(t)$ isn't even defined, since the input for function $g$ must be a number of cars, not a day of the week.

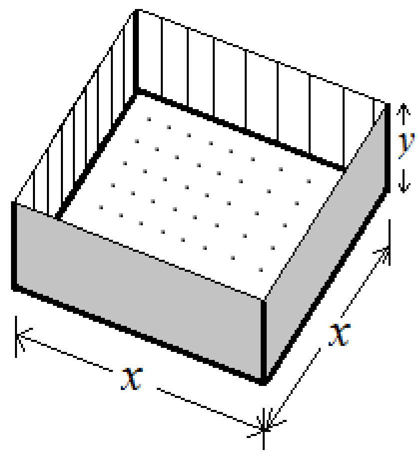
   (b) The number of Facebook stock shares that Mark owns at time $t$ is given by $f(t)$. The value per share of Facebook stock at time $t$ is $g(t)$ dollars. Find a function that would tell us the total value of Mark's Facebook shares at time $t$. Circle the correct choice, and explain your answer.

   (i) $f(g(t))$ (ii) $f(t) + g(t)$ (iii) $f(t)g(t)$ (iv) $\dfrac{f(t)}{g(t)}$

   Solution: The correct choice is (iii) $f(t)g(t)$. The total value of Mark's shares is the *product* we get when we multiply the number of shares he owns at time $t$ by the price per share at time $t$.

6. A box with an open top and a square base (see diagram below) must have a volume of 64 cubic inches. Let $x$ denote the length of one side of the base.

   (a) Find a function in the variable $x$ giving the surface area (that is, the sum of the areas of all five sides) of the box.




   Solution: The surface area is the sum of the areas of the base and the four sides. It is clear from the diagram that the base is a square with side length $x$, and each of the other four sides is a rectangle with dimensions $x$ and $y$. Therefore, the base has area $x^2$, and each of the other four sides has area $xy$, making the total surface area $x^2 + 4xy$.

   Now, we need to give this as a function of $x$, which means we need to find a way to rewrite $y$ in terms of $x$. This comes from the constraint that the volume of the box must be 64 cubic inches. Since the volume of the box must be $x^2y$, this tells us $x^2y = 64$. Solving this equation for $y$ gives us $y = \frac{64}{x^2}$. So, we just need to substitute this in place of $y$ in our expression for the surface area.

   Thus, our surface area, in terms of $x$, is $x^2 + 4x\left(\frac{64}{x^2}\right)$, which can (and should!) be simplified as follows:

$$\begin{aligned} x^2 + 4x\left(\frac{64}{x^2}\right) &= x^2 + \frac{4x}{1} \cdot \frac{64}{x^2} \quad \text{(cancel } x \text{ from the top and bottom!)} \\ &= x^2 + \frac{4}{1} \cdot 64x \\ &= x^2 + \frac{256}{x} \end{aligned}$$

   Note: the units of measurement for this expression would be square inches.

   (b) What is the domain of the function you found in part (a)? Write your answer using interval notation, and briefly explain your answer.

   Solution: The domain of this function is $(0, \infty)$. The width of the base obviously can't be zero or negative, so the lower endpoint of the domain is 0, and 0 is not included in the domain. On the other hand, there is no maximum value of $x$ in this setup – in theory, we could let $x$ be as large as we like, as long as $y$ is correspondingly small enough to make the volume 64 cubic inches.

7. In Delvamar City, the percentage of adults who smoke cigarettes stood at 25% at the beginning of 1995; this percentage decreased linearly to 16% at the beginning of 2010.

   (a) Find a linear function, $f(t)$, giving the percentage of adults in Delvamar City who smoked at the beginning of year $t$, where $t = 0$ corresponds to the beginning of 1995.

   Solution: To find our linear function, we need to find the slope, meaning the rate of change in the percentage of smokers with respect to time. The percentage decreased by 9% over 15 years, or $\frac{9}{15}$ of a percent per year. So, our slope is $-\frac{9}{15}$, or $-0.6$. Therefore, the linear function would be $f(t) = -0.6t + b$ for some number $b$. The value of $b$ is the value of $f(t)$ when $t = 0$; since we are defining $t = 0$ in 1995, and 25% of adults smoked in 1995, it follows that $f(0) = 25$; therefore, $f(t) = -0.6t + 25$.

   (b) What percentage of adults in Delvamar City smoked at the beginning of 2001?

   Solution: If $t = 0$ in 1995, then $t = 6$ in 2001, so we're being asked to find $f(6)$:

$$f(6) = -0.6 \cdot 6 + 25 = -3.6 + 25 = 21.4.$$

   Therefore, 21.4% of adults in Delvamar City smoked at the beginning of 2001.

   (c) If the percentage of adults who smoke continues to decrease linearly, by what year will all adults in Delvamar City quit smoking cigarettes? (Round your answer, if rounding is necessary, to the nearest year.).

   Solution: To answer this question, set $f(t)$ equal to zero and solve for $t$:

$$\begin{aligned} f(t) &= 0 \\ -0.6t + 25 &= 0 \\ 25 &= 0.6t \\ \frac{25}{0.6} &= t \\ t &= \frac{125}{3} \approx 41.67, \end{aligned}$$

   or $t = 42$ if we round off to the nearest year. Since $t = 0$ in 1995, $t = 42$ corresponds to the year 1995+42=2037. Therefore, if the percentage of adults who smoke continues to decrease linearly, then all adults in Delvamar City will quit smoking cigarettes by the year 2037.