Rogers-Ramanujan Identities: A Proof by Ramanujan

Introduction

The history of Rogers-Ramanujan identities is well described in various books and papers. In brief these identities were first discovered and proved by L. J. Rogers in 1894 and then later re-discovered (but not proved) by Ramanujan in 1913. Later in 1919 Ramanujan published a proof. It is this proof which will be described here.

Ramanujan was not used to publishing proofs of many of his discoveries and hence there is a feeling (even now) that his methods were mystical and often inspired by his dreams. However he did publish some proofs and when we study these it becomes at once very clear that Ramanujan possessed proofs of almost all the results he found, but it was just lack of time and resources due to which he did not record the proofs. Unfortunately this is a big loss for mathematics because from the nature of his formulas it seems that his methods were highly efficient and startling at the same time. The proof we present in this post also has the same qualities and I hope the reader will enjoy going through these.

Rogers-Ramanujan Identities

The famous Rogers-Ramanujan identities are stated below (in the style of Ramanujan, with minimal symbolism):

$$\begin{aligned}1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots\\ = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots} \quad (1)\\ 1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots\\ = \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})\cdots} \quad (2)\end{aligned}$$

In the modern notation this looks very high-brow and almost incomprehensible to anyone not well-versed in q-series:

$$\begin{aligned}1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}\\ 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}\end{aligned}$$

Ramanujan preferred to write his series or products by writing a few terms and letting the reader guess the pattern and thoroughly avoided the use of $\sum$ or $\prod$ symbols unless it was dictated by a lack of space.

Before we present Ramanujan's proof for the above identities we need to note down two corollaries of the Jacobi's Triple Product identity:

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n}) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+1)/2}\\ &= 1-q^2-q^3+q^9+q^{11}-\cdots \quad (3)\\ \prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})(1-q^{5n}) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+3)/2}\\ &= 1-q-q^4+q^7+q^{13}-\cdots \quad (4)\end{aligned}$$

Both of these easily follow from Jacobi's Triple Product identity:

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} z^n q^{n^2}$$

For example (3) is obtained by replacing q by $q^{5/2}$ and setting $z=-q^{1/2}$. Similarly (4) is obtained by replacing q by $q^{5/2}$ and setting $z=-q^{3/2}$.

With these prerequisites in place we are ready to give:

Ramanujan's Proof

Ramanujan multiplies both sides of the identities by $(q;q)_{\infty}=\prod(1-q^n)$ to get

$$\left(1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_n}\right)(q;q)_{\infty} = \prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n}) \quad (5)$$

$$\left(1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}\right)(q;q)_{\infty} = \prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})(1-q^{5n}) \quad (6)$$

Note that the RHS of (5) is same as LHS of (3) and RHS of (6) is same as LHS of (4). Again both the sums on the LHS are easily seen to be special values of the function

$$F(x) = 1+\frac{xq}{1-q}+\frac{x^2q^4}{(1-q)(1-q^2)}+\frac{x^3q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots \quad (7)$$

the first sum being $F(1)$ and the second sum being $F(q)$. It is easy to verify that $F(x)$ satisfies the following functional equation:

$$F(x) = F(xq)+xqF(xq^2) \quad (8)$$

Conversely $F(x)$ is the unique function satisfying the above equation with condition $F(0)=1$. Let us now define another function $G(x)$ by

$$G(x) = (xq;q)_{\infty}F(x) = F(x)(1-xq)(1-xq^2)(1-xq^3)\cdots$$

Then clearly we can see that LHS of (5) is $G(1)$ and LHS of (6) is $(1-q)G(q)$. Also the functional equation for $F(x)$ is transformed into

$$G(x) = (1-xq)G(xq)+xq(1-xq)(1-xq^2)G(xq^2) \quad (9)$$

The genius of Ramanujan is that he was able to find a series representation of $G(x)$ which can be shown to satisfy the above equation and is such that series for $G(1)$ matches RHS of (3) and series for $(1-q)G(q)$ matches RHS of (4). We now give series representation of $G(x)$ as follows:

$$
\begin{aligned}
G(x) &= 1 + \sum_{n=1}^{\infty}(-1)^n x^{2n} q^{n(5n-1)/2}(1 - xq^{2n})\frac{(xq;q)_{n-1}}{(q;q)_n}\\
&= 1 - x^2q^2(1-xq^2)\frac{1}{1-q} + x^4q^9(1-xq^4)\frac{1-xq}{(1-q)(1-q^2)} - \cdots \qquad (10)
\end{aligned}
$$

It is now a simple matter to verify that

$$
\begin{aligned}
G(1) &= 1 - q^2 - q^3 + q^9 + q^{11} - \cdots\\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+1)/2}\\
&= \prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n})
\end{aligned}
$$

and

$$
\begin{aligned}
(1-q)G(q) &= 1 - q - q^4 + q^7 + q^{13} - \cdots\\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+3)/2}\\
&= \prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})(1-q^{5n})
\end{aligned}
$$

The difficult part is to verify that $G(x)$ satisfies equation (9). To do so we recast (9) as follows:

$$
\frac{G(x)}{1-xq} - G(xq) = xq(1-xq^2)G(xq^2) \qquad (11)
$$

Also the series for $G(x)$ needs to be rearranged by splitting each term into two terms according to the equation:

$$
1 - xq^{2n} = 1 - q^n + q^n(1 - xq^n)
$$

and then associating second part of each term with first part of succeeding term. Doing this exercise we get

$$
\begin{aligned}
G(x) = (1 - x^2q^2) &- x^2q^3(1-x^2q^6)\frac{1-xq}{1-q}\\
&+ x^4q^{11}(1-x^2q^{10})\frac{(1-xq)(1-xq^2)}{(1-q)(1-q^2)} - \cdots \qquad (12)
\end{aligned}
$$

Next we calculate $G(x)/(1-xq)$ from (12) and use series (10) for $G(xq)$ and thereby evaluate:

$$
\begin{aligned}
H(x) &= \frac{G(x)}{1-xq} - G(xq)\\
&= xq - \frac{x^2q^3}{1-q}\{(1-q) + xq^4(1-xq^2)\}\\
&\quad + \frac{x^4q^{11}(1-xq^2)}{(1-q)(1-q^2)}\{(1-q^2) + xq^7(1-xq^3)\}\\
&\quad - \frac{x^6q^{24}(1-xq^2)(1-xq^3)}{(1-q)(1-q^2)(1-q^3)}\{(1-q^3) + xq^{10}(1-xq^4)\} + \cdots
\end{aligned}
$$

Again as before if we associate second part of each term with first part of the succeeding term then we get:

$$\begin{aligned}H(x) &= xq(1-xq^2)\left\{1 - x^2q^6(1-xq^4)\frac{1}{1-q}\right.\\&\quad + x^4q^{17}(1-xq^6)\frac{1-xq^3}{(1-q)(1-q^2)}\\&\quad \left.- x^6q^{33}(1-xq^8)\frac{(1-xq^3)(1-xq^4)}{(1-q)(1-q^2)(1-q^3)} + \cdots\right\}\\&= xq(1-xq^2)G(xq^2)\end{aligned}$$

and thus the proof of the Rogers-Ramanujan identities is completed.

Ramanujan does not offer any clue as to how he found out the definition of $G(x)$ given in (10) and given the complicated definition of $G(x)$ it appears that Ramanujan was probably trying to figure out the form of solution of the functional equation (9). But still this is the part of the proof which is highly non-obvious and ingenious.

Going one step further Ramanujan defines another function

$$K(x) = \frac{G(x)}{(1-xq)G(xq)}$$

and then the functional equation for $G(x)$ is transformed into

$$K(x) = 1 + \frac{xq}{K(xq)}$$

and from this we get the continued fraction expansion:

$$K(x) = 1 + \frac{xq}{1+}\,\frac{xq^2}{1+}\,\frac{xq^3}{1+}\cdots \tag{13}$$

Therefore by putting $x = 1$ we obtain the famous Rogers-Ramanujan continued fraction:

$$\begin{aligned}R(q) &= \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots\\&= q^{1/5}\cdot\frac{(1-q)G(q)}{G(1)}\\&= q^{1/5}\cdot\frac{\prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})}{\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})}\\&= q^{1/5}\cdot\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\end{aligned} \tag{14}$$

Out of the all the proofs given for Rogers-Ramanujan identities, I find the one by Ramanujan to be the simplest to understand as it requires no knowledge of high brow concepts and symbolism apart from Jacobi's Triple Product dentity.

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