Math 105, Spring 2017

Practice Exercises on Sequences and Recursion

Note: all terminology and notation used in this assignment is defined in the "sequences" handout on the class web page, as well as in recent class meetings.

1. List the first eight terms of the <u>arithmetic</u> sequence whose first two terms are 5, 9.

5 , 9 , ____ , ____ , ____ , ____ , ____ , ____

2. List the first eight terms of the <u>geometric</u> sequence whose first two terms are 48, 24.

48 , 24 , ____ , ____ , ____ , ____ , ____ , ____

3. Find the first eight terms of the sequence whose first two terms are 1, 3, which satisfies the following <u>two-term recursion</u> rule:

$$x_n = x_{n-1} + 2x_{n-2}.$$

(Hint: the third term is $3 + 2 \times 1 = 5$; the fourth term is $5 + 2 \times 3 = 11$.)

1 , 3 , 5 , 11 , ____ , ____ , ____ , ____

4. Figure out the next three terms in each of the following sequences. Explain your reasoning for each.

a. 1, 3, 7, 15, 31, ____ , ____ , ____

b. 1, 3, 6, 10, 15, ____ , ____ , ____

c. 8, 12, 18, 27, ________ , ________ , ________

d. 2, 9, 11, 20, 31, ____ , ____ , ____

5. Find the next few terms of each of the following sequences. For each, describe the recursive rule for the sequence in words. Then, let x_n denote the nth term of the sequence, and write an equation describing the recursive rule for the sequence.

a. 3, 7, 11, 15, 19, 23, 27, …
b. 2, 5, 11, 23, 47, 95, …
c. 1, 3, 4, 7, 11, 18, 29, …
d. 1, 3, 6, 10, 15, 21, …

6. Write the first ten terms of each sequence, given the recursive rule for the sequence.

a. $x_n = x_{n-1} + 2x_{n-2}$, with $x_1 = 1$ and $x_2 = 1$.
(Hint: this sequence starts with 1, 1, 3, 5, 11, …)
b. $x_n = 3x_{n-1} - 1$, with $x_1 = 1$
(Hint: this sequence starts with 1, 2, 5, 14, …)
c. $x_n = 3x_{n-1} - 1$, with $x_1 = 3$
(Hint: this sequence starts with 3, 8, 23, …
d. $x_n = x_{n-1} + x_{n-2} + x_{n-3}$, with x_1, x_2, and x_3 all equal to 1.
(Hint: this sequence starts with 1, 1, 1, 3, 5, …)

7. Let t_n stand for the number of ways to write a rhythm whose length is n beats using only quarter notes (1 beat) and/or eighth notes (1/2 beat).

For example: the only 1-beat rhythms we can write under this rule would be either a single quarter note, or two eighth notes. Since there are two ways to write a rhythm whose length is 1 beat, the value of t_1 is 2. In other words, $t_1 = 2$.

a. Find the values of t_2 and t_3 by listing all acceptable 2-beat and 3-beat rhythms, respectively. (Hint: your answer for t_3 should be between 10 and 20.)

b, (Optional – challenge question) Based on your observations for t_1, t_2, and t_3, see if you can find a recursion rule to predict how the sequence t_n will continue when n is greater than 3. See if your rule works when $n = 4$, by finding all acceptable 4-beat rhythms. (Hint: your answer for t_4 should be between 30 and 40.)

8. Let b_n stand for the number of n-beat rhythms we can write under the rule that every note is either a quarter note (one beat) or a dotted half note (3 beats).

a. Write out all possible rhythms for n = 1, 2, 3, 4, 5 and 6. Use your results to find the values of b_1, b_2, b_3, b_4, b_5 and b_6. (Hint: none of these should be a very large number.)
b. See if you can figure out, and explain, a recursive rule for the sequence of numbers b_n. (Hint: the reasoning for this one will be *similar* to the reasoning for the example from class in which we restricted ourselves to only quarter notes and half notes.)

SOLUTIONS
(Please do not read these until you've actually tried the problems yourself!)

1. 5, 9, 13, 17, 21, 25, 29, 33 (constant difference of 4 between terms)

2. 48, 24, 12, 6, 3, $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}$ (constant ratio of ½ between terms)

3. 1, 3, 5, 11, 21, 43, 85, 171

a. 1, 3, 7, 15, 31, 63, 127, 255
Rule: $x_n = 2x_{n-1} + 1$ (each term is double the preceding term, plus one)

b. 1, 3, 6, 10, 15, 21, 28, 36
Rule: $x_n = x_{n-1} + n$
In this case, verbal may be better: the key observation is that the differences between terms are increasing by 1 at each step: 3-1=2, 6-3=3, 10-6=4, etc. That's the pattern we need to continue the list.

Alternate solution: The nth number in this sequence is the sum 1+2+3+…+n.
That is: 1+2=3; 1+2+3=6; 1+2+3+4=10; 1+2+3+4+5=15; etc.
(These numbers are known in mathematics as the "triangular numbers.")

Alternate solution #2: Take a look at Pascal's triangle and see if you can find this sequence (it should jump right out at you if you're looking for it…)

c. 8, 12, 18, 27, $\frac{81}{2}, \frac{243}{4}, \frac{729}{8}, \frac{2187}{16}$ (or: 8, 12, 18, 27, 40.5, 60.75, 91.125, 136.6875)
Rule: $x_n = \frac{3}{2}\, x_{n-1}$ … this is just a geometric sequence with common ratio 3/2

d. 2, 9, 11, 20, 31, 51 , 82 , 133

Rule: $x_n = x_{n-1} + x_{n-2}$ … this is a "Fibonacci-type" sequence, with the same two-term recursive rule (but different initial values!)

5. Find the next few terms of each of the following sequences. For each, briefly describe the recursive rule for the sequence:

Answers:
3, 7, 11, 15, 19, 23, 27, **31, 35, 39, …**
Recursive Rule: Each term is four more than the preceding term; that is, it's an arithmetic seqeuence with a common difference of 4.
Equation: $x_n = x_{n-1} + 4$

2, 5, 11, 23, 47, 95, **191, 383, 767, …**
Recursive Rule: Each term is one more than double the preceding term.
Equation: $x_n = 2x_{n-1} + 1$

1, 3, 4, 7, 11, 18, 29, **47, 76, 123, …**
Recursive Rule: This is another "Fibonacci-type" sequence, with the same two-term recursive rule.
Equation: $x_n = x_{n-1} + x_{n-2}$

1, 3, 6, 10, 15, 21, **28, 36, 45, …**
(This one is a re-run from #4. Oops!)

6. Write the first ten terms of each sequence, given the recursive rule for the sequence

1, 1, 3, 5, 11, 21, … where $x_n = x_{n-1} + 2x_{n-2}$
Answer: The recursive rule is telling us that each term is the sum of the preceding term and twice the term that came before that. So, for example, the next term in the sequence after the terms 11, 21, would be $21 + 2 \cdot 11 = 43$. The next term, which comes after 21, 43, would be $43 + 2 \cdot 21 = 85$. Continuing with this rule, here are the first ten terms of the sequence… 1, 1, 3, 5, 11, 21, 43, 85, 171, 341

1, 2, 5, 14, … where $x_n = 3x_{n-1} - 1$
Answer: The recursive rule here is that each term is one less than three times the preceding term. (Note that the only subscript involved is n-1 – there is no n-2 term this time – so each new term in the sequence depends only on the immediately preceding term, and not on any terms that came before that.) So, for example, the term following 14 in this sequence would be $3 \cdot 14 - 1 = 41$. The term after 41 would be $3 \cdot 41 - 1 = 122$. Continuing with this rule, here are the first ten terms of the sequence… 1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842

1, 1, 1, 3, 5, 9, ..., where $x_n = x_{n-1} + x_{n-2} + x_{n-3}$
Answer: The recursive rule here is that each new term is the sum of the preceding three terms. (That's what the subscripts n-1, n-2, and n-3 are telling us.) So, for example, the next term after 3,5,9 in the sequence will be $3 + 5 + 9 = 17$. The next term after that, which will come after 5, 9, 17, will be $5 + 9 + 17 = 31$. And so on; continuing with this rule, here are the first ten terms of the sequence… 1, 1, 1, 3, 5, 9, 17, 31, 57, 105

7. [Will be discussed in class…]

8. Let b_n stand for the number of n-beat rhythms (not melodies – just rhythms this time) we can write under the rule that every note is either a quarter note (one beat) or a dotted half note (3 beats).

a) Write out all possible rhythms for n = 1, 2, 3, 4, 5 and 6. Use your results to find the values of b_1, b_2, b_3, b_4, b_5 and b_6. (Hint: none of these should be a very large number.)

Answers: 1, 1, 2, 3, 4, 6

b) See if you can figure out, and explain, a recursive rule for the sequence of numbers b_n. (Hint: the reasoning for this one will be *similar* to the reasoning for the example from class in which we restricted ourselves to only quarter notes and half notes.)

Answer: $b_n = b_{n-1} + b_{n-3}$ -- that is, each term is equal to the preceding term plus the term that came three places earlier. (This is similar to the Fibonacci rule, except that instead of using the term that came two places earlier in the sequence, we use the term that came three places earlier.

Rationale: to write an n-beat rhythm, either add a quarter note to the end of an n-1 beat rhythm, or add a dotted half note to the end of a n-3 beat rhythm.