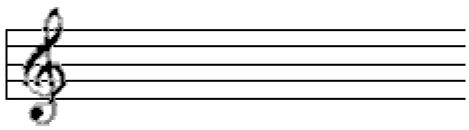
Math 105 – Variations
Supplemental Practice Exercises II

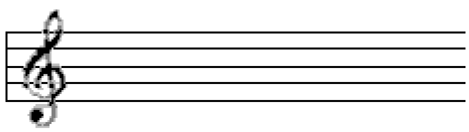
A few more examples that you should work out for yourself. We'll go over these in class, as needed.

1. For the melody shown to the right, apply each of the following variations, or combinations of variations. Write your answer as a list of notes, and also by writing the variation on a music staff.

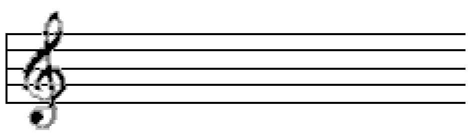
a) T_6



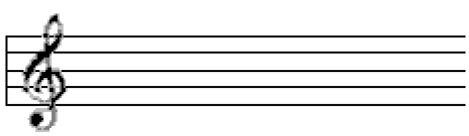
b) T_4R



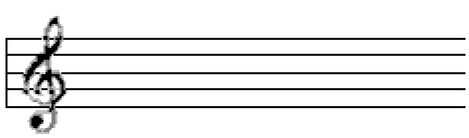
c) T_3RT_3



d) IT_3I



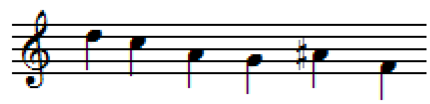
e) T_5IRT_7



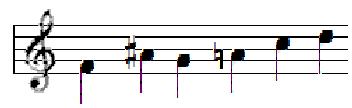
2. Simplify each of the following combinations of variations. That is, rewrite the combination in one of the following forms – T_n, T_nR, T_nI, or T_nIR - where $0 \leq n \leq 11$).

a) IIIII b) $IT_4RT_3IT_2$ c) $T_9RT_6T_8T_3RT_4R$ d) $\mathrm{T_5RIR}$

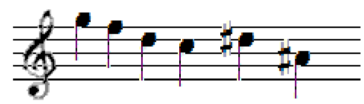
3. Starting with the "melody" D, C, A, G, A#, F (shown to the right), determine which variation (from the group of 48 variations discussed in class) gives us each of the following results. (Hint: consider the interval formed by consecutive notes in each example, and try to match that up with the melody.



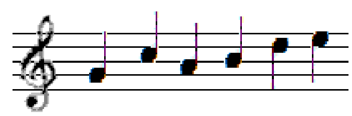
a) F, A#, G, A, C, D



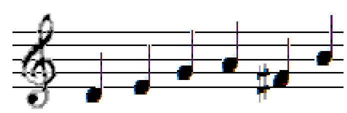
b) G, F, D, C, D#, A#



c) G, C, A, B, D, E



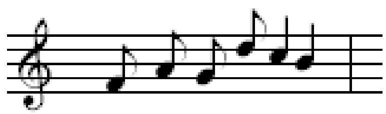
d) D, E, G, A, F#, B



Variations Practice Exercises, Part II – Solutions

Note: for each of the following solutions, a list of notes as well as an equivalent list of numbers (according to the "musical clock" as described in class) is given.

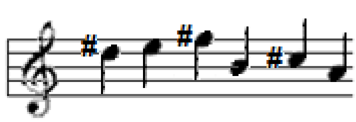
1. This melody consists of the notes F, A, G, D, C, B (or, "by the numbers:" 5, 9, 7, 2, 0, 11)



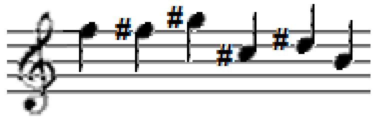
a) T_6: **B, D#, C#, G#, F#, F** (by the numbers: 11, 3, 1, 8, 6, 5)

(Comment: the ♮ symbol next to the second "F" is there to indicate that it's an F "natural" rather than an F "sharp." In musical composition, if two F's are played back to back and the first is marked as a "sharp," it's understood that the next F will also be a sharp unless there's something written to indicate otherwise. If you're unfamiliar with reading/writing music then don't worry, this isn't a vital detail for the purpose of this class; that is, I won't take points off if you write something incorrectly based on it. However, for those who do read music, the ♮ is necessary to make the above notation technically correct.)

b) T_4R: **D#, E, F#, B, C#, A** (by the numbers: 3, 4, 6, 11, 1, 9)



c) T_3RT_3: **F, F#, G#, C#, D#, B** (by the numbers: 5, 6, 8, 1, 3, 11)
Comment: Note that this is just the retrograde of the variation found in part (a) – this is because T_3RT_3 is equal to T_6R, or the retrograde of T_6.



d) IT_3I: **D, F#, E, B, A#, G#** (by the numbers: 2, 6, 4, 11, 9, 8)
Comment: While it's a good exercise to work this out in steps (first the inversion, then T_3 of that result, then the inversion of the second result), it's worth noting that we could also use the rules discussed in class to show that IT_3I is equal to T_9; thus, this variation is simply the original melody transposed upwards by 9 semitones. (Note: in the solution below, we actually see that variation lowered by an octave, so that all the notes end up on the staff.

e) This one can be simplified slightly by working out that T_5IRT_7 is equal to $T_{10}IR$…

T_{10}: D#, G, F, C, A#, A   (by the numbers: 3, 7, 5, 0, 10, 9)
$T_{10}I$: A, F, G, C, D, D# (by the numbers: 9, 5, 7, 0, 2, 3)
$\boldsymbol{T_{10}IR}$: **D#, D, C, G, F, A** (by the numbers: 3, 2, 0, 7, 5, 9)

2. Simplify each of the following combinations of variations.
a) IIIII   b) $IT_4RT_3IT_2$ c) $T_9RT_6T_8T_3RT_4R$ d) $\mathrm{T_5RIR}$

a) Since $II = T_0$ - that is, two consecutive inversions "cancel," we can remove two pairs of I's, leaving us with just a single I as our answer.

b) $\underbrace{IT_4}_{T_8I} RT_3 \underbrace{IT_2}_{T_{10}I} = T_8IR \underbrace{T_3T_{10}}_{T_1} I = T_8I \underbrace{RT_1}_{T_1R} I = T_8 \underbrace{IT_1}_{T_{11}I} \underbrace{RI}_{IR} = \underbrace{T_8T_{11}}_{T_7} \underbrace{II}_{T_0} R = \boldsymbol{T_7R}$

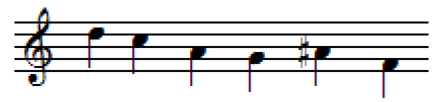
Answer: $\boldsymbol{T_7R}$

c) $T_9RT_6T_8T_3RT_4R$ – note that there are no inversions in this example. Therefore, it is acceptable here to rearrange the variations in whatever order we like (since transpositions and retrogrades "commute" with each other – i.e. the order doesn't affect the outcome). This allows us to rewrite $T_9RT_6T_8T_3RT_4R$ as $T_9T_6T_8T_3T_4RRR$. Since 9+6+8+3+4=30, which reduces to 6 (mod 12), and two of the three R's will "cancel," our result is just $\boldsymbol{T_6R}$.

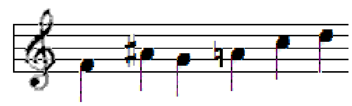
Note: Make sure you understand why it's okay to just rearrange the T's and R's in this example, while a similar strategy would not have worked in part (b) due to the presence of inversions.

d) $T_5RIR = T_5 \underbrace{RR}_{T_0} I = T_5I$

3. Starting with the "melody" D, C, A, G, A#, F (shown to the right), determine which variation (from the group of 48 variations discussed in class) gives us each of the following results. (Hint: consider the interval formed by consecutive notes in each example, and try to match that up with the melody. Also, it may help to rewrite the melody and each of the following variations numerically – i.e,. C=0, C#=1, etc.)

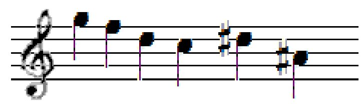


a) F, A#, G, A, C, D



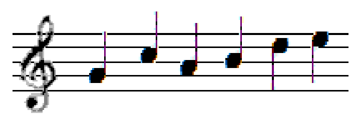
Answer: R. This is the retrograde of the original melody.

b) G, F, D, C, D#, A#



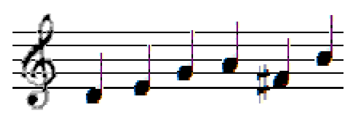
Answer: T_5: This is the original melody transposed by five semitones (a.k.a. a perfect fourth).

c) G, C, A, B, D, E



Answer: T_2R. This is the retrograde (see part (a)) of the original melody transposed upwards by two semitones. (To see this, compare the "shapes" of variation (a) and (c) as shown on the staff. Alternative, you can check intervals between consecutive notes, and find that the intervals between consecutive notes are the same in part (c) as they were in part (a).)

d) D, E, G, A, F#, B



Answer: T_8I.

This is a little harder to see, since it involves an inversion. To see that an inversion is involved, note that the "shape" of this variation is, roughly speaking, an upside-down "flip" of the original melody. (That is, it rises where the original melody falls, and vice-versa.) Alternatively, look at intervals between consecutive notes – they are the same size for the original melody and for this variation, but in opposite directions. For example, the first interval in the melody is D-C, which falls (goes down) by two semitones; on the other hand, the first interval in this variation is D-E, which rises (goes up) by two semitones. This pattern holds for the entire melody.

Once we see that this variation involves an inversion somehow, a logical first step is to find the inversion of the original melody: this is A#, C, D#, F, D, G. Then, we note that A# (the first note of the inversion) is four semitones below D (the first note of the variation in part (d), so we'd need to transpose the inversion up by four semitones. This works for all six notes; therefore, the variation in part (d) is found by using IT_4. Finally, to put this in the form we've been using in class (where transpositions are listed first), we rewrite IT_4 as T_8I, using the rule for switching inversions with transpositions. (Note: you should check this on your own, to make sure that applying first T_8, then I, to the original melody gives you the inversion in part (d). It should work!)

Additional comment on 3(d) – we could also view this variation as an inversion "centered at D." That is, it's an inversion that fixes the note D (rather than C) in place and inverts the other notes based on their distances from D (rather than from C). For example: the second note of the original melody is a C, which is two semitones below a D; correspondingly, the second note of the variation is an E, which is two semitones above a D. This pattern continues for the other four notes as well.

It's interesting to note that we don't need to define "inversion about D" as a separate variation – instead, we can just use $T_8 I$ for this, as shown in #3(d) above. (Or equivalently, we can use IT_4 for the same thing, as shown in class.) Similarly, inversion about any other note can be defined in terms of the inversion (about C) used in class, combined with a transposition.