Introduction to Change Ringing

Change ringing is a centuries-old art by which arrays of large bells are rung in various orderings, known as “changes.” The bells are mounted in high towers on circular mounts which allows them to swing through a full circle in order to ring. The ringers stand far below (the size and volume of the bells necessitates this) and ring the bells by pulling on ropes to swing them through a full rotation.

In practice, several bells are rung in sequence. The sequence is then changed slightly (hence the term “change”), and this process is repeated. An objective of change ringing is to carry out many different changes, without repetition. There are limitations to the changes that can be made from one sequence to the next, since the large size of each bell requires some time to elapse between ringings. The effect of this limitation is that the only changes allowed between sequences of bells are those that can be described as “adjacent swaps;” that is, the relative location of a bell in a sequence must be kept the same, or it may trade (“swap”) with the bell that came immediately before or immediately after it in the preceding sequence.

A simple example which illustrates this rule is a change ringing method known as “Plain Hunt” on four bells. In the following diagram, each number represents a distinct bell – we may imagine that they are numbered in decreasing order of pitch, so that 1 stands for the highest-pitched bell (the “treble”) and 4 for the lowest (the “tenor”). Each line describes a sequence of bells, or “change;” note how each change varies from the previous one:

1 2 3 4
2 1 4 3
2 4 1 3
4 2 3 1
4 3 2 1
3 4 1 2
3 1 4 2
1 3 2 4
1 2 3 4

We can see a repetitive pattern. First, the first two and last two bells in the sequence are swapped; then, the middle two are swapped. This process is repeated four times, giving us a total of eight different changes. (Note that the last line is the same as the first line; this implies that a continuation of the same process will lead to the same sequence of changes over and over again; this would create a cycle.) Are these all the changes that are possible? Or, might we be able to modify this method slightly, resulting in even more different changes on four bells?

To describe these changes in a systematic way, we introduce some notation. First, we'll index the entries in each change using letters, rather than numbers, to avoid confusion. So, in any given change, we refer to the first position as position "a," the second position as "b," the third as "c," and so on. (We're never going to look at changes involving anywhere near twenty-six positions, so this system will be fine.)

So, for example, in the initial list 1 2 3 4 from the previous page, we'd say that bell 1 is in position a, 2 is in position b, 3 is in position c, and 4 is in position d.

Now, in order to describe the change from 1 2 3 4 to 2 1 43, we'll refer to the positions whose entries are being interchanged. First, we're swapping the 1 and the 2; more generally, this change swaps positions a & b. Similarly, swapping the 3 and 4 is described as swapping positions c & d. A standard notation for each of these swaps is the following:
(ab) – the action of swapping the entries in positions a and b
(cd) – the action of swapping the entries in positions c and d

So, the action that rearranges 1 2 3 4 into 2 1 4 3 can be described as $(ab)(cd)$. The point of describing it in this way is that, no matter what the contests of the list are at any given time, $(ab)(cd)$ always refers to the action of swapping positions a & b, and then swapping positions c & d.

For further illustration, we'll use our notation to describe the next few changes in the list on the preceding page. The second change rearranges 2 1 4 3 into 2 4 1 3. In this rearrangement, we are swapping the second (position "b") and third (position "c") entries in the list. We can describe this action using the notation (bc).

From the second change to the third change on the preceding page, 2 4 1 3 is rearranged into 4 2 3 1. The action here is the same action as the one we started with – it interchanges the first two entries (positions a and b), and it interchanges the last two entries (positions c and d). Thus, once again we are seeing the action $(ab)(cd)$; this time it's applied to 2 4 1 3 to give us 4 2 3 1.

The following change, 4 2 3 1 to 4 3 2 1, is another (bc) swap. Indeed the entire procedure on the preceding page can be described as alternating between the actions $(ab)(cd)$ and (bc). We find that we can iterate (repeat) this process four times before our results start to cycle, giving us a total of eight changes in all.

This leads naturally to a related question – how many different changes, or rearrangements, are possible on four bells? Is this list of eight changes exhaustive? We can quickly see that the answer to this second question is no, since many rearrangements of 1 2 3 4 do not appear in our list – for example, 2 1 3 4 and 1 2 4 3 are both rearrangements of 1 2 3 4, but these are not generated by our procedure. So, how many (other) rearrangements did we miss? And, if we wanted to, could we find a relatively simple procedure by which we'd generate all possible rearrangements of 1 2 3 4? Let's try to answer these questions…

Problem: To count the rearrangements (changes) of n distinct objects (bells)

It would take a very long time to list all of the rearrangements of five objects, so let's not start there. When addressing a large problem, it is often advisable to start by tackling something similar but smaller, both to get a feel for the more general problem and also (in some cases) to find a way to generalize solutions from smaller problems to larger ones.

In this case, let's start by listing all of the rearrangements of two items, say 1 and 2. These would be: [1,2] and [2,1]. (Comment: often square brackets are used in place of rounded brackets to distinguish between ordered and unordered lists, respectively.) Obviously there are two ways to rearrange two objects; no surprise there.

Next, let's list all of the rearrangements of three items: 1, 2, and 3. Again, not difficult to do:

[1, 2, 3]	[1, 3, 2]	[3, 1, 2]
[2, 1, 3]	[2, 3, 1]	[3, 2, 1]

etc… will finish later. (We did this in class on Thursday!)