Solutions to Cosets Practice Exercises

1. The set of whole numbers $\{0, 1, 2, \dots, 12, 13, 14\}$ under addition (mod 15) turns out to be a group.

a) Show that the set {0, 5, 10} is a subgroup of this group.

b) Find the coset generated by 1.

c) Find the coset generated by 2.

d) Does this subgroup have any other cosets? If so, find them. If not, how do you know?

Solutions:

a) From the following table, we can see that the set is closed under addition (mod 15). Also, we see that 0 is the identity of the group, and each element has an opposite (0 is its own opposite; 5 and 10 are opposites of each other.)

+	0	5	10
0	0	5	10
5	5	10	0
10	10	0	5

(Note: 10 +5 = 15, which is equivalent to 0 under (mod 15) addition. 10+10=20, which is equivalent to 5 under (mod 15) addition.)

b) To find the coset generated by 1, add 1 to each number in the subgroup:

1+0=1; 1+5=6; 1+10=11

Thus, the coset generated by 1 is {1, 6, 11}.

c) Similar to part (b) – the coset generated by 2 is {2, 7, 12}.

d) There are two more cosets – they can be generated by 3 and by 4. These are (respectively) {3, 8, 13} and {4, 9, 14}. This gives us a total of five distinct cosets:

(0, 5, 10} (Note: technically a subgroup is considered to be one of its own cosets)
{1, 6, 11} (This is the coset generated by 1, by 6, or by 11)
{2, 7, 12} (This is the coset generated by 2, by 7, or by 12)
{3, 8, 13} (This is the coset generated by 3, 8, or 13)
{4, 9, 14} (This is the coset generated by 4, 9, or 14)

2. The set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} under multiplication (mod 11) turns out to be a group.

a) Show that the set {1, 3, 9, 5, 4} is a subgroup of this group.

b) Find the coset generated by 2.

c) Are there any other cosets of this subgroup? If not, why not? Otherwise, find the other coset(s).

Answers:

a) The following table demonstrates that this set has closure, identity and opposites under multiplication (mod 11). (Make sure you get the same results – multiplication (mod 11) can be a little tricky, so take this opportunity to practice it)

×	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

Comment: This is also the cyclic subgroup generated by 3 – repeatedly multiplying by 3 (mod 11) gives us the results 3, then 9, then 5, then 4, then 1.

b) The coset generated by 2 is found by multiplying each number in the subgroup- by 2 (mod 11):

$2 \times 1 = 2;$

$2 \times 3 = 6;$

$2 \times 4 = 8;$

$2 \times 5 = 10;$

$2 \times 9 = 18 \equiv 7 \ (mod\ 11)$

So, the coset generated by 2 is $\{2, 6, 8, 10, 7\}$.

c) There are only ten elements in the original group, each of which is either in the subgroup {1,3,4, 5, 9} or in the coset {2,6, 8, 10, 7}. This accounts for all elements of the original group; therefore, this subgroup can have no other cosets.

Comment: As discussed in class, the cosets of a subgroup will always “partition” the larger group, meaning that each element of the group will be in one, and only one, coset. That fact is illustrated by this example.

3. Consider the set of variations $\{T_0, T_3R, T_6, T_9R\}$.

a) Verify that this is a group.

b) Find the coset generated by T_2R.

c) Find the coset generated by IR.

d) How many other cosets of this subgroup should there be? (Hint: recall that there are a total of 48 different variations.)

Solutions:

a) This set has identity, opposites and closure, so it's a group. The table to the right demonstrates this. Note that T_3R and T_9R are opposites, and T_6 is its own opposite.

	T_0	T_3R	T_6	T_9R
T_0	T_0	T_3R	T_6	T_9R
T_3R	T_3R	T_6	T_9R	T_0
T_6	T_6	T_9R	T_0	T_3R
T_9R	T_9R	T_0	T_3R	T_6

b) The coset generated by T_2R is found by combining T_2R with each element of the subgroup, as follows:

$$T_2R\ T_0 = T_2R$$
$$T_2R\ T_3R = T_2T_3RR = T_5$$
$$T_2R\ T_6 = T_2T_6R = T_8R$$
$$T_2R\ T_9R = T_2T_9RR = T_{11}$$

Therefore, the coset generated by T_2R is $\{T_2R, T_5, T_8R, T_{11}\}$.

c) The coset generated by IR is found by combining T_2R with each element of the subgroup.

(Note: Don't forget the rule for combining inversions with transpositions: $IT_n = T_{12-n}I$)

$$IR\ T_0 = IR$$
$$IR\ T_3R = \underbrace{IT_3}_{T_{12-3}I}\ \underbrace{RR}_{T_0} = T_9I$$
$$IR\ T_6 = \underbrace{IT_6}_{T_{12-6}I}\ R = T_6IR$$
$$IRT_9R = \underbrace{IT_9}_{T_{12-9}I}\ RR = T_3I$$

Therefore, the coset generated by IR is $\{IR,\ T_9I,\ T_6IR,\ T_3I\}$.

d) Since there are 48 variations in all, and the subgroup we're dealing with has 4 variations, then every coset will contain exactly four variations. (This is illustrated by the two cosets we found explicitly here). Thus, we're partitioning the full group of 48 variations into cosets of size 4, which means there will be $48 \div 4 = 12$ distinct cosets, including the subgroup itself. Since two more cosets were found in parts (b) and (c) above, we've accounted for three cosets so far; this means there are nine other cosets that we haven't found here. (Optional extended exercise – find the other nine cosets!)