Lab #3: Quadratic Approximation – due Monday, November 9

Note: In Section 3.10, we will learn about "linear approximation" techniques, which are applications of the idea that functions and their tangent lines are approximately the same near the tangency point. This lab expands upon this idea by considering what are effectively "tangent parabolas" rather than tangent lines.

In this introduction, we will demonstrate how to find the "quadratic approximation" to the cosine function. Read this example, and then follow the instructions for the lab assignment on the third page of this handout.

Definition: The "quadratic approximation" to the function $f(x)$ at $x = a$ is the unique quadratic function, $P(x) = A + Bx + Cx^2$, that satisfies all three of the following conditions:

i) $P(a) = f(a)$ ii) $P'(a) = f'(a)$ iii) $P''(a) = f''(a)$

(Note: tangent lines, also known as "linear approximations," satisfy the first two of these three conditions.)

Part of the advantage of working with polynomial functions is that they are so easy to differentiate: if $P(x) = A + Bx + Cx^2$, then we can quickly compute $P'(x) = B + 2Cx$ and $P''(x) = 2C$. This will make it pretty straightforward to solve for A, B, and C. Let's see how this works with an example…

<u>EXAMPLE</u>: Find the "quadratic approximation" to the function $f(x) = \cos x$ at $x = 0$.

This will be the quadratic function, $P(x) = A + Bx + Cx^2$, that satisfies each of the following conditions:

i) $P(0) = f(0)$ ii) $P'(0) = f'(0)$ iii) $P''(0) = f''(0)$

Answer: First, note that if $f(x) = \cos x$, then $f'(x) = -\sin x$ and $f''(x) = -\cos x$.
Therefore,

$$f(0) = \cos 0 = 1, f'(0) = -\sin 0 = 0, \text{ and } f''(0) = -\cos 0 = -1.$$

Also, as noted above,

$$P(x) = A + Bx + Cx^2, P'(x) = B + 2Cx, \text{ and } P''(x) = 2C.$$

Substituting in 0 for x gives us

$$P(0) = A, \quad P'(0) = B, \quad \text{and} \quad P''(0) = 2C.$$

Therefore, conditions (i), (ii), and (iii) for a quadratic approximation give us the following results:

i) $P(0) = f(0)$ ii) $P'(0) = f'(0)$ iii) $P''(0) = f''(0)$
i) $A = 1$ ii) $B = 0$ iii) $2C = -1$, which means $C = -\frac{1}{2}$

Therefore, $P(x) = A + Bx + Cx^2 = 1 + 0x - \frac{1}{2}x^2$, or just $P(x) = 1 - \frac{1}{2}x^2$.

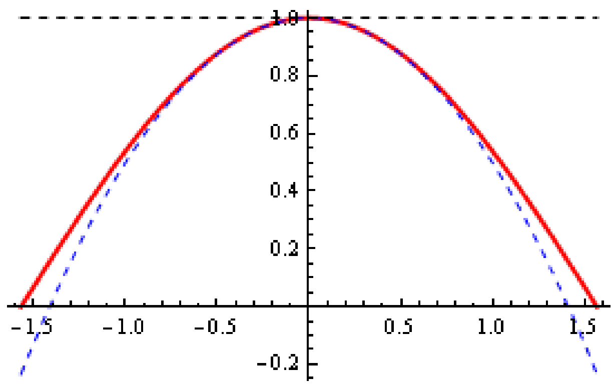
Thus, the quadratic approximation to the function $f(x) = \cos x$ at $x = 0$ is $P(x) = 1 - \frac{1}{2}x^2$.

Note: the following is not a part of the lab assignment; it's just a side note to illustrate why one would be interested in a quadratic approximation (or "tangent parabola") to a continuous function.

Below is a graph of $y = \cos x$, $y = 1$, and $y = 1 - \frac{1}{2}x^2$:

```
In[10]:= Plot[{Cos[x], 1, 1 - x^2 / 2}, {x, -Pi / 2, Pi / 2},
          PlotStyle → {{Red, Thick}, {Black, Dashed}, {Blue, Dashed}}]
```

Out[10]=



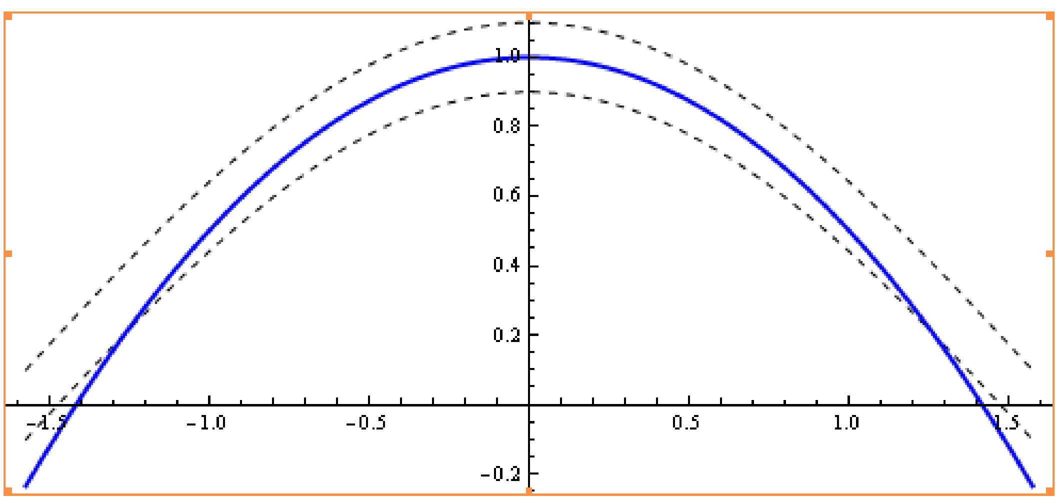
Observe that the quadratic function $P(x) = 1 - \frac{1}{2}x^2$ (blue in the above diagram) is a *much* better approximation of $\cos x$ near $x = 0$ than the linear approximation $y = 1$.

<u>Example</u> (continued):
Determine the values of x for which the quadratic approximation $P(x) \approx \cos x$ is accurate to within 0.1.

To do this, we'll adapt a technique that is also demonstrated in the textbook (p. 253), which is to plot "boundaries" that appear 0.1 units above and below the graph of $f(x)$…

```
In[11]:= Plot[{Cos[x] + 0.1, Cos[x] - 0.1, 1 - x^2 / 2}, {x, -Pi / 2, Pi / 2},
          PlotStyle → {{Black, Dashed}, {Black, Dashed}, {Blue, Thick}}]
```

Out[11]=



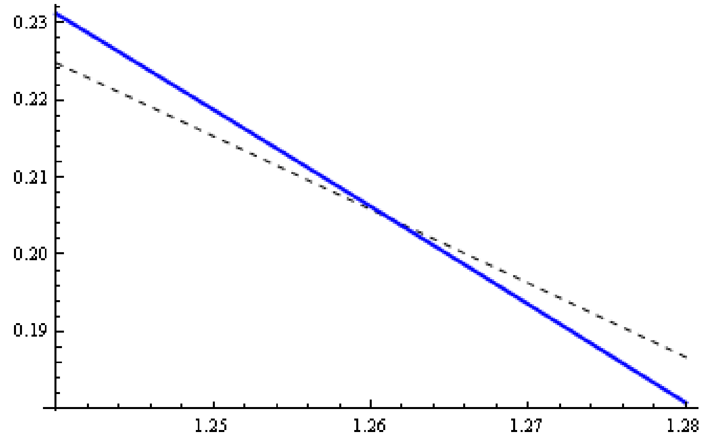
Note: the dashed curves in the above graph represent $\cos x + 0.1$ and $\cos x - 0.1$.

We see that the graph of $P(x)$ falls between the two dashed "boundary" curves when x is between -1.3 and 1.3 (approximately). So, the approximation $P(x) \approx \cos x$ is apparently accurate to within 0.1 between $x = -1.3$ and $x = 1.3$; that is, on the interval (-1.3, 1.3).

To get a more precise answer, we can "zoom in" on the points where $y = P(x)$ intersects the lower boundary curve, $y = \cos x - 0.1$:

```
In[13]:= Plot[{Cos[x] - 0.1, 1 - x^2/2}, {x, 1.24, 1.28},
          PlotStyle → {{Black, Dashed}, {Blue, Thick}}]
```

Out[13]=

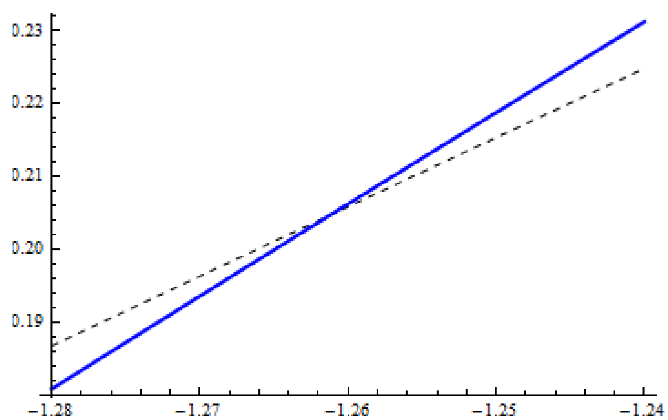


(Note that in this graph, we *only* included P(x) and the "boundary" curve that P(x) actually intersects. If we include any other curves in our graph, we will not be able to see the intersection point clearly.)

This suggests that the actual intersection point is close to $x = 1.26$; similar investigation reveals that these curves also intersect close to $x = -1.26$:

```
In[30]:= Plot[{Cos[x] - 0.1, 1 - x^2/2}, {x, -1.28, -1.24},
          PlotStyle → {{Black, Dashed}, {Blue, Thick}}]
```

Out[30]=



So, **the approximation $P(x) \approx \cos x$ appears to be accurate to within 0.1 on the interval (-1.26, 1.26).**

(Caution: don't *assume* that the left and right endpoints of these intervals will be opposites of one another. That happens in this case because $\cos(x)$ happens to be an even function; this will *not* always be the case!)

<u>Lab Assignment</u>:

Repeat the above steps at $x = 0$ for each of the following functions:

$g[x] = e^x$
(Reminder: The expression e^x is entered as **E^x** in Mathematica)

$h[x] = \ln(1 + x)$
(Reminder: In Mathematica, the natural log function is entered as **Log,** so this will be **Log[1+x].**)

$j[x] = \sqrt{4 + x}$
(Reminder: In Mathematica, this will be entered as **Sqrt[4+x].**)

For *each* of these functions, complete the following steps…

1. Find a quadratic approximation, $P(x) = A + Bx + Cx^2$, as demonstrated in the example. Your goal is to find the right coefficients A, B, C to give you a quadratic approximation near x=0 for that particular function. (You should mimic the example to the extent possible, modifying it to suit the function you're trying to approximate.)

2. Still following the example, find an interval of x-values for which the quadratic approximation, $P(x)$, is within 0.1 of the function being approximated. Estimate the endpoints of this interval accurate to the nearest hundredth, as shown at the end of the example.

What to turn in:

- Your quadratic approximation for each function – show your work! Either write your work out by hand or include it in the Mathematica worksheet (see below).

- A printout of your Mathematica worksheet that includes your graphs, similar to the graphs shown in the example on the preceding pages, which you used to find the precise endpoints of the interval of x-values for which the quadratic approximation is accurate to within 0.1.

DUE DATE: Monday, November 9.