Math 160, Fall 2015
Collected Homework #5 Solutions & Comments

- Section 4.1, #70

  Find all relative minimum and maximum values of $f(x) = x + \frac{9}{x} + 2$.

  Solution: We begin by finding the "critical numbers" of f - numbers in the domain of f at which $f'(x) = 0$ or f' is undefined. To this end, we'll need to find f':

$$f(x) = x + \frac{9}{x} + 2$$
$$f'(x) = 1 - \frac{9}{x^2}$$

  Note that f' is undefined at $x = 0$; however, since 0 isn't in the domain of f ($f(0)$ is also undefined), this does not count as a "critical number." However, it is an x-value where $f'(x)$ could change sign, so we'll have to account for it when we test intervals (see below):

  Next, we'll set $f'(x) = 0$ and solve for x:

$$1 - \frac{9}{x^2} = 0 \to 1 = \frac{9}{x^2} \to x^2 = 9 \to x = \pm 3$$

  So, our critical numbers are 3 and -3.

  If we had a graph, we could easily locate the relative maximum/minimum points with this information. Since we don't have a graph handy, we'll instead test for intervals of increase/decrease.

Interval	Test Value	Result	
$(-\infty, -3)$	$x = -4$	$f'(-4) = 1 - (9/16) > 0$	(positive)
$(-3, 0)$	$x = -1$	$f'(-1) = 1 - (9/1) < 0$	(negative)
$(0, 3)$	$x = 1$	$f'(-1) = 1 - (9/1) < 0$	(negative)
$(3, \infty)$	$x = 4$	$f'(-4) = 1 - (9/16) > 0$	(positive)

  This tells us that f is increasing when $x < -3$, and decreasing when $-3 < x < 0$. Therefore, f has a relative maximum at $x = -3$. Similarly, f has a relative minimum at $x = 3$. Since $f(-3) = -3 - 3 + 2 = -4$, (-3,-4) is a relative maximum of f; similarly, since $f(3) = 3 + 3 + 2 = 8$, (3,8) is a relative minimum of f.

- Section 4.1, #92

  Determine the values of t at which $C(t)$ is increasing, and the values of t at which $C(t)$ is decreasing, on the interval $0 \le t \le 4$, where

$$C(t) = \frac{t^2}{2t^3 + 1}.$$

  First, we need to find the derivative of this function. We'll need to use the Quotient Rule:

$$\begin{aligned} C'(t) &= \frac{(2t^3+1)(2t) - (t^2)(6t^2)}{(2t^3+1)^2} \\ &= \frac{4t^4 + 2t - 6t^4}{(2t^3+1)^2} = \frac{2t - 2t^4}{(2t^3+1)^2} = \frac{2t(1-t^3)}{(2t^3+1)^2} \end{aligned}$$

  Note that the only t-values where $C'(t)$ can change sign are those where either its numerator or denominator is equal to zero. Also, don't forget that the domain of C is $0 \le t \le 4$, so the only values of t we're concerned with here are those between 0 and 4.

  The numerator of $C'(t)$ is zero when $t = 0$ or $t = 1$. Since we're assuming t is at least 0, it follows that $2t^3 + 1$ is positive, which means the denominator is never negative. So the only time in the interval $0 \le t \le 4$ when C could switch from increasing to decreasing, or vice-versa, is at $t = 1$.

  So, we need to test the sign of $C'(t)$ on the intervals (0,1) and $(1, \infty)$. Note (again) that the denominator of $C'(t)$ will be positive whenever t is positive, which means we only really have to check the numerator, $2t(1-t^3)$. We can use $t = 1/2$ and $t = 2$ as test points for the intervals (0,1) and $(1, \infty)$, respectively:

Interval	Test Value	Result	
$(0, 1)$	$t = 1/2$	$2t(1-t^3) = 2(1/2)(1/8) > 0$	(positive)
$(1, \infty)$	$t = 2$	$2t(1-t^3) = 2(2)(-7) < 0$	(negative)

  So, $C'(t)$ is positive when $0 < t < 1$, and negative when $t > 1$.

  Thus, $C(t)$ is increasing when $0 < t < 1$, and decreasing when $t > 1$.

- Section 4.2, #50

  Find the inflection point(s) of $g(x) = x^3 - 6x$.

  Inflection points are points on a graph at which the second derivative changes sign. To find potential inflection points, we must find all values of x at which $g''(x) = 0$. (More generally, we'd look for values of x in the domain of g at which $g''(x)$ is zero *or* undefined; since g is a polynomial in this example, $g''(x) = 0$ is the only possibility we need to consider.)

  If $g(x) = x^3 - 6x$, then $g''(x) = 6x$. Clearly, then, $g''(x)$ is negative when x is negative, and $g''(x)$ is positive when x is positive. So, the inflection point of $g(x)$ occurs when $x = 0$. Since $g(0) = 0$, the single inflection point of $g(x)$ is the point (0,0).

- Section 4.2, #64

  First, we'll set $g'(x) = 0$ to find the critical numbers of g. Since $g'(x) = 3x^2 - 6 = 3(x^2 - 2)$, our critical numbers are $x = \pm\sqrt{2}$. To apply the Second Derivative Test, we must find the sign of g'' at each of these numbers.

  Note that $g''(x) = 6x$. Therefore, $g''(-\sqrt{2}) = -6\sqrt{2}$, which is negative, and $g''(\sqrt{2}) = 6\sqrt{2}$, which is positive.

  Now we apply the Second Derivative Test:

  Since $g'(\sqrt{2}) = 0$ and $g''(\sqrt{2}) < 0$, g has a relative minimum at $\sqrt{2}$.

  Since $g'(-\sqrt{2}) = 0$ and $g''(-\sqrt{2}) < 0$, g has a relative maximum at $-\sqrt{2}$.

  To find the value of g at each of these points, first note that

  $$(\sqrt{2})^3 = (\sqrt{2})^2 \cdot \sqrt{2} = 2\sqrt{2}.$$

  Therefore,

  $$g(\sqrt{2}) = 2\sqrt{2} - 6\sqrt{2} = -4\sqrt{2},$$

  and

  $$\begin{aligned} g(-\sqrt{2}) &= 2(-\sqrt{2}) - 6(-\sqrt{2}) \\ &= -2\sqrt{2} + 6\sqrt{2} \\ &= 4\sqrt{2} \end{aligned}$$

  Thus, g has a relative minimum at $(\sqrt{2}, -4\sqrt{2})$, and g has a relative maximum at $(-\sqrt{2}, 4\sqrt{2})$.