# Irrationality of ζ(2) and ζ(3): Part 2

In the last post we proved that $\zeta(2)$ is irrational. Now we shall prove in a similar manner that $\zeta(3)$ is irrational. Note that this proof is based on Beukers' paper *"A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$."*

### Irrationality of $\zeta(3)$

Like the case of $\zeta(2)$ we first establish certain formulas concerning some double integrals which are related to $\zeta(3)$. The derivation of these formulas is based on the integral formulas established in last post.

#### Preliminary Results

*Let $r, s$ be non-negative integers with $r > s$. Then we have*

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}x^r y^r\,dx\,dy = 2\sum_{n=1}^{\infty}\frac{1}{(n+r)^3} \tag{1}$$

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}x^r y^s\,dx\,dy = \frac{1}{r-s}\left\{\frac{1}{(s+1)^2}+\frac{1}{(s+2)^2}+\cdots+\frac{1}{r^2}\right\} \tag{2}$$

Using equation $(2)$ from the last post we get

$$\int_0^1\int_0^1 \frac{x^{r+\alpha}y^{r+\alpha}}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty}\frac{1}{(n+r+\alpha)^2}$$

Differentiating the above relation with respect to $\alpha$ we get

$$\int_0^1\int_0^1 \frac{x^{r+\alpha}y^{r+\alpha}\log xy}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty}\frac{-2}{(n+r+\alpha)^3}$$

Now putting $\alpha = 0$ the first result is established. This means that

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}\,dx\,dy = 2\zeta(3)$$

and if $r$ is a positive integer then

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}x^r y^r\,dx\,dy = 2\left\{\zeta(3)-\left(\frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{r^3}\right)\right\}$$

Next from equation $(3)$ of last post we have

$$\int_0^1\int_0^1 \frac{x^{r+\alpha}y^{s+\alpha}}{1-xy}\,dx\,dy = \frac{1}{r-s}\left\{\frac{1}{s+\alpha+1}+\frac{1}{s+\alpha+2}+\cdots+\frac{1}{r+\alpha}\right\}$$

Differentiating the above relation with respect to $\alpha$ we get

$$\begin{aligned}&\int_0^1\int_0^1 \frac{x^{r+\alpha}y^{s+\alpha}\log xy}{1-xy}\,dx\,dy\\&\quad=\frac{1}{r-s}\left\{\frac{-1}{(s+\alpha+1)^2}+\frac{-1}{(s+\alpha+2)^2}+\cdots+\frac{-1}{(r+\alpha)^2}\right\}\end{aligned}$$

Putting $\alpha = 0$ in the above equation we obtain equation $(2)$.

From the above results it is now clear that if $P(x), Q(x)$ are polynomials of degree $n$ with integer coefficients then

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy}P(x)Q(y)\,dx\,dy=\frac{a\zeta(3)+b}{d_n^3}$$

where $a, b$ are some integers dependent on polynomials $P(x), Q(x)$ and $d_n$ denotes the LCM of numbers $1, 2, \ldots, n$ (aslo for completeness we can assume $d_0 = 1$).

**Strategy of the Proof**

Now we choose a specific polynomial $P_n(x)$ defined by

$$P_n(x)=\frac{1}{n!}\frac{d^n}{dx^n}\{x^n(1-x)^n\}$$

Since $P_n(x)$ is a polynomial of degree $n$ with integer coefficients it follows that the integral defined by

$$I_n=\int_0^1\int_0^1 \frac{-\log xy}{1-xy}P_n(x)P_n(y)\,dx\,dy$$

can be expressed in the form

$$I_n=\frac{a_n\zeta(3)+b_n}{d_n^3}$$

where $a_n, b_n$ are integers dependent on $n$.

We will establish that

1. $I_n \neq 0$ for all positive integers $n$.
2. $d_n^3 I_n \to 0$ as $n \to \infty$.

This will imply that the expression $a_n\zeta(3)+b_n \to 0$ as $n \to \infty$ and is never zero for any value of $n$. If $\zeta(3)$ were rational, say $p/q$, then we would have $|a_n\zeta(3)+b_n| \geq 1/q$ and hence $a_n\zeta(3)+b_n$ would not tend to zero. This contradiction proves that $\zeta(3)$ is irrational.

**Estimation of $I_n$**

Now we come to the proof of the two claims mentioned above which are vital to obtain a contradiction needed to prove the irrationality of $\zeta(3)$.

First we need to observe that

$$\begin{aligned}\int_{0}^{1}\frac{dz}{1-az} &= \left[\frac{-1}{a}\log(1-az)\right]_{z=0}^{z=1}\\ &= -\frac{\log(1-a)}{a}\end{aligned}$$

hence on putting $a = 1 - xy$ we get

$$\frac{-\log xy}{1-xy} = \int_{0}^{1}\frac{dz}{1-(1-xy)z}$$

Using the above equation we can write the integral $I_n$ as a triple integral

$$I_n = \int_{0}^{1}\int_{0}^{1}\int_{0}^{1}\frac{P_n(x)P_n(y)}{1-(1-xy)z}\,dx\,dy\,dz \tag{3}$$

Using integration by parts $n$ times with respect to $x$ we get

$$I_n = \int_{0}^{1}\int_{0}^{1}\int_{0}^{1}\frac{(xyz)^n(1-x)^nP_n(y)}{\{1-(1-xy)z\}^{n+1}}\,dx\,dy\,dz$$

Following Beukers, we apply the substitution

$$w = \frac{1-z}{1-(1-xy)z}$$

so that

$$z = \frac{1-w}{1-(1-xy)w},\ 1-z = \frac{xyw}{1-(1-xy)w}$$

Hence

$$dz = \frac{-xy}{\{1-(1-xy)w\}^2}dw$$

and

$$\begin{aligned}\frac{z^n}{\{1-(1-xy)z\}^{n+1}} &= \frac{(1-w)^n}{\{1-(1-xy)w\}^n}\frac{w^{n+1}}{(1-z)^{n+1}}\\ &= \frac{(1-w)^n(1-(1-xy)w)}{(xy)^{n+1}}\end{aligned}$$

Also note that as $z$ moves from $0$ to $1$, $w$ moves from $1$ to $0$.

After substituting these expressions we get

$$I_n = \int_{0}^{1}\int_{0}^{1}\int_{0}^{1}\frac{(1-x)^n(1-w)^nP_n(y)}{1-(1-xy)w}\,dx\,dy\,dw$$

Using integration by parts $n$ times with respect to $y$ we get

$$I_n = \int_{0}^{1}\int_{0}^{1}\int_{0}^{1}\frac{x^n(1-x)^ny^n(1-y)^nw^n(1-w)^n}{\{1-(1-xy)w\}^{n+1}}\,dx\,dy\,dw \tag{4}$$

and from this expression it is clear that $I_n > 0$ as the integrand is positive for all

$x, y, w \in (0, 1)$.

We next need to find an estimate for the function $f(x, y, w)$ defined by

$$f(x, y, w) = \frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w}$$

for $x, y, w \in (0, 1)$. Finding the maximum value using first and second partial derivatives seems a bit complicated hence it is better to go for a simpler approach based on inequalities.

The denominator of $f(x, y, w)$ is $1 - w + xyw$ and clearly we have

$$1 - w + xyw \geq 2\sqrt{(1-w)xyw}$$

and hence we have

$$f(x, y, w) \leq \frac{1}{2}\sqrt{x}(1-x)\sqrt{y}(1-y)\sqrt{w(1-w)}$$

If we put $x = t^2$ then $\sqrt{x}(1-x) = t(1-t^2) = t - t^3$ which is maximum when $t = 1/\sqrt{3}$ and the maximum value is $2/(3\sqrt{3})$. Similar is the case for $\sqrt{y}(1-y)$. The maximum value of $\sqrt{w(1-w)}$ is clearly $(w + 1 - w)/2 = 1/2$. Hence we have

$$f(x, y, w) \leq \frac{1}{2}\frac{2}{3\sqrt{3}}\frac{2}{3\sqrt{3}}\frac{1}{2} = \frac{1}{27}$$

Therefore by equation $(4)$ we get

$$\begin{aligned} I_n &= \int_0^1\int_0^1\int_0^1 \frac{\{f(x, y, w)\}^n}{1-(1-xy)w}\,dx\,dy\,dw \\ &\leq \left(\frac{1}{27}\right)^n \int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-xy)w}\,dx\,dy\,dw \\ &= \left(\frac{1}{27}\right)^n \int_0^1\int_0^1 \frac{-\log xy}{1-xy}\,dx\,dy \\ &= 2\zeta(3)\left(\frac{1}{27}\right)^n \end{aligned}$$

From the last post we know that if $K > e$ is a fixed number then $d_n < K^n$ for all sufficiently large values of $n$. Hence it follows that

$$d_n^3 I_n < 2\zeta(3)\left(\frac{K^3}{27}\right)^n \tag{5}$$

for all sufficiently large values of $n$. If we choose $K$ such that $e < K < 3$ then we can see that the right hand side of equation $(5)$ above tends to zero as $n \to \infty$. Therefore $d_n^3 I_n \to 0$ as $n \to \infty$. We have thus completed the proof of irrationality of $\zeta(3)$.

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Thursday, October 10, 2013

Labels: Irrational Numbers , Mathematical Analysis