Section 1.2 Just Intonation

Recall that the Pythagorean tuning system is designed to preserve octaves and perfect fifths. However, there are some drawbacks to this tuning system:

- There will always be (at least) one perfect fifth that is not tuned correctly. The mathematical reason for this is that twelve fifths are approximately, but not exactly, equal to seven octaves.
- The frequency ratios of several intervals other than perfect fifths are close to, but not quite, other ratios that are also pleasing to the ear. For example, the C-E interval has a frequency ratio of $\frac{81}{64} \approx 1.266$ which is very close to the "nicer" ratio of $\frac{5}{4} = 1.25$. Other such examples:
  - $\frac{32}{27} \approx 1.185$ is close to the "nicer" ratio of $\frac{6}{5} = 1.2$.
  - $\frac{27}{16} \approx 1.688$ is close to the "nicer" ratio of $\frac{5}{3} \approx 1.667$.

Generally speaking, the ear prefers intervals whose frequency ratios (higher frequency divided by lower frequency) are "nicer" fractions, which essentially means fractions with smaller denominators.

The second of these drawbacks ("close… but not quite") can be addressed through a process called "**just intonation**." This is a tuning system designed to "preserve" certain preferred intervals, at the expense of other less important intervals. (Note that terms such as "preferred" and "less important" are subjective and could be defined in ways other than those we'll elect to use in our treatment.)

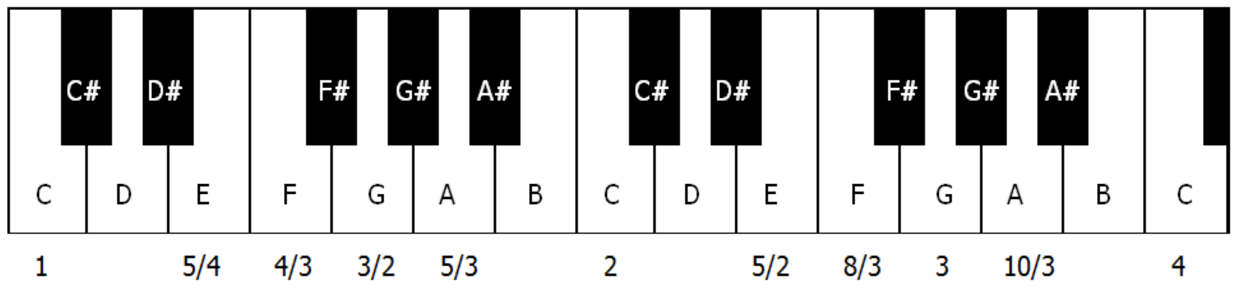
We'll develop a system consistent with the one in the text (p. 27) based on the following principles:

- All octaves will be preserved
- The following frequency ratios for intervals whose lower note is the "base" note (C, for our example) will be preserved:
  - Perfect fifth: 7 semitones, frequency ratio = 3/2
  - Perfect fourth: 5 semitones, frequency ratio = 4/3
  - Major third: 4 semitones, frequency ratio = 5/4 (rather than the Pythagorean ratio 81/64)
  - Major sixth: 9 semitones, frequency ratio = 5/3 (rather than the Pythagorean ratio 27/16)

  Note: the names "third," "fourth," "fifth" and "sixth" correspond to the third, fourth, fifth and sixth notes of what is called the "major scale." For example, starting from C, these notes are, respectively: E, F, G and A.
- Frequency ratios for other notes will be selected in such a way as to preserve fifths, fourths, thirds and/or sixths wherever possible (with the understanding that we'll sometimes be forced to *choose* between one interval and another – there is no single "right" choice in these cases.)

Note: the "just intonation tuning system" is not unique – there is a wide variety of ways to tune the octave using the just intonation process. In the following, the word "decide" is used to indicate steps at which we are <u>selecting</u> one method (from among several possibilities) to carry out the process.

The first two guidelines above give us the following start to our just intonation system:



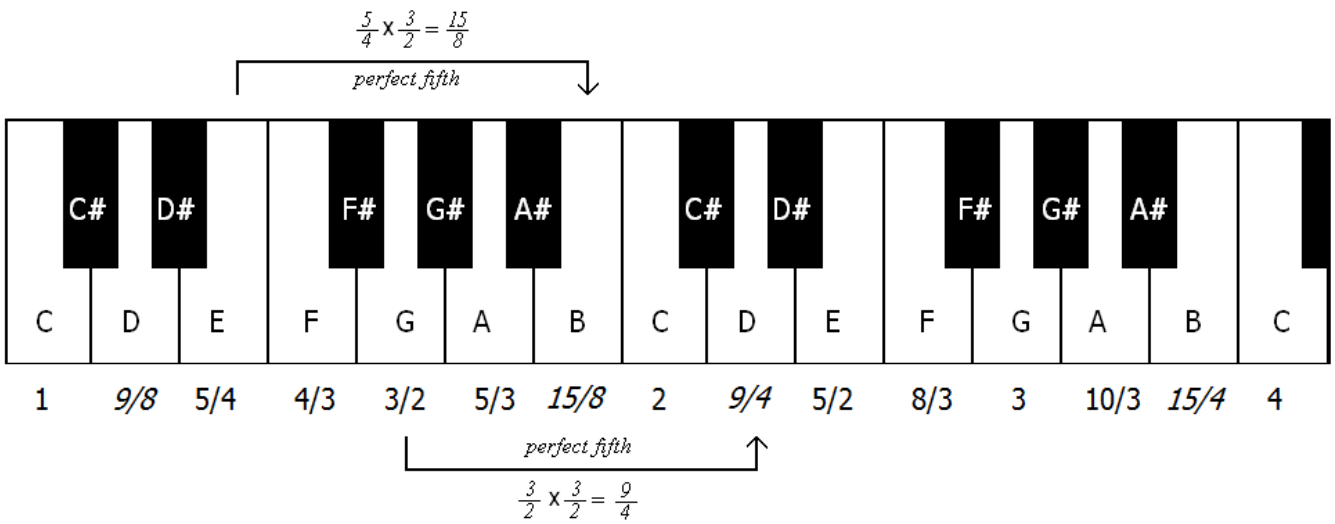
Next, we will *decide* to preserve the fifths G-D and E-B:

Raise the G by a perfect fifth to tune the D: $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$

So, the higher D is tuned to 9/4 of the base frequency, implying the lower D – one octave lower - is tuned to 9/8 of the base frequency.

Raise the E by a perfect fifth to tune the B: $\frac{5}{4} \times \frac{3}{2} = \frac{15}{8}$

So, the E is tuned to 15/8 of the base frequency, implying the higher E – one octave higher – is tuned to 15/4 of the base frequency.

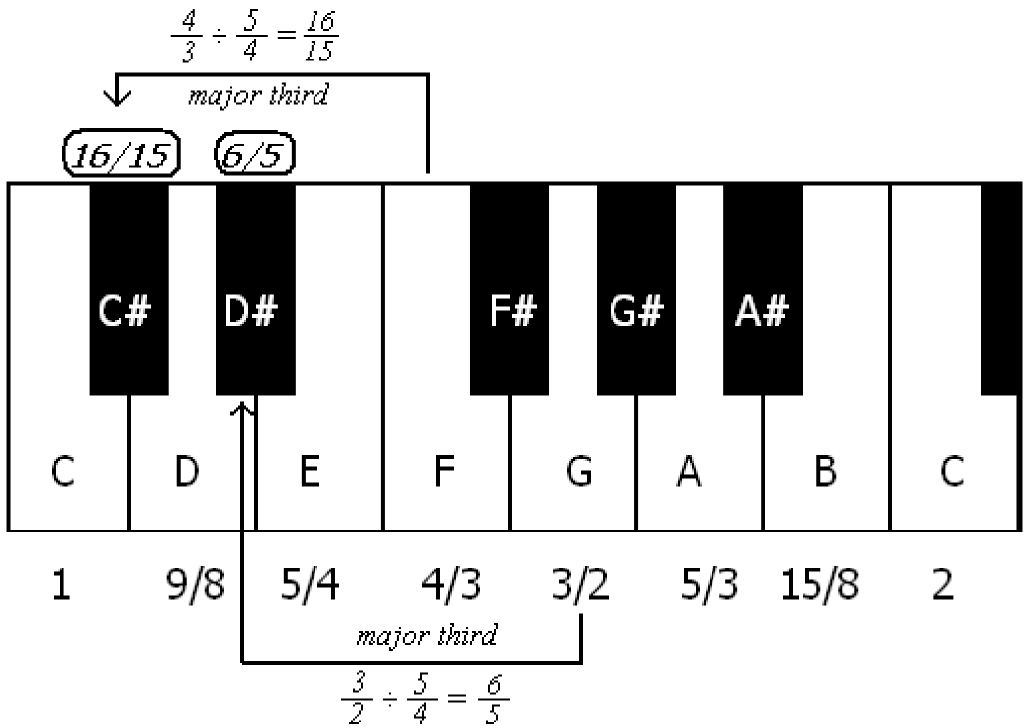


(Note: for the remainder of this section, we'll focus on just one octave of the twelve-tone scale.)

Now we'll tune the other five notes of the scale – the ones corresponding to the "black keys" on the standard keyboard. To begin this process, we'll decide to preserve the intervals C#-F and D#-G; note that both pairs of notes are separated by four semitones, so both intervals should be major thirds (with a frequency ratio of 5/4):

Lower the G by a major third to tune D#: $\frac{3}{2} \div \frac{5}{4} = \frac{3}{2} \times \frac{4}{5} = \frac{6}{5}$

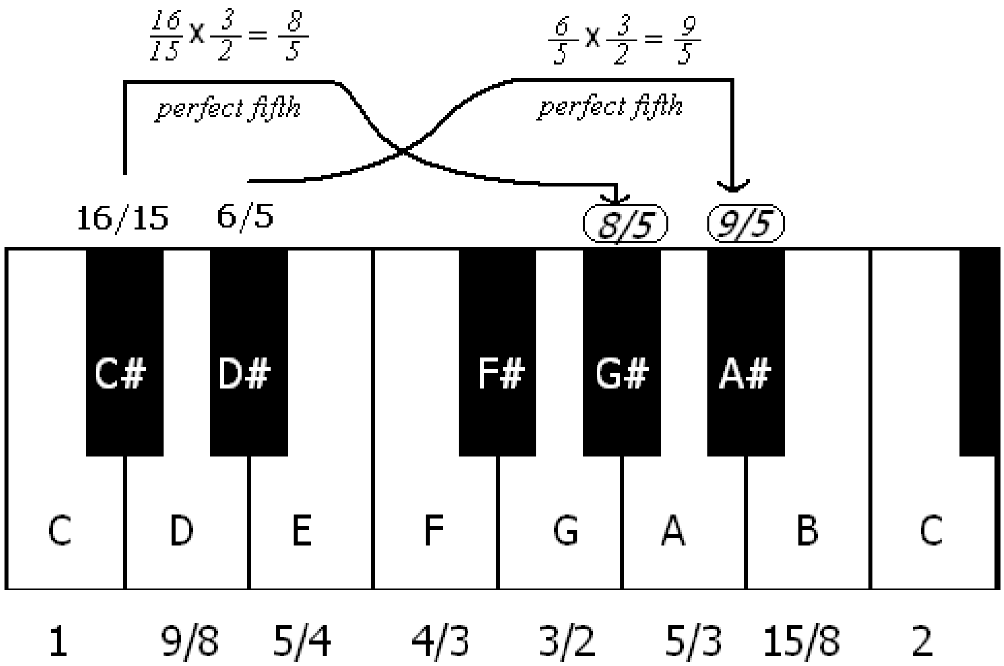
Lower the F by a major third to tune C#: $\frac{4}{3} \div \frac{5}{4} = \frac{4}{3} \times \frac{4}{5} = \frac{16}{15}$.



Next, we will *decide* to tune G# and A# by preserving the intervals C#-G# and D#-A#. Note that both pairs of notes are 7 semitones apart, so both intervals should be perfect fifths (with a 3/2 frequency ratio):

Raise the C# by a perfect fifth to tune G#: $\frac{16}{15} \times \frac{3}{2} = \frac{8}{5}$

Raise the D# by a perfect fifth to tune A#: $\frac{6}{5} \times \frac{3}{2} = \frac{9}{5}$.



Finally, it remains to tune the sixth note of the scale, F#. Recall that F# is the note we get by raising C by six fifths (that is, "R6") – or, alternately, by lowering it six fifths ("L6"). In a sense, this makes F# the tone that's farthest removed from C, and – it turns out – the hardest one to tune. You can actually hear this for yourself -- if you play C and F# on a keyboard simultaneously, they do not sound good together at all. It turns out that there's no good way to "fix" this – that is, it's not possible to find a "nice" frequency ratio for F# that corresponds closely to its Pythagorean frequency ratio of 729/512 (or about 1.424). There's no one standard "just intonation" frequency ratio to use for F#; presented below are a few options, based on which interval we might want to preserve:

Lower B by a perfect fourth (5 semitones): $\frac{15}{8} \div \frac{4}{3} = \frac{15}{8} \times \frac{3}{4} = \frac{45}{32} \approx 1.406$
(Note: raising D by a major third gives us the same result)

Raise C# by a perfect fourth (5 semitones): $\frac{16}{15} \times \frac{4}{3} = \frac{64}{45} \approx 1.422$

Lower A# by a major third (4 semitones): $\frac{9}{5} \div \frac{5}{4} = \frac{9}{5} \times \frac{4}{5} = \frac{36}{25} = 1.44$

None of these frequency ratio options is particularly pleasing to the ear; or, mathematically, note that each has a denominator that is much larger than the ones we found for the other notes in the scale. Again, it's important to note that Just Intonation is not one specific set of frequency ratios; rather, it's a process one may use to tune the octave in such as way that certain desired intervals are preserved.

There are many varieties of just intonation – in fact, many versions divide the octave into some number of notes other than the usual twelve! We won't go into those here, but it is an interesting exercise to try to divide the octave into (for example) 17 distinct tones using just intonation.

For a 12-tone scale, we could have changed several of our above results by making different decisions along the way. As just one example, refer back to the first decision we made: preserving the D-G perfect fourth interval. This decision led to a "broken" fifth, D-A. Recall that we tuned A to 5/3 of the "base" frequency (that is, C), and we tuned D to 9/8 of the base frequency. Therefore, the frequency ratio of the D-A interval is $\frac{5}{3} \div \frac{9}{8} = \frac{5}{3} \times \frac{8}{9} = \frac{40}{27} \approx 1.481$, which is not equal to 3/2. If, rather than tuning the D-G perfect fourth, we had decided instead to tune the D-A perfect fifth to a 3/2 frequency ratio, then we would have had a "broken" D-G perfect fourth:

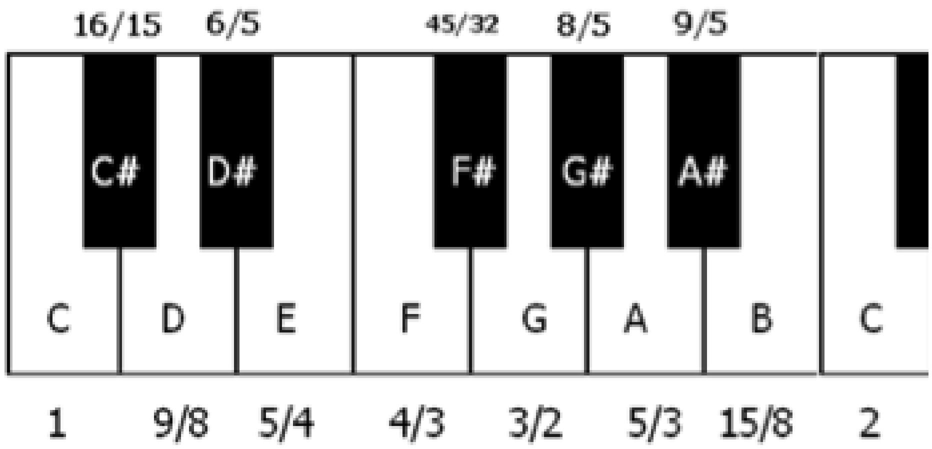
Alternate tuning (to preserve the D-A perfect fifth rather than the D-G perfect fourth):
Lower A by a perfect fifth to tune D: $\frac{5}{3} \div \frac{3}{2} = \frac{5}{3} \times \frac{2}{3} = \frac{10}{9} \approx 1.111$
(Compare this to the previous frequency ratio for D of 9/8, or 1.25.)

This gives us a "broken" D-G interval: $\frac{3}{2} \div \frac{10}{9} = \frac{3}{2} \times \frac{9}{10} = \frac{27}{20} = 1.35$
(This is close to, but different from, the desired perfect fourth frequency ratio of $\frac{4}{3} \approx 1.333$).

Again: neither of these choices is "right" or "wrong;" they are just different results due to different decisions. The main ideas here are that just intonation is a decision making process that can be used to tune the octave, and that any decisions we may make to preserve certain intervals will always have the effect of "breaking" other intervals. In short, there is no perfect tuning system – any method of tuning the octave will have its own pros and cons.

Just Intonation Practice Exercises. (Answers are on the next page.)

For all exercises, use these Just Intonation Frequency Ratios (shown here relative to C)



Remembe rthat, as is the case with Pythagorean tuning, all octaves are preserved. For example, the frequency of the next higher D (to the right of the high C in the above column) will be twice the frequency of the D in the diagram, giving it a frequency of $\frac{9}{8} \times \frac{2}{1} = \frac{9}{4}$ that of the base frequency at C.

1. In a just intonation tuning system based on C:500 Hz, what is the frequency of the next higher A#? The next lower A#?

2. In a just intonation tuning system based on C:500 Hz, what note would have a frequency of 800 Hz?

3. In a just intonation tuning system based on C:500 Hz, what note would have a frequency of 375 Hz?

4. In a just intonation tuning system based on C:500 Hz, what note would have a frequency of 3750 Hz?

5. In a just intonation tuning system based on D:300 Hz, what would be the frequency of the next higher F#? (Note: the above diagram's frequency ratios are based on "C." To figure out frequencies for a tuning system based on D:300, you need to figure out what type of interval D-F# will be, and then find the corresponding frequency ratio…)

6. In a just intonation tuning system based on D:300 Hz, what note would have a frequency of 375 Hz? How about 4000 Hz?

Practice Exercise Solutions and Comments

1. The next higher A# has frequency $\frac{500}{1} \times \frac{9}{5} = 900$ Hz. The next lower A# will be one octave lower than 900 Hz, so divide by 2 to get 450 Hz.

2. To answer this question, figure out the frequency ratio between the “mystery” note and our base note: 800/500 reduces to 8/5, which is the frequency ratio of the C-G# interval (a.k.a. a “minor sixth”). So, our answer is G#.

3. This is similar to #2, except our target note is below 375 Hz, so it’s in the next lower octave. As long as we remember that we can raise/lower by octaves by multiplying/dividing frequencies by 2, this will be fine…

The frequency ratio (as found in #2 above) is 375/500, which reduces to ¾ (or 0.75). We can raise this note by an octave by multiplying by 2: ¾ times 2 = 3/2, which is the frequency ratio of the interval C-G (a perfect fifth). So, the note whose frequency is 375 Hz is the G below C:500.

(Alternate solution: first raise 375 Hz by an octave to get 750 Hz. Then find the corresponding frequency ratio: 750/500 reduces to 3/2, giving us the C-G interval. Since 750 Hz is the frequency of G above C:500 Hz, it follows that 375 Hz is the frequency of the next lower G.)

4. The frequency ratio formed by 3750 Hz and our base of 500 Hz is 3750/500, which reduces to 15/2 (divide on top and bottom by 250), or 7.5. We can lower this by octaves by multiplying by ½; if we do this twice, we get a frequency ratio of 15/8, which is the C-B interval (or “major seventh”) frequency ratio. So, 3750 Hz is the frequency of the B two octaves above C:500.

Similar solution: First lower 3750 Hz by octaves twice; this gives us a result of 937.5 Hz. The frequency ratio 937.5/500 reduces to 1.875, or 15/8. This is another (equivalent) way to figure out that 3750 Hz corresponds to a B in this tuning sytem.

Yet another solution: Start by finding another C that is closer to 3750 Hz. If we raise C:500 by octaves a couple of times, we get first C:1000, then C:2000. Now, the ratio 3750/2000 reduces to 15/8 (divide the top and bottom by 250 to get this), or 1.875 as a decimal; this, as noted twice already, is the frequency ratio for a major seventh (C-B interval), so our answer is B.

Note: Any of the approaches shown here would be fine; one is not better or worse than the other. Really they are essentially the same, except for the order in which the steps are carried out. No matter how we get our answer, at some point we have to raise or lower a note by two octaves, so that we’re comparing frequencies of two notes in the same octave with a frequency ratio.

5. If our just intonation tuning system is based on a D, rather than C, then all intervals will be tuned relative to D:300. In this case, the interval we're interested in is D-F#, which is a major third (four semitones). So, the frequency ratio between the D and the F# must be a 5/4 ratio. Therefore, the frequency for the next higher F# will be $\frac{5}{4} \times 300 = 375$ Hz.

6. In a just intonation tuning system based on D:300, the note with frequency 375 Hz will be an F#, as shown in #5. If we hadn't just worked that out, we would find this result by looking at the frequency ratio 375/300, which reduces to 5/4 (or 1.25); since this is the ratio for a major third, we'd conclude that the corresponding note is four semitones above D, or F#.

To determine the note with a frequency of 4000 Hz, we'd proceed as in #4 above. As in #4, there are several ways to work this out. One way is as follows:

Since our base note is D:300, the next few D's in the tuning system would be D:600, D:1200, D:2400, and D:4800. Since 4000 is between 2400 and 4800, we'll look at the frequency ratio formed by D:2400 and the 4000 Hz note. This ratio is 4000/2400, which reduces to 5/3 (divide both numbers by 800 to get this), or 1.666… as a decimal. The diagram at the top of this handout tells us this is the frequency ratio for a nine-semitone interval, or "major sixth." The note nine semitones above a D is a B; therefore, the note tuned to 4000 Hz would be a B – specifically, the B three octaves above D:300.

Note: Alternately, we could have first lowered the 4000 Hz note by a few octaves to start; dividing by 2 three times gives us 2000, then 1000, then 500. Comparing this to D:300 gives us a frequency ratio of 500/300 = 5/3, and our conclusion follows as in the preceding paragraph.