Thoughts On Ramanujan

Of late I had been reading Ramanujan's *Collected Papers* and based on my understanding of it (and inputs from works of Borwein brothers, Bruce C. Berndt) I wrote a series of posts explaining some of Ramanujan's discoveries (see 10 posts starting from here and 4 posts beginning from here). While studying Ramanujan's Papers I could not help myself being astounded by the depth of his discoveries and the ingenuity of the proofs he provided for some of his results.

Reading Papers has not been an easy job for me and seems like an unending task if I wish to have a complete and thorough understanding of it. Hence I decided to take a break for sometime and dedicate one of my posts about my thoughts on Ramanujan, his works, abilities and methods. Needless to say whatever I present here would be a personal view and may differ from general perception a reader might have of Ramanujan and his works. Because of the same reason this post is bound to be of somewhat personal nature.

My Introduction with Ramanujan

Ramanujan is not so famous in Indian textbooks of primary and secondary classes. On the other hand Aryabhatta (also Indian), Pythagoras, Euclid and Newton are quite famous especially in the secondary and higher secondary classes. The first time I read about Srinivasa Ramanujan was in 1994 when I was in 9th grade. In the NCERT 9th grade mathematics textbook there was a chapter on real numbers and during the introduction to irrational numbers π was presented as the most popular example. In the same article it was mentioned that:

"Using an identity of Indian mathematician Srinivasa Ramanujan the value of π has been calculated to several million places of decimals."

This was the only thing I knew about Ramanujan during 1994. At that time I could not stop wondering about the "identity of Ramanujan" used to calculate the value of π. Later in higher secondary classes I got acquainted with *infinite series* and found various arctan formulas to calculate the value of π. None of these were related to Ramanujan. Thus the "identity of Ramanujan" remained a mystery. At the same time (around 1996) I got hold of Hardy's *A Course of Pure Mathematics* and found another mention of Ramanujan:

"A large number of curious approximations [to π] will be found in Ramanujan's Collected Papers, pp. 23-29. Among the simplest are

$$\frac{19}{16}\sqrt{7}, \ \frac{7}{3}\left(1 + \frac{\sqrt{3}}{5}\right), \ \left(9^2 + \frac{19^2}{22}\right)^{1/4}, \ \frac{63}{25}\left(\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}\right)$$

these are correct to 3, 3, 8 and 9 places respectively."

Thus even being so closely associated with Ramanujan, Hardy fails to mention the "identity of Ramanujan". A probable reason could be that the calculation of π to million places of decimals was done using digital computers which were not available during Hardy's time.

Then in my college years I got access to a very good library and internet which opened up avenues for all kinds of information. There I got hold of the literary masterpiece *A Mathematician's Apology* by Hardy. In it I found details of the Hardy-Ramanujan connection in the foreword by C. P. Snow. The Hardy-Ramanujan story seems to be the most romantic one in the history of mathematical collaboration and has been amply described by Hardy in his other works and by various biographers of Ramanujan. This story itself created a very great impression of Ramanujan (and Hardy too) on me.

After college I read Robert Kanigel's *The Man Who Knew Infinity* (to date it seems to be the best and authentic biography of Ramanujan) and then I began to appreciate Ramanujan even more. And with access to Wikipedia and Wolfram I finally came to know of the "identity of Ramanujan" used to calculate the value of π to several million places of decimals (equation (1) of [this post](#)) namely:

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\frac{1}{2}\frac{1\cdot 3}{4^2} + \frac{53883}{99^{10}}\frac{1\cdot 3}{2\cdot 4}\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} + \cdots$$

In the beginning of last year (2012) I finally got *Collected Papers* of Ramanujan and then understood the theory behind the "identity of Ramanujan" presented above. The only missing part is the calculation needed to obtain 1103 in above formula. Nobody knows the way Ramanujan calculated it, but it has been calculated using other methods by Borwein brothers.

Going through *Collected Papers* gave me a deeper understanding of the mathematics developed by Ramanujan and some inkling of his methods. In a way I believe it is impossible to understand Ramanujan deeply without understanding some of his work. I now present my thoughts on Ramanujan.

Ramanujan's Work

Ramanujan's work primarily consists of formulas and formulas and formulas. He brought back the golden age of formulas championed by Euler and Jacobi. Ramanujan's approach to mathematics was shaped up by the kind of mathematical atmosphere during early 20th century in India. Mathematics was just formulas and calculations. The books like Loney's *Trigonometry* and Hall and Knight's *Algebra* were the most revered ones. Ramanujan however got inspired by Carr's *Synopsis* containing a list of formulas from algebra and calculus.

Ramanujan worked extensively with *infinite series, infinite products, continued fractions and radicals*. However all his work was based on algebraic manipulations rather than analytical considerations of theoretical nature (with some exceptions though). His most significant contribution however I think was in the field of elliptic and theta functions. In this field he rediscovered almost all the formulas of Euler and Jacobi and discovered far more new results of his own. All of this was obtained by just algebraic manipulation and hence we don't find any

mention of double periodicity in his development of elliptic functions.

Another significant work he did was in the theory of partitions and thanks to Hardy's guidance he also used analysis together with algebra and thereby obtained formula for calculating number of unrestricted partitions of a number. This was a formula of asymptotic nature but gave exact results on calculation. Later it was modified by Rademacher to obtain a theoretically exact formula.

In the theory of theta functions he developed his own approach which is much more useful in obtaining formulas compared to the approach used by Jacobi. Using his theta functions Ramanujan obtained so many modular equations that "modular equations" became a new field of study and research. He also used theta functions to obtain a number of results in number theory.

Continued fractions were his another favorite topic and he found many formulas related to Rogers-Ramanujan continued fraction. Some of the results in this area were already discovered and proved by British mathematician L. J. Rogers but his work was forgotten by the mathematical community until Ramanujan communicated these results to Hardy. Regarding his continued fraction formulas Hardy remarked that *"they must be true because, if they were not true, no one would have had the imagination to invent them."* The following two samples should give the reader a similar feeling:

(a) If
$$u = \frac{x}{1+} \frac{x^5}{1+} \frac{x^{10}}{1+} \frac{x^{15}}{1+} \cdots \quad \text{and}$$
$$v = \frac{\sqrt[5]{x}}{1+} \frac{x}{1+} \frac{x^2}{1+} \frac{x^3}{1+} \cdots,$$
then
$$v^5 = u \cdot \frac{1 - 2u + 4u^2 - 3u^3 + u^4}{1 + 3u + 4u^2 + 2u^3 + u^4}$$
(b) $$\frac{1}{1+} \frac{e^{-2\pi}}{1+} \frac{e^{-4\pi}}{1+} \frac{e^{-6\pi}}{1+} \cdots = \left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \right) \sqrt[5]{e^{2\pi}}$$

Another field where Ramanujan was at his best is definite integrals. He provided lots of formulas for various kinds of definite integrals. Almost all of these formulas look very complicated, unusual and probably weird to a student studying definite integrals. A few samples below should be able to convince readers of their unusual nature:

(a) $$\int_0^\infty \frac{dx}{(1 + x^2)(1 + r^2 x^2)(1 + r^4 x^2)(1 + r^8 x^2) \cdots} = \frac{\pi}{2(1 + r + r^3 + r^6 + r^{10} + \cdots)}$$
where $1, 3, 6, 10, \ldots$ are sums of natural numbers.

(b) If
$$\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}} - 1} \, dx = \phi(n),$$
then
$$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}} - 1} \, dx = \phi(n) - \frac{1}{2n} + \phi\left(\frac{\pi^2}{n}\right) \sqrt{\frac{2\pi^3}{n^3}}$$

3

$\phi(n)$ is a complicated function. The following are certain special values:

$$\phi(0) = \frac{1}{12}, \phi\left(\frac{\pi}{2}\right) = \frac{1}{4\pi}, \phi(\pi) = \frac{2-\sqrt{2}}{8}, \phi(2\pi) = \frac{1}{16}$$

$$\phi\left(\frac{2\pi}{5}\right) = \frac{8-8\sqrt{5}}{16}, \phi\left(\frac{\pi}{5}\right) = \frac{6+\sqrt{5}}{4} - \frac{5\sqrt{10}}{8}, \phi(\infty) = 0$$

$$\phi\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \sqrt{3}\left(\frac{3}{16} - \frac{1}{8\pi}\right)$$

Ramanujan's Abilities

A disclaimer must be put in place before I proceed further. No mathematician in current times has the audacity to provide a judgement of Ramanujan's abilities because still many of them are struggling to prove some of his results in the manner in which Ramanujan might have proved them. Whatever we find in literature about Ramanujan's abilities are merely expressions of wonder and awe at his mathematical powers. As for my case I am more like a beginner studying *Collected Papers* and hence my views regarding Ramanujan's abilities are of extreme amazement.

The first thing one observes while studying Ramanujan's Papers is that almost all the results are of algebraical nature and proofs wherever provided consist of algebraical manipulations without deep analytical considerations. Thus Ramanujan appears to be a master algebraist with great powers of symbolic and numerical manipulation. A word of caution here: the algebra we talk here is the *classical one* dealing with symbol manipulation based on certain rules rather the *modern* or *abstract* algebra dealing with groups, rings and fields. Ramanujan had an uncanny ability to find algebraical relationship between two expressions (be it infinite series, infinite product or a continued fraction) with very minimal effort. It is almost as if he was able to see connection between such expressions. Naturally this boils down to extraodinary powers of calculating various coefficients of such expressions without any mistakes and probably doing this with minimal use of pen and paper.

Another ability of Ramanujan which is mentioned by various present day authors is the capability of manipulating radicals (including denesting complicated nested radicals). Technically this is equivalent to calculating integral powers of various radical expressions in head and in very fast manner. Probably Ramanujan possessed general formulas for symbolic manipulation of radicals and applied them for denesting radicals which he encountered in his researches.

While dealing with infinite series, products and continued fractions he could use them interchangeably. Thus converting a series into a product and then into a continued fraction was something like a routine activity for Ramanujan. In case of continued fractions his insights were far ahead of any other mathematician till date. Using algebraical relationship between two given series he would express their ratio in the form of a continued fraction.

Finally Ramanujan had an innate aesthetic sense of form. The way he presented his formulas is so unlike other mathematicians that one could recognize whether a formula was written by him or not just by physically looking at the formula. He wrote his formulas so that they possessed

the following properties:

- meaning of the formula could be understood by anyone with basic knowledge of algebra and calculus
- focus was on special cases of general formulas with actual numbers rather than general formula itself
- minimal use of symbolism and wherever possible indicate a pattern by exhibiting it numerically or by writing about the pattern in English rather than describing pattern via a formula.
- each formula had a certain unexpectedness providing a shock treatment to the reader
- most of the formulas had deep theories behind them
- avoiding use of Σ and Π symbols to represent infinite series and products

Overall the main goal was to make his results exciting, accessible and avoiding any high-brow stuff. As an example I quote from *Collected Papers*:

$$\frac{32}{\pi} = (5\sqrt{5} - 1) + \frac{47\sqrt{5} + 29}{64}\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{5} - 1}{2}\right)^8$$
$$+ \frac{89\sqrt{5} + 59}{64^2}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3\left(\frac{\sqrt{5} - 1}{2}\right)^{16} + \cdots$$

here $5\sqrt{5} - 1, 47\sqrt{5} + 29, 89\sqrt{5} + 59, \ldots$ *are in arithmetical progression.*

A modern author would have written the above formula with Σ notation and the arithmetical progression would have been an expression of the form $an + b$. On the other hand the formula as provided by Ramanujan is such that any 12 year old kid with knowledge of square roots would understand it in some sense and, if curious enough, would probably sum the given 3 terms of the series by using a calculator. This is something I really love about Ramanujan and I wish the modern mathematicians tried to follow him in this regard.

Although Ramanujan's results were often expressed in simplest possible mathematical forms so as to make them accessible to a wider audience, but proving any such result was altogether a different matter. Each result probably was the outcome of deep theoretical research and formidable calculations done over a period of time.

Ramanujan's Methods

Ramanujan's methods were a direct reflection of his abilities thus most of the proofs he gave consisted of algebraical manipulation of various expressions. These were done in the most efficient and non-obvious manner thereby producing results or derivations never heard of. The most common tools Ramanujan used are manipulation of infinite series and products and applying the processes of differentiation and integration wherever possible.

He treated the processes of calculus as mere algebraic rules to get one expression from another. Often his calculations consisted of differentiating complicated expressions and then simplifying the results. Owing to a very powerful memory he could do a lot of these calculations without

noting them down.

Ramanujan obtained his results normally by looking at particular numerical examples and analyzing them on empirical basis. With a given set of numerical data he was able to put forth various hypotheses depicting various algebraical relations and then he would establish the relations via symbolic manipulation. For example he found the congruence properties of partition by looking at the table of partitions developed by P. A. MacMahon and then subsequently obtained a proof by manipulation of eta functions (in the infinite product form).

Since most of his results were discovered by observation of numerical data, he himself used to calculate various numerical data (like calculating several coefficients of various complicated series and products) and I can say without doubt that he was a master calculator when it came to numerical calculations. For him multiplying two series, or finding integral powers of a series was a routine job. Same is the case with infinite products.

No one knows how he found modular equations of higher degrees, but it seems he had some general formula for finding relations between various theta functions and then he could transcribe the relation into a modular equation. Probably he used these modular equations to find expressions for the multiplier and other related stuff. Modular equations were also used by him to compile a long table of class invariants. A single look at the table is enough to convince the reader that the calculations must have been very tedious and time taking, and no one has tried to hand-calculate these invariants to this day. I believe that apart from having great powers of calculation, Ramanujan also devoted a hell lot of time doing these laborious calculations. One sample class invariant definitely needs to be mentioned in this regard:

$$G_{1645}^2 = (2 + \sqrt{5})\sqrt{(3 + \sqrt{7})\left(\frac{7 + \sqrt{47}}{2}\right)}\left(\frac{73\sqrt{5} + 9\sqrt{329}}{2}\right)^{1/4}$$
$$\times \left\{\sqrt{\frac{119 + 7\sqrt{329}}{8}} + \sqrt{\frac{127 + 7\sqrt{329}}{8}}\right\}$$
$$\times \left\{\sqrt{\frac{743 + 41\sqrt{329}}{8}} + \sqrt{\frac{751 + 41\sqrt{329}}{8}}\right\}$$

Last but not the least, I want to emphasize one point about Ramanujan. A lot of mathematicians and biographers have written that Ramanujan obtained most of his results via empirical observations and did not have any rigorous proofs for many of the results. I fully agree that many of his results were obtained by an empirical process (generally backed by a large amount of numerical evidence), but Ramanujan was very honest in expressing whether he had a rigorous proof of a formula or not. He did not provide many proofs in his papers, but I strongly believe that he did have those proofs with him.

In the instances in which he did provide a proof I have found his proofs to be much better (in terms of economy and the surprise element) than the proofs which modern authors have provided. In fact some of his results were so astounding that it is silly to think that they could

have been obtained via empirical processes without the backing of a solid proof. The kind of language he used in his papers reflects a kind of honest confidence in his methods and results which he obtained. For instance in one paper titled *"Algebraic Relations between Certain Infinite Products"* he writes as follows:

"I have now found an algebraic relation between $G(x)$ and $H(x)$ viz.

$$H(x)\{G(x)\}^{11} - x^2 G(x)\{H(x)\}^{11} = 1 + 11x\{G(x)H(x)\}^6$$

Another noteworthy formula is

$$H(x)G(x^{11}) - x^2 G(x)H(x^{11}) = 1$$

Each of these formulae is the simplest of a large class."

Here $G(x)$ and $H(x)$ are infinite products given by

$$G(x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^{5n-1})(1 - x^{5n-4})}$$
$$H(x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^{5n-2})(1 - x^{5n-3})}$$

How can someone discover a large class of formulas involving such complicated functions with just empirical observation? It must have been backed by a thorough process of research of which we have no inkling now. And just because we are not aware of his methods does not mean that we can say that he did not have a valid and rigorous method. Instead of downplaying the nature of Ramanujan's methods by branding them as *"empirical"* and somehow favoring the complicated and unintuitive modular form approach (which can only be used to verify such results but not find them in the first place) modern mathematicians should try to uncover his methods.

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