# Cavalieri's Principle and its Applications

### Introduction

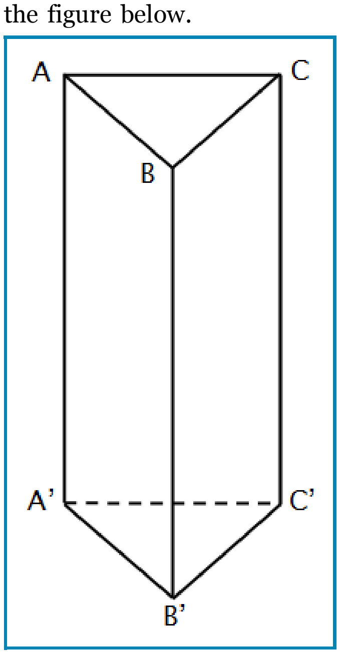
In this post we will discuss something is which is very elementary and fascinating, yet not available in a high school curriculum. More precisely we will study a part of solid geometry related to calculation of volume of solids. In so doing we will need the famous *Cavalieri's Principle* which relates volumes of two solids under certain conditions.

### Cavalieri's Principle

The Cavalieri's Principle states that:
**If two solids lie between two parallel planes and any plane parallel to these planes intersects both the solids into cross sections of equal areas then the two solids have the same volume.**

Intuitively the principle is easy to understand once one recognizes that the volume of a solid can be found by slicing the solid into multiple parts through a set of parallel planes and then adding the volume of each small part. The idea is that if parts are sufficiently thin (distance between the parallel planes is very small) then the volume of each part can be approximated by "area of cross section of each part" times "height of each part". The approximation becomes better and better as the parts become thinner and thinner. The formal justification of this principle lies in Integral Calculus but we won't go into that and assume that reader is content with the intuitive idea of cutting solids into thin parts.

As a first application we will establish the formulas for volume of a pyramid and a cone. The basic idea is to first establish the formula for the volume of prism. A *prism* is a simple solid consisting of two congruent triangular bases parallel to each other and and remaining three surfaces are obtained as planes passing through corresponding sides of the triangular bases. A prism $ABCA'B'C'$ is shown in the figure below.



The volume of a prism is easy to find. We just need to construct a square of area equal to the triangular base of the prism and develop a right cuboid on top of it with height same as that of the prism. Clearly the cuboid and the prism lie between same parallel planes (planes passing through their bases) and each plane parallel to the base cuts cross sections of equal areas from the prism and cuboid. Therefore by Cavalieri's principle the volume of the prism and the cuboid are equal. Thus the volume of a prism is given as

$$V_{\text{prism}} = A_{\text{prism}} \times h_{\text{prism}}$$

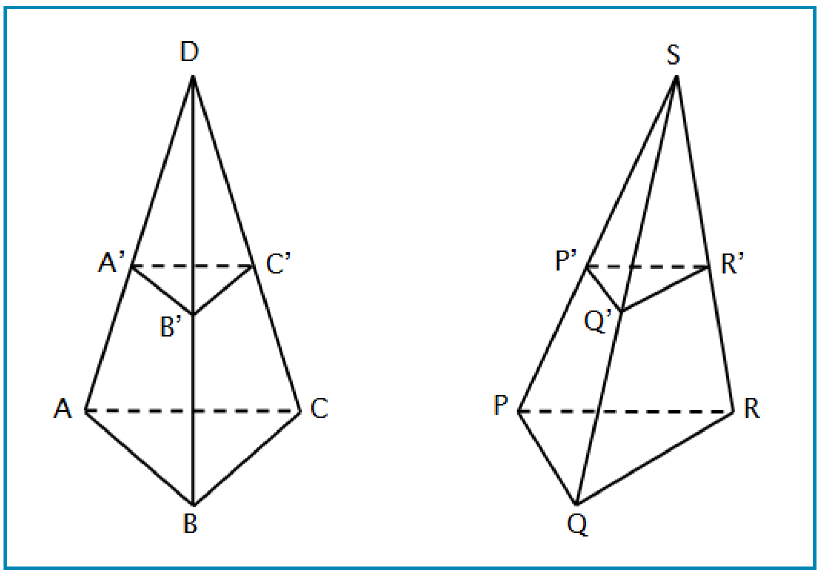
where $V, A, h$ denote the volume, area of base and height of the prism respectively. Next we derive the formula for the volume of a pyramid.

### Volume of a Pyramid and a Cone

We first start with the simplest case where the base of pyramid is a triangle. Such a solid is more properly called a *tetrahedron* which consists of four vertices, six edges and four triangular faces. We first establish the following property of a tetrahedron.

*Two tetrahedrons with same base area and same height are of equal volume.*

In the figure below we have two tetrahedrons $ABCD$ and $PQRS$ such that base triangles $ABC$ and $PQR$ have same area. Also the height of both tetrahedrons is same i.e. the distance of point $D$ from plane of $ABC$ is same as that of point $S$ from plane of $PQR$.




We have to notice that if a plane parallel to the base $ABC$ cuts the tetrahedron $ABCD$ in a triangular cross section $A'B'C'$ then the triangle $A'B'C'$ is similar to triangle $ABC$. Also the ratio of the sides of $A'B'C'$ to those of $ABC$ depends upon the distance of this cutting plane from the plane containing base $ABC$. This implies that the area of triangle $A'B'C'$ bears a constant ratio to that of triangle $ABC$ which depends only on the distance between planes containing $A'B'C'$ and $ABC$. Similar is the case with areas of triangles $P'Q'R'$ and $PQR$ coming from tetrahedron $PQRS$.

Since the height of both the tetrahedrons is same both of them can be brought to lie between two parallel planes (one plane containing their bases $ABC$ and $PQR$ and the other containing vertices $D$ and $S$). If an intermediate plane parallel to the base plane cuts these tetrahedrons in triangular cross sections $A'B'C'$ and $P'Q'R'$ then we have

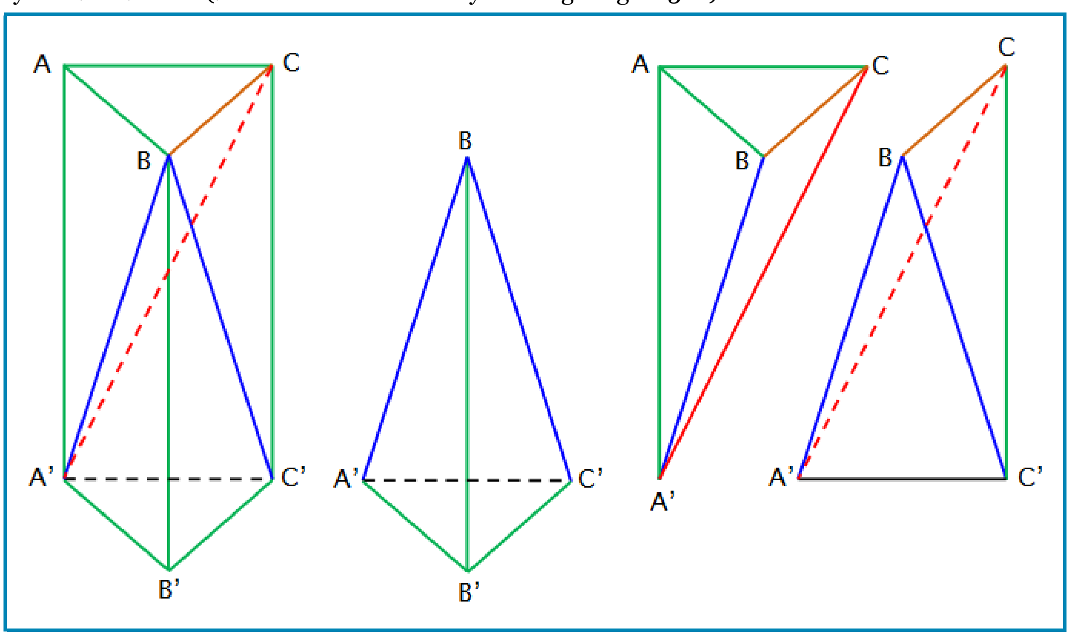
$$\frac{\text{area } A'B'C'}{\text{area } ABC} = \frac{\text{area } P'Q'R'}{\text{area } PQR}$$

and this constant ratio is only dependent on the distance of the cutting plane from the plane containing the bases $ABC$ and $PQR$. Since the areas of bases $ABC$ and $PQR$ are equal it follows that the areas of triangles $A'B'C'$ and $P'Q'R'$ are equal. It now follows from Cavalieri's principle that both the tetrahedrons $ABCD$ and $PQRS$ have the same volume.

We now establish the following formula for the volume of a tetrahedron

$$V_{\text{tetra}} = \frac{1}{3} \times A_{\text{tetra}} \times h_{\text{tetra}}$$

where $A$ represents the area of a base of tetrahedron and $h$ represents the height of the tetrahedron relative to the chosen base. If we compare this formula with the formula for volume of a prism, it is obvious that the area of a tetrahedron is one-third that of a prism with the same base area and same height. This suggests that we should be able to able cut a solid prism into three tetrahedrons of equal volume (similar to the way we can cut a parallelogram into two triangles of equal area through the diagonal which leads to formula of area of triangle as half of product of base and height). It turns out that this is possible and simple enough but may not be obvious (because of the difficulty of imagining in 3-D).




In the above figure we have a prism $ABCA'B'C'$ and we show to how to cut it into three tetrahedrons of equal volume. First cut is simple: it is made by a plane passing through the two blue diagonals $A'B$ and $C'B$. The result of this cut is a tetrahedron $A'B'C'B$ and another figure (which is a pyramid $A'C'CAB$ with base $A'C'CA$ and vertex $B$, this is not shown in figure above). Next it is possible to cut the remaining pyramid $A'C'CAB$ by a plane passing through the red diagonal $A'C$ and the triangle $A'BC$. This cut results into two tetrahedrons

$ABCA'$ and $A'C'CB$. Thus we have cut the original prism $ABCA'B'C'$ into three tetrahedrons $A'B'C'B$, $ABCA'$ and $A'C'CB$.

What remains to be shown is that the three tetrahedrons obtained by this process are of equal volume. First we can easily see that the tetrahedrons $A'B'C'B$ and $ABCA'$ have bases $A'B'C'$ and $ABC$ of same area and clearly their height is same as it is the distance between the planes containing $ABC$ and $A'B'C'$ in the original prism. Hence both these tetrahedrons have the same volume. Next we can see that the tetrahedrons $ABCA'$ and $A'C'CB$ have bases $A'CA$ and $A'C'C$ which are of equal area as these are obtained by cutting the parallelogram $A'C'CA$ through the diagonal $A'C$. Also the height of these tetrahedrons is same as it is the distance of vertex $B$ from the plane containing $A'C'CA$ in the original prism. It follows that the tetrahedrons $ABCA'$ and $A'C'CB$ have the same volume.

It now follows that the volume of the tetrahedron $A'B'C'B$ is one-third that of the prism $ABCA'B'C'$. Since both of them have bases (namely $A'B'C'$) of same area and same height, it follows that the volume of a tetrahedron is one-third the product of area of its base and its height relative to the chosen base.

For a general pyramid we can cut its polygonal base into a number of triangles via diagonals and joining each of these triangles to the vertex of the pyramid we get a number of tetrahedrons of same height. The volume of the pyramid is now the sum of volumes of these tetrahedrons. Since each of these tetrahedrons is of same height, while adding the volumes we effective add the areas of their bases and multiply by the common height. Thus we obtain the following formula

$$V_{\text{pyr}} = \frac{1}{3} \times A_{\text{pyr}} \times h_{\text{pyr}}$$

where $A$ represents the area of base of pyramid and $h$ represents the height of pyramid.

In the case of the cone we can think of the cone as the limiting case of a pyramid whose polygonal base tends to the curved based of the cone as the number of sides of polygonal base approaches infinity (imagine $n$ points on the periphery of the curved base of the cone and join them to make a polygonal base and join this base with vertex of cone to generate a pyramid). Therefore the same formula applies in case of a cone too. Thus we obtain

$$V_{\text{cone}} = \frac{1}{3} \times A_{\text{cone}} \times h_{\text{cone}}$$

In the case of a right circular cone of radius $r$ and height $h$ we know that $A_{\text{cone}} = \pi r^2$ and we obtain the familiar formula

$$V_{\text{right circular cone}} = \frac{1}{3}\pi r^2 h$$

When I studied this formula in $8$th grade in my school days I found the factor $1/3$ totally mysterious and no explanation was given apart from the experimental verification of pouring liquid from a conical flask into a corresponding cylindrical flask. Later when I studied calculus and "solids of revolution" I got the formula by a simple integration. But somehow the mystery of $1/3$ remained in my thoughts (compare the case of a triangle where the factor $1/2$ in the formula for area of a triangle can be explained both by integral calculus as well as from elementary geometry by dissecting a parallelogram through a diagonal into two equal triangles).

Sometime later (but luckily in the pre-college years of $12$th grade) I got hold of a book on solid geometry which taught solid geometry via the old and *golden* approach of Euclid axioms (compared to the usual and boring coordinate geometry). In this book I found the first mention of Cavalieri principle and the proofs which I have presented above. Its rather unfortunate that I can't recall the name of that book for the benefit of readers.

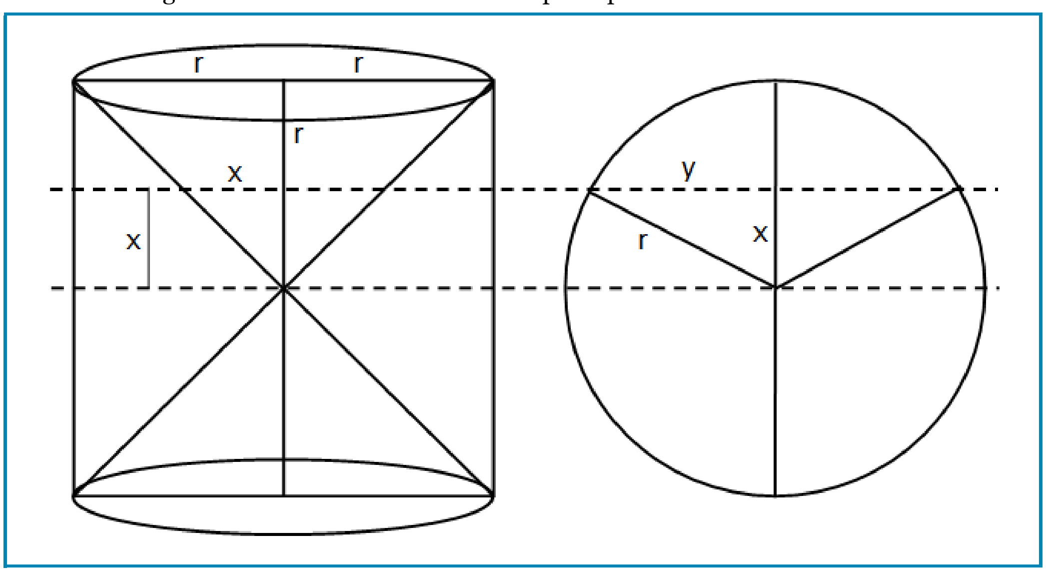
### Volume of a Sphere

Next we will use the Cavalieri Principle to establish the following formula for volume of sphere

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

where $r$ is the radius of the sphere. We start with a non-obvious technique. Let's us have a right circular cylinder of radius $r$ and height $2r$ so that its volume is $V_{\text{cyl}} = \pi r^2(2r) = 2\pi r^3$ and remove from it two cones having same base as the cylinder and height as $r$. The volume of each such cone is $V_{\text{cone}} = (1/3)\pi r^2 r = \pi r^3/3$ . The volume of remaining solid is therefore $2\pi r^3 - 2(\pi r^3/3) = (4/3)\pi r^3$ .

We will show that the volume of a sphere of radius $r$ is equal to the volume of a solid obtained by removing two right circular cones of radius $r$ and height $r$ from a right circular cylinder of radius $r$ and height $2r$. This we do via Cavalieri's principle.




In the above figure a cylinder of radius $r$ and height $2r$ and a sphere of radius $r$ are placed side by side. Also two cones of radius $r$ and height $r$ are removed from the cylinder (one from above and one from below). Clearly both solids are of same height $2r$ and hence are contained between two parallel planes (each passing through the circular base of the cylinder). The center of the cylinder and that of the sphere are at the same height. Let a plane parallel to the base of the cylinder cut both the solids and let this plane be above the center of cylinder at a height $x$. The argument for the case when cutting plane is below the center is same due to symmetry.

Clearly in case of the sphere the cross section is a circle whose area is $\pi y^2 = \pi(r^2 - x^2)$. In the case of solid derived from cylinder by removing two cones, the cross section is not a circle but a ring (as the central circular portion is removed during removal of cone). The external radius of the ring is $r$ and the internal radius is $x$ so that the total area of the ring shaped cross section is $\pi r^2 - \pi x^2 = \pi(r^2 - x^2)$. Thus the areas of the cross sections of two solids is same. By Cavalieri's principle the two solids have the same volume.

The above proof is straight from $8$th grade NCERT mathematics textbook (obviously the old edition which I studied in $1993$) except for the fact that the NCERT textbook does not mention the name Cavalieri's principle nor does provide an intuitive argument as to why the volume of two such solids having same cross section area and same height should be equal. But I think this is the best a non-extraordinary student of $8$th grade can handle at that tender age. This is one of the reasons why I hold NCERT textbooks in high esteem.

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Labels: Geometry

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