Math 210, Test #3
Solutions

1. (30 pts) Complete *one* of the following inductive proofs. (Your choice!) If you attempt both, make it clear which *one* you want to have graded. You will *not* get extra credit for trying both.

OPTION #1:
Use induction to prove: for all $n \geq 1$,

$$\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \frac{1}{3^n}.$$

Proof: Let p_n denote the above proposition for $n \in \mathbb{N}$. Then, p_1 is the proposition $\frac{2}{3} = 1 - \frac{1}{3}$, which is clearly true.

For the inductive step, assume that p_n is true for all values of n from 1 up to k, for some positive integer k. (We want to show that $p_{k+1} : \sum_{i=1}^{k+1} \frac{2}{3^i} = 1 - \frac{1}{3^{k+1}}$ follows from this assumption.)

Our inductive hypothesis allows us to assume that p_k is true:

$$\frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^k} = 1 - \frac{1}{3^k}.$$

The statements listed below follow from this assumption:

$$\begin{aligned}
\frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^k} &= 1 - \frac{1}{3^k} \\
\left(\frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^k}\right) + \frac{2}{3^{k+1}} &= \left(1 - \frac{1}{3^k}\right) + \frac{2}{3^{k+1}} \\
\frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^k} + \frac{2}{3^{k+1}} &= 1 - \frac{3}{3^{k+1}} + \frac{2}{3^{k+1}} \\
\sum_{i=1}^{k+1} \frac{2}{3^i} &= 1 - \frac{1}{3^{k+1}}.
\end{aligned}$$

The final statement above is the proposition p_{k+1}.

We've shown that p_1 is true and $p_k \to p_{k+1}$ for each positive integer k.

Therefore,

$$p_n : \sum_{i=1}^{n} \frac{2}{3^i} = 1 - \frac{1}{3^n}$$

is true for all positive integers n. ■

1. (continued)

OPTION #2:
Use induction to prove that every power of 6 (6, 36, 216, etc.) has a units digit of 6. (In other words, prove that for all $n \geq 1$, $6^n - 6$ is divisible by 10.)

Proof: Let p_n denote the proposition: $6^n - 6$ is divisible by 10.

For our initial step, note that p_1 is the proposition that 6-6 is divisible by 10; this is true, since $6 - 6 = 10 \cdot 0$.

For the inductive step, assume that p_n is true for all values of n from 1 up to k, for some positive integer k. (We want to show that $p_{k+1} : 6^{k+1} - 6$ is divisible by 10 follows from this assumption.)

Our inductive hypothesis allows us to assume that p_k is true; that is, $6^k - 6$ is divisible by 10. Therefore, $6^k - 6 = 10j$ for some integer j. The statements listed below follow from this assumption:

$$\begin{aligned} 6^k - 6 &= 10j \\ 6\left(6^k - 6\right) &= 6(10j) \\ 6^{k+1} - 36 &= 60j \\ 6^{k+1} - 36 + 30 &= 60j + 30 \\ 6^{k+1} - 6 &= 10(6j + 3) \end{aligned}$$

Thus, 6^{k+1} is an integer multiple of 10; that is, $6^{k+1} - 6$ is divisible by 10. This is proposition p_{k+1}.

We've shown that p_1 is true and $p_k \to p_{k+1}$ for each positive integer k.

Therefore,

$$p_n : 6^n - 6 \text{ is divisible by } 10$$

is true for all positive integers n. ■

2. (20 pts) In the game "three-card poker," a "hand" consists of an unordered selection of three cards from a standard fifty-two card deck.

(a) (4 pts) How many different ways are there to select a three-card poker hand?

Solution: A "hand" is an unordered, non-repeating selection of cards, so it is a combination. The number of ways to select a combination of 3 cards from a deck of 52 cards is

$$C(52,3) = \frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 50, \text{ or } 22,100.$$

For (b) and (c), assume a three-card poker hand is selected at random. Find the number of different ways in which each event can occur, and find the probability of each event.

(b) (8 pts) All three cards are of the same suit.

Solution: Our selection process for such a hand is as follows:

- Select a suit - 4 options
- Select three cards of this suit. The number of ways in which this can be done is

$$C(13,3) = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} = 13 \cdot 2 \cdot 11$$

Therefore, there are $4 \cdot 13 \cdot 2 \cdot 11$, or 1144, ways to select such a hand. The probability that this will occur at random is

$$\frac{4 \cdot 13 \cdot 2 \cdot 11}{26 \cdot 17 \cdot 50} = \frac{2 \cdot 11}{17 \cdot 25}, \text{ or } \frac{22}{425}.$$

Comment: The reciprocal of this probability is approximately 19.32. So, if you repeatedly select randomized three-card hands from a standard deck, you would expect to see a "flush" roughly once every 19 hands.

(c) (8 pts) Exactly one card is a face card. (Note: a "face card" is a Jack, Queen, or King.)

Solution: Our selection process for such a hand is as follows:

- Select one face card - 12 options
- Select two non-face cards - $C(40,2) = \dfrac{40 \cdot 39}{2 \cdot 1} = 20 \cdot 39$ options

Thus, there are $12 \cdot 20 \cdot 39$, or 9360, ways to select such a hand. The probability that this will occur at random is

$$\frac{12 \cdot 20 \cdot 39}{26 \cdot 17 \cdot 50} = \frac{12 \cdot 3}{17 \cdot 5}, \text{ or } \frac{36}{85}.$$

3. (15 pts) Find the number of different rearrangements of each of the following nine-letter words.

(a) (5 pts) BALTIMORE Answer: Since this word has nine *distinct* letters, it has 9! distinct rearrangements.

(b) (10 pts) TENNESSEE Solution: Two valid selection processes are shown below (others are also possible). For each, we proceed by deciding what set of locations in our rearrangement of TENNESSEE will be assigned which set of letters – the four E's, the two N's, the two S's, or the one T.

Selection Process #1:

- Select four (out of nine) locations for the four E's: $C(9,4) = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 3 \cdot 7 \cdot 6$, or 126 options
- Select two (out of five remaining) locations for the two N's: $C(5,2) = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$ options
- Select two (out of three remaining) locations for the two S's: $C(3,2) = 3$ options
- Place the single T in the remaining location (1 option)

This gives us a total of $126 \cdot 10 \cdot 3$, or 3780, rearrangements of TENNESSEE.

Selection Process #2:

- Select one (out of nine) locations for the single T: 9 options
- Select two (out of eight remaining) locations for the two N's: $C(8,2) = \dfrac{8 \cdot 7}{2 \cdot 1} = 4 \cdot 7 = 28$ options
- Select two (out of six remaining) locations for the two S's: $C(6,2) = \dfrac{6 \cdot 5}{2 \cdot 1} = 15$ options
- Place the four E's in the remaining four locations: $C(4,4) = 1$ option

This gives us a total of $9 \cdot 28 \cdot 15$, or 3780, rearrangements of TENNESSEE.

Just for fun, here is yet another alternate solution (similar to an example in section 3.1 of the text):

Start by indexing the repeated letters of TENNESSEE: $TE_1N_1N_2E_2S_1S_2E_3E_4$.

With this, the nine "letters" are distinct, so there are 9! rearrangements. However, this counts the rearrangements we want - that is, those with normal, unindexed letters – multiple times. In particular, there are $4! = 24$ ways to rearrange the four E's, so every rearrangement of normal letters is being counted 24 times due to the 24 arrangements of indexed E's. Similarly, the two N's and the two S's both lead to double-counting.

Cumulatively, the 9! rearrangements of indexed letters will include each unindexed rearrangement a total of $4! \cdot 2! \cdot 2! = 24 \cdot 2 \cdot 2$, or 96, times. So, the total number of rearrangements of normal, unindexed letters is given by

$$\frac{9!}{4!\ 2!\ 2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 2} = 9 \cdot 2 \cdot 7 \cdot 6 \cdot 5 = 3780.$$

4. (10 pts)

(a) In the space below, write out the first eight rows of Pascal's Triangle. (Recall that the first row consists of two 1's.)

Answer: The first eight rows of Pascal's Triangle are as follows: Note that the third entry of the sixth row is circled, since it is the number corresponding to $C(6,3)$ (see part (b)).

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

(b) Identify (circle) the entry in the triangle that corresponds to $C(6,3)$. Use the combinations formula to verify that this entry in the triangle has the correct value.

Answer: See above. To verify: $C(6,3) = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20.$

5. (10 pts) Use the Binomial Theorem to find the coefficient of x^3 in the expansion of $(2x+5)^6$.

Solution: The Binomial Theorem gives us the expansion

$$(2x+5)^6 = \sum_{k=0}^{6} \binom{6}{k} (2x)^k (5)^{6-k}.$$

The requested coefficient comes from the term in this sum corresponding to $k=3$:

$$\binom{6}{3} (2x)^3 (5)^{6-3} = 20(2^3)(5^3)x^3.$$

Thus, the coefficient of x^3 in the expansion of $(2x+5)^6$ is $20 \cdot 2^3 \cdot 5^3$, or 20000.

6. (15 pts) There are 110 students in this year's incoming freshman class at Arrakis University. Every one of these incoming freshmen is taking at least one of the introductory courses in architecture, celestial mechanics, or ecology.

   There are 60 freshmen taking architecture, 75 taking celestial mechanics, and 45 taking ecology. Many students are taking more than one of these courses – in particular, 30 freshmen are taking both architecture and celestial mechanics, 25 are taking both architecture and ecology, and 20 are taking both celestial mechanics and ecology.

   (a) How many freshmen are taking *all three* introductory classes at the same time? Solution: Let A, C, and E denote the sets of students taking architecture, celestial mechanics, and ecology, respectively. We want to find $|A \cap C \cap E|$.

   We are given the following information, in terms of A, C, and E:

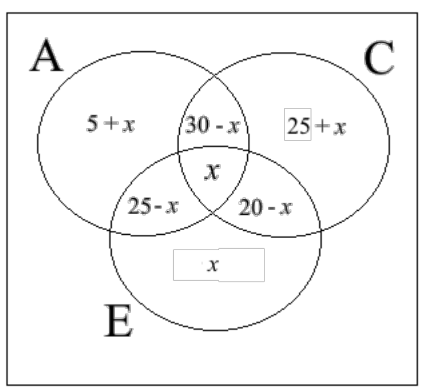
   - $|A \cup C \cup E| = 110$
   - $|A| = 60$, $|C| = 75$, and $|E| = 45$
   - $|A \cap C| = 30$, $|A \cap E| = 25$, and $|C \cap E| = 20$

   By the Principle of Inclusion and Exclusion (for three sets), the following must be true:

$$
\begin{aligned}
|A \cup C \cup E| &= |A| + |C| + |E| - (|A \cap C| + |A \cap E| + |C \cap E|) + |A \cap C \cap E| \\
110 &= 60 + 75 + 45 - (30 + 25 + 20) + |A \cap C \cap E| \\
110 &= 180 - 75 + |A \cap C \cap E| \\
110 &= 105 + |A \cap C \cap E| \\
5 &= |A \cap C \cap E|
\end{aligned}
$$

   Thus, there are five freshmen taking all three courses at the same time.

   Comment: A Venn diagram *could* be used here, but it's harder to set up the problem that way, since every number in the diagram depends on the value of the (unknown) number of elements on the intersection of all three sets. To avoid a great deal of "guess-and-check," the setup, with x denoting the unknown quantity, would have to look something like this:



   Working from the outside in, each entry is determined in such a way that the total for each set, or intersection of sets, is as given in the instructions. As a result, the sum of the individual entries is $105 + x$, which must equal 110; thus, $x = 5$.

(b) How many freshmen are taking architecture *or* ecology?

Solution: Here we are being asked to find $|A \cup E|$. By the Principle of Inclusion and Exclusion (for two sets),

$$\begin{aligned} |A \cup E| &= |A| + |E| - |A \cap E| \\ &= 60 + 45 - 25 \\ &= 80 \end{aligned}$$

Thus, 80 freshmen are taking architecture or ecology.

(c) How many freshmen are taking *only* celestial mechanics?

Solution: Since all 110 of our freshmen are taking architecture, ecology, or celestial mechanics, the set of those taking only celestial mechanics must be the *complement* of the set we enumerated in part (b). Therefore, there are $110 - 80 = 30$ freshmen taking only celestial mechanics.