Math 105 – Music & Mathematics – Introduction to Pythagorean Tuning

We begin with some terminology:

* “**Interval**” – the “distance” between two tones. Usually the tones are played either simultaneously (at the same time) or consecutively (one right after the other).
* “**Frequency**” – the number of vibrations per second (or “*Hertz*”) corresponding to a certain tone
* “**Frequency Ratio**” (of an interval) – the frequency of the higher tone divided by the frequency of the lower tone in an interval. Note that by this definition (higher divided by lower), the frequency ratio of an interval will always be greater than one.

The sound of an interval is determined primarily by its frequency ratio. Two of the most frequently occurring intervals in music are the “octave” and the “perfect fifth:”

* **Octave**: An octave is an interval whose frequency ratio is exactly 2. For example, an interval consisting of tones with frequencies 440 Hz and 880 Hz would be an octave. Tones that are separated by an octave create a sort of auditory illusion, in which two notes sound “the same” even while one is clearly higher than the other. For an example of this, pick out any two C’s on a piano, and play one followed by the other.
* **Perfect Fifth:** A perfect fifth is an interval whose frequency ratio is exactly 3/2, or 1.5. For example, an interval consisting of tones with frequencies 800 Hz and 1200 Hz would be a perfect fifth. As another example, the interval C-G on a standard piano is (approximately) a perfect fifth.

We will learn about other types of intervals later.

Every interval has its own unique sound. Musical instruments are built based largely on which frequencies its designer would like for it to be able to produce, which in turn dictates which intervals that instrument can produce. This statement does not apply to all instruments, however; certain instruments, such as string instruments (violins, cellos, guitars, etc.), the human voice, and the trombone, may generate practically any frequency within their range. However, even with this increased flexibility, it is standard practice to define a set of “allowable” frequencies which may be used to create music; we call such a set, together with the process by which it is devised, a “tuning system.”

**Tuning Systems**

A “tuning system” is a set of frequencies that may be used to produce music. Every tuning system has a foundational tone that’s called the “base tone,” or just the “base;” other tones to be included in the tuning system are determined based on the “base.”

There are many different ways to create a tuning system. We’ll be exploring just a few of these.

A primary goal in creating a tuning system is to “preserve” a variety of “nice” intervals – that is, to have frequencies that produce intervals that we would like to use in creating music. Note that there is some subjectivity involved here; however, there are at least a few types of interval that are universally perceived as “nice,” and these are the ones we’ll focus on.

Perhaps the most important of these “nice” intervals is the “octave” (as defined above), which is the foundation of most tuning systems. Every tuning system example that we’ll look at will preserve octaves – this means that for every frequency used in a tuning system, the frequency an octave higher (found by doubling the original frequency) and an octave lower (found by dividing the original frequency by 2) will automatically be included as well. This process of multiplying or dividing by 2 can, in theory, be continued indefinitely. In practice, there are limits to this – either physical limitations of the instrument being played, or (if that’s not a factor) the limits of human hearing. The human ear can typically detect frequencies in the 20 Hz – 20,000 Hz range, although this varies from person to person.

It turns out that, in order to fully develop a tuning system, all we need to explicitly work out is the set of frequencies between the base frequency and the tone one octave above the base frequency (i.e., the base frequency times two). Once this is accomplished, all other frequencies can be found, if needed, by raising/lowering each of these frequencies by octaves. This observation actually simplifies the study of tuning systems, since it allows us to define an entire tuning system simply by specifying the tones that will occur in a specific one-octave range. We’ll demonstrate in the following example.

Example: Let’s design our own simple tuning system, starting with a “base” of 450 Hz.

If we want to preserve octaves, we must include all frequencies obtained when we raise by an octave – that is, when we double the frequency. This requires us to include tones with frequencies of 900 Hz, 1800 Hz, 3600 Hz, etc.

Also in the interest of preserving octaves, we must include all frequencies obtained when we lower by an octave – that is, when we divide the frequency by 2. As a result, we also include tones with frequencies of 225 Hz, 112.5 Hz, 56.25 Hz, etc.

If we wanted to, we could stop here; however, a tuning system consisting entirely of octaves is not particularly interesting. Every tone in this system would sound very much the same!

To add some variety, let’s include a perfect fifth in our tuning system. A way to do so is to raise the base, 450 Hz, by a perfect fifth. Since the “frequency ratio” of a perfect fifth is 3/2 (or 1.5), we can multiply our base by 3/2 to create a perfect fifth: $450 ×\frac{3}{2}=675$. Thus, if we add a tone with frequency 675 Hz to our tuning system, then we’ll have a perfect fifth at our disposal, since $675/450$ reduces to 3/2.

Now that we’ve added 675 Hz to our tuning system, we must also include all tones which are obtained by raising or lowering this new tone by octaves. Raising gives us 1350 Hz, 2700 Hz, 5400 Hz, etc.; lowering gives us 337.5 Hz, 168.75 Hz, 84.375 Hz, and so on.

Let’s look at a complete list of frequencies that we’ve included so far. To keep it manageable, we’ll set a lower and upper cutoff of 50 Hz and 20,000 Hz. (All frequencies below are measured in Hz.)

{56.25, 84.375, 112.5, 168.75, 225, 337.5, 450, 675, 900, 1350, 1800, 2700, 3600, 5400, 7200, 10800, 14400}

Note that every frequency in this list is included as a result of raising/lowering either 450 Hz or 675 Hz by octaves. As noted earlier, this will always be the case for tuning systems that preserve octaves. Therefore, it’s not always necessary to list all frequencies in a tuning system, since most of them are implied by a relatively small set of numbers.

For our current example, it’s enough to state that we’re including 450 Hz and 675 Hz; the rest of the numbers in the above list are implied as a result. This principle will work for us in general – when describing tuning systems, we will typically only list the frequencies that between the base (450 Hz, in the preceding example) and the tone one octave above the base (900 Hz in the preceding example).

So, the tuning system in the preceding example could be completely described as follows: it has a base of 450 Hz, it also includes 675 Hz, and it preserves octaves. (That’s it!)

Note that we’ve also “accidentally” added some other intervals to our tuning system. For example, notice that 675 Hz and 900 Hz are both included. These two tones form a frequency ratio of $\frac{900}{675}=\frac{4}{3}=1.333…$ An interval with this ratio is known as a “perfect fourth.”

* A “perfect fourth” is an interval whose frequency ratio is exactly 4/3, or 1.333… . For example, an interval consisting of tones with frequencies 600 Hz and 800 Hz would be a perfect fourth, since the higher frequency is 4/3 of the lower frequency. Another (approximate) example of a perfect fourth is the C-F interval on a standard piano.

For any two frequencies in our tuning system so far, the interval they form can be described in terms of octaves, fifths, and/or fourths. For example, suppose we look at the interval consisting of the tones with frequencies 225 Hz and 1350 Hz. Since these are more than an octave apart, think of the “distance” between these tones as follows. First, if we raise 225 Hz by octaves, we get 450 Hz (one octave), then 900 Hz (two octaves). Now, we observe that 1350 Hz is 3/2 of 900 Hz (in other words, 1350/900=1.5, or 3/2), which means the interval formed by 900 Hz and 1350 Hz is a perfect fifth. Therefore, the interval between 225 Hz and 1350 Hz would be best described as “two octaves plus a perfect fifth.”

Now we’re going to design a more sophisticated tuning system, still using octaves and fifths as our building blocks. Before delving further into these examples, though, it will help to have the following notation at our disposal…

**NOTATION: R**$n$ **and L**$n$(where $n$” is a counting number: 1, 2, 3, etc.)

Given a “base” tone with frequency $f$, we define “R$n$” to be the frequency of the tone which is found by raising the “base” tone by a perfect fifth $n$ times, then lowering by octaves (as necessary) to obtain a result between $f$ and $2f$. Similarly, we define “L$n$” to be the frequency of the tone which is found by *lowering* the “base” tone by a perfect fifth $n$ times, then raising by octaves (as necessary) to obtain a result between $f$ and $2f$.

Examples: If we select a base tone with a frequency of 450 Hz, then R1 would be found by raising the base tone by a perfect fifth; numerically, this means we multiply by 3/2, or 1.5: $R1=450×\frac{3}{2}=675 Hz$.

To find R2, we’d multiply by 1.5 again: $675×\frac{3}{2}=1012.5 Hz$; however, since 1012.5 is greater than twice the base frequency, we’d lower it by an octave; so, $R2=1012.5×\frac{1}{2}=506.25 Hz$.

We’ll discuss L1, L2, etc. a little bit later.

**Pentatonic tuning system**

A “pentatonic” tuning system is a system in which each octave is divided into five steps; in other words, we develop a five-tone scale. A standard way of doing this is to first select a base tone, and then add the tones whose frequencies are **R1, R2, R3 and R4** (based on the above definition). These five notes form a “pentatonic scale;” the corresponding tuning system would then include these five tones, along with any other tone obtained through raising/lowering by octaves.

Example: If we select a base tone with frequency $f=320 Hz$, then our “pentatonic scale” includes the following frequencies:

R1: We raise the base frequency by a perfect fifth: $320×\frac{3}{2}=480$ **Hz.** (Note: 480 is less than twice the base frequency, so it’s part of our scale)

R2:We start by raising R1 by a perfect fifth: $480×\frac{3}{2}=720$. Since this is more than twice the base frequency, we need to lower it by an octave. So, R2 = $720×\frac{1}{2}=360$ **Hz.**

R3: We raise R2 by a perfect fifth: $360×\frac{3}{2}=540$ Hz. This is less than double the base frequency, so **R3** $= 540$ **Hz.**

R4: We raise R3 by a perfect fifth: $540×\frac{3}{2}=810$ Hz. This is more than twice the base frequency, so we lower it by an octave: $810×\frac{1}{2}=405$ **Hz.**

Thus, our “pentatonic scale” includes the frequencies: 320 Hz, 360 Hz, 405 Hz, 480 Hz, and 540 Hz. (Optional: the “top” of the scale is usually taken to be the base tone raised by an octave; so, we could list 640 Hz as the “top” of the scale. This would technically give us a sixth note in our pentatonic scale, but as notes separated by octaves are considered to be “equivalent,” we still only have five truly distinct notes in our scale.)

Note: the calculation of R1, R2, R3, etc. is a repetitive process; practice it to make sure you get the hang of it. For example, if we continue raising by perfect fifths from the preceding example, we’d get the following results (make sure you can duplicate these results on your own):

R5: 607.5 Hz; R6: 455.625 Hz; R7: 341.71875 Hz; R8: 512.578125 Hz; R9: 384.43359375 Hz

Pythagorean Tuning System
The Pythagorean system is the original basis of the “twelve-tone scale,” which is commonly used in western music. The standard piano keyboard is based on a twelve-tone scale (as opposed to a pentatonic scale). The Pythagorean tuning system is based on the twelve-tone scale consisting of the following frequencies:

The “base” frequency; **R1, R2, R3, R4, R5** and **R6**; and also **L1, L2, L3, L4** and **L5**.

Example: Let’s develop a Pythagorean tuning system based on a base frequency of 320 Hz (as used in the preceding example). We’ve already worked out several of the frequencies (see above); all we still need to do is find L1, L2, L3, L4 and L5.

To find L1, we first lower the base frequency by a perfect fifth. That is, we must find a frequency, call it $x$, such that the frequency ratio $320÷x$ equals 3/2. That is, $\frac{320}{x}=\frac{3}{2}.$ There are (at least) two ways to solve for x:

1. Cross-multiply: $320⋅2=3x$, so $x=\frac{640}{3}≈213.333$.

2. Divide (rather than multiply) by the frequency ratio 3/2. The rationale here is that lowering is the opposite of raising; thus, if we multiply to raise by a perfect fifth, then we should divide in order to lower by a perfect fifth.

This gives us: $x=320÷\frac{3}{2}$. Recall now that dividing by a fraction is equivalent to multiplying by its reciprocal; therefore, $x=\frac{320}{1}÷\frac{3}{2}=\frac{320}{1}×\frac{2}{3}=\frac{640}{3}≈213.33$ Hz.

(Note: in practice, it’s easier to just “invert and multiply” when dividing by a fraction, which is the second method shown above. That’s what we’ll usually do, rather than the “cross-multiplication” method shown previously. We showed two methods here, though, to make the point that dividing by a frequency ratio – in this case, 3/2 – does actually solve the problem we’re interested in.)

This isn’t L1 yet – remember, we want frequencies between the base frequency (320 in this example) and twice the base frequency (640). Our answer above, 640/3 is too low, so we must raise it by an octave:

$L1= \frac{640}{3}×\frac{2}{1}=\frac{1280}{3}≈426.67 $ Hz.

Proceeding similarly, next we find L2. Note that in order to avoid introducing unnecessary rounding errors into your calculations, you should use fractions rather than decimals to calculate each frequency. Only round off when writing your final answer (usually two decimal places is fine) for each frequency, but use the fraction – not the decimal – to calculate additional frequencies. (Note: this wasn’t as big of a deal for R1, R2, etc., because those frequencies worked out to exact values; we didn’t need to round off any repeating decimals. Here, though, the decimals repeat, so we have to be more careful about rounding…)

So, to find L2, L3, etc., we proceed as follows.

L2: $\frac{1280}{3} ÷\frac{3}{2}=\frac{1280}{3}×\frac{2}{3}=\frac{2560}{9}≈284.44444…$. Now, since this is less than the base frequency of 320 Hz, we need to raise it by an octave again: $\frac{2560}{9}×\frac{2}{1}=\frac{5120}{9}≈568.8888….$, or about $568.89 Hz. $

So, $L2≈568.89 Hz$.

Proceed similarly to find L3, L4 and L5. You should get the following results. (As before, make sure that you can duplicate these results on your own.)

L3: Exact answer: $\frac{10240}{27}$. Decimal approximation: 379.26 Hz

L4: Exact answer: $\frac{40960}{81}$. Decimal approximation: 505.68 Hz

L5: Exact answer: $\frac{81920}{243}$. Decimal approximation: 337.12 Hz.

Note: don’t be intimidated by the large numerators and denominators in these fractions; you can use a calculator to find each of these. (For example: in L3, the numerator 10240 is just 2560 multiplied by 2 twice.) Finding these fractions correctly is really just a matter of keeping your work organized! After that, use a calculator to find a decimal approximation to that fraction; but, again, remember to use the fraction, not the decimal, to find the next frequency in the scale. This is the only way to ensure answers which are exactly correct.

Thus, our Pythagorean “scale” consists of the following frequencies. These are the base frequency followed by R1 through R6 and L1 through L5 (rearranged from smallest to largest, and each rounded to two decimal places):

**320 (base), 337.12, 360, 379.26, 405, 426.67, 455.63, 480, 505.68, 540, 568.89, 607.5, 640 (octave)**

So, if a keyboard were tuned in such a way that one of the keys had a frequency of 320 Hz, then the next twelve keys would be tuned to approximately the frequencies shown in the list above. This is an example of a Pythagorean scale, or tuning system.

After you’ve read this handout, along with the corresponding class notes and section of the text, start working on some of the Pythagorean Tuning practice exercises (under the “homework” link on the web page).