Math 105 – Test #3 Review

Test #3 will be given in class on Friday, May 5. Calculators will be allowed; however, all work must be shown in order to receive credit for any answers. This document summarizes concepts you should understand, and some of the types of problems you should be able to solve, for the test.

CHANGE RINGING and PERMUTATIONS

A "permutation" is a method of rearranging the entries in an ordered list. We write permutations using "cycle notation," as demonstrated in class. The set of permutations of n objects (where n is any fixed counting number) forms a group, since there is an identity permutation (the one which doesn't move anything), every permutation has an opposite, and the combination of any two permutations is another permutation.

"Cycle notation" allows us to represent any permutation in terms of "cycles" – in particular, the set(s) of positions that are interchanged by the permutation. A cycle's length corresponds to the number of positions in the list it involves – e.g., "2-cycle" (swaps two positions), "3-cycle" (involves 3 positions in the list, "4-cycle," etc. The length of a cycle indicates the number of repetitions it would take to "reset" all of the positions involved in that cycle.

"Change Ringing" is a musical application of permutations. Essentially "changes" are permutations that consist only of disjoint "adjacent swaps" – that is, permutations that only exchange adjacent (or consecutive) positions in the list. A swap is also called a "2-cycle" since it only involves two positions.

One goal of "change ringing" is to use a simple repeating pattern (i.e., repeating two or more distinct permutations) to generate a large number of different "changes," or rearrangements, of the bells. One trick used to accomplish this is to occasionally interrupt the repeating pattern with a permutation not already included. This is an application of the idea of "subgroups" (for the initial change ringing pattern) and "cosets" (induced by the changes to the pattern), each of which were introduced in the Variations and Groups unit earlier in the course.

If a change ringing strategy for n bells produces every possible change, without repeating any of them, then it is called an "extent" on n bells. Note that there are $n!$ (n factorial) possible permutations for n bells, so an extent on more than 4 bells (which we actually found in class) would take a very long time to carry out.

COUNTING PROBLEMS

We discussed a variety of different types of counting problems in class. Here's a review of some of the main ideas:

Multiplication Principle:
If a process or task can be completed in a sequence of steps, and if the number of available options for completing each step is fixed (i.e., doesn't depend on choices made at preceding steps), then the number of ways to complete the process or task is found by listing the number of options at each step, and multiplying these numbers together.

Permutations:
A permutation of a given set is an ordered selection, without repetition, of objects from that set. Examples: rearrangements of a word with no repeated letters; selection of officers (president, vice-president, etc.) from the members of a club; melodies with no repeated notes selected from a specific set of notes.

The number of permutations of k elements, selected from a set of n elements, is given by the formula

$$P(n,k) = \underbrace{n \times (n-1) \times \ldots \times (n-k+1)}_{product\ of\ k\ different\ numbers}.$$

Combinations:
A combination is an unordered selection, without repetition, of objects from a given set. Examples: selection of a *committee* (no specific ranks or titles) from the members of a club; choosing toppings for a pizza; chords selected from a specific set of notes.

The number of combinations of k elements, selected from a set of n elements, is equal to $P(n,k)$ divided by $k!$:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n \times (n-1) \times \ldots \times (n-k+1)}{k \times (k-1) \times \ldots \times 2 \times 1}.$$

Note that both the top and the bottom of this fraction are "descending products," each with exactly k factors. The product on the top starts from n, and the one on the bottom starts from k.

This formula works because each combination of k objects could be arranged into $k!$ different permutations; so, there are $k!$ times as many permutations as there are combinations.

Combinations (cont.)
We discussed the following “special cases,” for which the combinations formula isn’t necessary:

- $C(n,0)$ and $C(n,n)$ are both always equal to 1, for any counting number n
- $C(n,1)$ and $C(n,n-1)$ are both always equal to n, for any counting number n

More generally, for any two whole numbers n and k (as long as k is between 0 and n), we have

$$C(n,k) = C(n,n-k)$$

This is the “symmetry” rule that we discussed in class. It’s based on the idea that selecting which k objects (out of a set of n objects) to include in a combination is equivalent to selecting which $n-k$ objects to exclude from that combination.

Note: This symmetry rule, $C(n,k) = C(n,n-k)$, comes in handy whenever k is a “large” number relative to n. As an example, suppose we want to find the value of $C(50{,}48)$. This is difficult to compute directly with the combinations formula:

$$C(50{,}48) = \frac{50 \times 49 \times 48 \times \ldots \times 5 \times 4 \times 3}{48 \times 47 \times 46 \times \ldots \times 3 \times 2 \times 1}$$

However, the symmetry rule tells us that $C(50{,}48) = C(50{,}2)$ (since 50-48-2), which is relatively easy to compute:

$$C(50{,}48) = C(50{,}2) = \frac{50 \times 49}{2 \times 1} = 1225.$$

Other types of counting problems
Many counting problems are not straightforward permutation/combination problems, but rather require the use of both ideas at different steps in a selection process – for example, “word rearrangements” (finding the number of ways to rearrange the letters in a given word), or melodies in which one or more notes must occur a specified number of times. You can find several such examples in the homework and in the practice exercises. You can reasonably expect to see at least one problem along these lines on the third test and/or on the final exam.

PASCAL'S TRIANGLE

Pascal's Triangle is an array of numbers, arranged in a triangular format, as an alternative method for counting combinations. The triangle's first row consists of two 1's; beyond that, each row begins and ends with a 1, but other numbers are filled in by adding consecutive numbers in the preceding row, and writing their sum beneath and between those two numbers. This has the effect of "staggering" the numbers in each row relative to the row above. The first six rows of the triangle are shown below:

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

As an example, the two 10s in the fifth row are added to get 20; this number is placed in the sixth row, aligned horizontally between the two 10s. Similarly, we could add, say, the 15 and 20 in the sixth row to get 35; this 35 would be a number in the seventh row, placed "between" the 15 and 20.

It turns out that $C(n,k)$ is the k^{th} entry in the nth row of Pascal's triangle, provided that we start counting entries in each row from zero – that is, the leftmost 1 is the "zeroth" entry in each row. This works in general due to a rule known as "Pascal's Identity:"

$$C(n,k) = C(n-1,k-1) + C(n-1,k)$$

We discussed a few other interesting patterns that can be found in Pascal's triangle. An important one of these patterns is that Pascal's Triangle is symmetric, in the sense that each row reads the same left-to-right or right-to-left. More specifically, in row n of the triangle, if we count k places in from the left to find $C(n,k)$, we get the same result by counting k places in from the right to find $C(n,n-k)$. This is how the symmetry rule for combinations (see above under "special cases") shows up in Pascal's Triangle.

SEQUENCES

- Be familiar with the types of sequences seen in class: arithmetic, geometric, or recurrence relations. These are all discussed in the “sequences handout,” and several examples can be found in the practice exercise set on sequences.
- An application of sequences that we spent some time discussing in class was counting the number of n-beat rhythms (or even melodies) that can be written under certain restrictions – for example, using only half-notes and quarter-notes (the Fibonacci sequence example).

FINAL THOUGHTS

- This document is meant to *summarize* what we’ve covered to this point. While it is pretty thorough, I do not guarantee that it is exhaustive. In general, anything we covered in class, or anything that was covered in the assigned reading, after the second test is fair game for the third test.

- Calculators will be allowed for this test; however, you must show all of your work on each problem. If you write an answer without any work or explanation showing where it came from, then you will not receive credit for it.